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THE LEBESGUE SUMMABILITY OF DOUBLE TRIGONOMETRIC INTEGRALS

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Abstract. We recall that the Lebesgue summability of trigonometric series (see [6]) or trigonometric integrals (see [4] and [2]) is defined in terms of the symmetric differentiability of the formally integrated series or integral, respectively. In the present paper we define the Lebesgue summability of double trigonometric integrals, and extend a previous theorem from single to double trigonometric integrals.

1. Introduction: the Lebesgue summability of single trigonometric integrals

Motivated by the notion of Lebesgue summability of trigonometric series (see [6, Vol. I, pp. 321–322]), O. Szász [4] defined the Lebesgue summability of single trigonometric integrals as follows.

Let $f:\mathbb{R}\to\mathbb{C}$ be such that it is integrable in Lebesgue's sense over any bounded interval, in symbols: $f\in L^1_{\text{loc}}(\mathbb{R})$. We consider the trigonometric integral

$$\int_{\mathbb{R}} f(s)e^{isx}ds, \quad x\in\mathbb{R}, \tag{1.1}$$

with the symmetric partial integrals

$$I_S(x) := \int_{|s|<S} f(s)e^{isx}ds, \quad S>0.$$

The integral (1.1) is said to converge at a point $x\in\mathbb{R}$ to the finite limit ℓ if

$$\lim_{S\to\infty} I_S(x) = \ell.$$

A formal integration of the integrand in (1.1) with respect to x gives

$$\int_{\mathbb{R}} f(s)\frac{e^{isx}}{is}ds =: L(x), \quad x\in\mathbb{R}. \tag{1.2}$$

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The definition of $L(x)$ is interpreted formally, since the integral in (1.2) may not exist in Lebesgue's sense.

We recall (see [4]) that the Lebesgue summability of the integral (1.1) is formally defined in terms of the symmetric differentiability of L. We say that the integral (1.1) is Lebesgue summable at some point $x \in \mathbb{R}$ to the finite limit $\ell \in \mathbb{C}$ if

$$\frac{\Delta L(x;h)}{2h} := \frac{L(x+h)-L(x-h)}{2h} = \int_{\mathbb{R}} f(s)e^{isx}\frac{\sin sh}{sh}ds \to \ell \quad \text{as} \quad 0<h\to 0. \tag{1.3}$$

The following theorem was proved in [4, Theorem 2′].

THEOREM 1. *If $f:\mathbb{R}\to\mathbb{C}$ is such that $f\in L^1_{\mathrm{loc}}(\mathbb{R})$ and*

$$\lim_{S\to\infty}\frac{1}{S}\int_{|s|<S}|sf(s)|ds = 0, \tag{1.4}$$

then the integral in (1.3) exists in Lebesgue's sense and we have uniformly in x that

$$\lim_{h\to 0}\left(\frac{\Delta L(x;h)}{2h} - I_{1/h}(x)\right) = 0, \quad h>0.$$

In other words, under condition (1.4) the integral (1.1) is Lebesgue summable at a point $x\in\mathbb{R}$ to some finite limit if and only if (1.1) converges at x to the same limit.

REMARK 1. We recall (see, e.g., [6, Vol. II, p. 246]) that the Fourier transform $\hat{f}$ of a function $f\in L^1(\mathbb{R})$ is defined by

$$\hat{f}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} f(s)e^{-isx}ds, \quad x\in\mathbb{R} \tag{1.5}$$

(cf. (1.1)). Clearly, Theorem 1 can be reformulated in terms of the Lebesgue summability of the integral in (1.5) under the same condition (1.4).

REMARK 2. We also note that related results have just proved by J. Vindas [5] on the relation between Lebesgue summability and some other summation methods.

2. New result: the Lebesgue summability of double trigonometric integrals

Let $f:\mathbb{R}^2\to\mathbb{C}$ be such that it is integrable in Lebesgue's sense over any bounded rectangle of $\mathbb{R}^2$, in symbols: $f\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. We consider the double trigonometric integral

$$\int\int_{\mathbb{R}^2} f(s,t)e^{i(sx+ty)}dsdt, \quad (x,y)\in\mathbb{R}^2, \tag{2.1}$$

with the symmetric rectangular partial integrals

$$I_{S,T}(x,y) := \int_{|s|<S}\int_{|t|<T} f(s,t)e^{i(sx+ty)}dsdt, \quad S,T>0. \tag{2.2}$$

We say that the double integral (2.1) converges in Pringsheim's sense at a point $(x,y) \in \mathbb{R}^2$ to the finite limit ℓ, in symbols:

$$\lim_{S,T\to\infty} I_{S,T}(x,y) = \ell,$$

if for every $\varepsilon > 0$ there exists $\rho_1 = \rho_1(\varepsilon) > 0$ such that

$$|I_{S,T}(x,y) - \ell| < \varepsilon \quad \text{if} \quad S,T > \rho_1.$$

This notion of convergence was introduced by Pringsheim [3] for double series of numbers; see for multiple series , e.g., in [6, Vol. II, p. 303, just after formula (1.18)] by Zygmund, without indication of the term 'in Pringsheim's sense'. As to this convergence notion for double integrals over $\mathbb{R}^2_+$, see for example in [1].

A formal integration of the integrand in (2.1) with respect to both x and y gives

$$\int\int_{\mathbb{R}^2} f(s,t)\frac{e^{i(sx+ty)}}{i^2 st} dsdt =: L(x,y), \quad (x,y) \in \mathbb{R}^2. \tag{2.3}$$

Again the definition of $L(x,y)$ is interpreted formally (as in the case of (1.2)), since the double integral in (2.3) may not exist in Lebesgue's sense.

Motivated by (1.3), we say that the integral (2.1) is Lebesgue summable at some point $(x,y) \in \mathbb{R}^2$ to the finite limit $\ell \in \mathbb{C}$ if

$$\begin{aligned} \frac{\Delta L(x,y;h,k)}{4hk} &:= \frac{1}{4hk}\Big(L(x+h,y+k) - L(x-h,y+k) \\ &\quad - L(x+h,y-k) + L(x-h,y-k)\Big) \\ &= \int\int_{\mathbb{R}^2} f(s,t)e^{i(sx+ty)}\frac{\sin sh}{sh}\,\frac{\sin tk}{tk} dsdt \to \ell \\ &\quad \text{as } 0 < h,k \to 0. \end{aligned} \tag{2.4}$$

We note that $\Delta L(x,y;h,k)$ may be formally called the symmetric difference of the function L at the point $(x,y) \in \mathbb{R}^2$ with the mesh sizes $h,k > 0$, while the limit in (2.4) (if it exists) may be formally called the symmetric mixed derivative of L at (x,y).

Now, our main result is formulated in the following

THEOREM 2. *If* $f : \mathbb{R}^2 \to \mathbb{C}$ *is such that* $s\int_{\mathbb{R}} f(s,t)dt \in L^1_{\text{loc}}(\mathbb{R},ds)$ *and*

$$\lim_{S,T\to\infty} \frac{1}{S}\int_{|s|<S}\int_{|t|<T} |sf(s,t)|dtds = 0 \tag{2.5}$$

as well as $t\int_{\mathbb{R}} f(s,t)ds \in L^1_{\text{loc}}(\mathbb{R},dt)$ *and*

$$\lim_{S,T\to\infty} \frac{1}{T}\int_{|s|<S}\int_{|t|<T} |tf(s,t)|dtds = 0, \tag{2.6}$$

then the double integral in (2.4) *exists in Lebesgue's sense and we have uniformly in* (x,y) *that*

$$\lim_{h,k\to 0}\Big(\frac{\Delta L(x,y;h,k)}{4hk}-I_{1/h,1/k}(x,y)\Big)=0,\quad h,k>0. \tag{2.7}$$

In other words, under conditions (2.5) and (2.6) the double integral (2.1) is Lebesgue summable at some point $(x,y)\in\mathbb{R}^2$ to a finite limit if and only if (2.1) converges in Pringsheim's sense at (x,y) to the same limit.

REMARK 3. Analogously to (1.5), the double Fourier transform $\hat{f}$ of a function $f\in L^1(\mathbb{R}^2)$ is defined by

$$\hat{f}(x,y):=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}f(s,t)e^{-i(sx+ty)}dsdt,\quad (x,y)\in\mathbb{R}^2 \tag{2.8}$$

(cf. (2.1)). Clearly, Theorem 2 can be reformulated in terms of the Lebesgue summability of the double integral in (2.8) under the same conditions (2.5) and (2.6).

3. Auxiliary results

Conditions (2.5) and (2.6) in our main Theorem 2 resemble the Tauberian hypothesis in Tauber's first classical Tauberian theorem for power series. The technique employed in the proof of Theorem 2 is a two-dimensional generalization of that employed by the second aouthor in [2]. The proof is based on Lemmas 1–5 below. Lemma 1 is taken from [2]. Lemmas 2–4 connect the Tauberian assumptions with the growth order of the tails of certain related integrals, while Lemma 5 is folklore.

LEMMA 1. (see in [2, Lemma 2′]) *If* $g:\mathbb{R}\to\mathbb{C}$ *is such that* $g\in L^1_{\rm loc}(\mathbb{R})$ *and*

$$\lim_{S\to\infty}\frac{1}{S}\int_{|s|<S}|sg(s)|ds=0, \tag{3.1}$$

then

$$\lim_{S\to\infty}S\int_{|s|>S}\Big|\frac{g(s)}{s}\Big|ds=0. \tag{3.2}$$

LEMMA 2. *If* $f:\mathbb{R}^2\to\mathbb{C}$ *is such that* $s\int_{\mathbb{R}}f(s,t)dt\in L^1_{\rm loc}(\mathbb{R},ds)$ *and*

$$\lim_{S\to\infty}\frac{1}{S}\int_{|s|<S}\int_{\mathbb{R}}|sf(s,t)|dtds=0, \tag{3.3}$$

then

$$\lim_{S\to\infty}S\int_{|s|>S}\int_{\mathbb{R}}\Big|\frac{f(s,t)}{s}\Big|dtds=0. \tag{3.4}$$

Proof. Lemma 2 is a trivial consequence of Lemma 1 in the case when

$$g(s):=s\int_{\mathbb{R}}f(s,t)dt,\quad s\in\mathbb{R}.\quad\square$$

The symmetric counterpart of Lemma 2 reads as follows.

LEMMA 3. *If* $f:\mathbb{R}^2\to\mathbb{C}$ *is such that* $t\int_{\mathbb{R}} f(s,t)ds \in L^1_{\mathrm{loc}}(\mathbb{R},dt)$ *and*

$$\lim_{T\to\infty}\frac{1}{T}\int_{\mathbb{R}}\int_{|t|<T}|tf(s,t)|dtds=0, \tag{3.5}$$

then

$$\lim_{T\to\infty} T\int_{\mathbb{R}}\int_{|t|>T}\Big|\frac{f(s,t)}{t}\Big|dtds=0. \tag{3.6}$$

LEMMA 4. *If* $f:\mathbb{R}^2\to\mathbb{C}$ *is such that* $f\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ *and*

$$\lim_{S,T\to\infty}\frac{1}{ST}\int_{|s|<S}\int_{|t|<T}|stf(s,t)|dsdt=0, \tag{3.7}$$

then

$$\lim_{S,T\to\infty} ST\int_{|s|>S}\int_{|t|>T}\Big|\frac{f(s,t)}{st}\Big|dsdt=0. \tag{3.8}$$

Proof. By (3.7), for every $\varepsilon>0$ there exists $\rho_2=\rho_2(\varepsilon)>0$ such that

$$\frac{1}{ST}\int_{|s|<S}\int_{|t|<T}|stf(s,t)|dsdt<\varepsilon \quad \text{if} \quad S,T>\rho_2. \tag{3.9}$$

For the sake of brevity in writing, we will use the following notation for the dyadic blocks of numbers:

$$D_p(S):=\{s\in\mathbb{R}:2^pS<|s|\leqslant 2^{p+1}S\}, \quad \text{where} \quad S>0 \quad \text{and} \quad p=0,1,2,\ldots.$$

Clearly, we have

$$\int_{D_p(S)}\int_{D_q(T)}|stf(s,t)|dsdt\geqslant 2^{p+q}ST\int_{D_p(S)}\int_{D_q(T)}|f(s,t)|dsdt$$

and

$$\int_{D_p(S)}\int_{D_q(T)}\Big|\frac{f(s,t)}{st}\Big|dsdt\leqslant\frac{1}{2^{p+q}ST}\int_{D_p(S)}\int_{D_q(T)}|f(s,t)|dsdt.$$

It follows from these two inequalities and (3.9) that

$$\int_{D_p(S)}\int_{D_q(T)}\Big|\frac{f(s,t)}{st}dsdt\leqslant\frac{1}{(2^{p+q}ST)^2}\int_{D_p(S)}\int_{D_q(T)}|stf(s,t)|dsdt$$

$$<\frac{1}{(2^{p+q}ST)^2}2^{p+q+2}ST\varepsilon=\frac{4\varepsilon}{2^{p+q}ST} \quad \text{if} \quad S,T>\rho_2 \quad \text{and} \quad p,q=0,1,2,\ldots.$$

Hence we conclude that

$$ST\int_{|s|>S}\int_{|t|>T}\Big|\frac{f(s,t)}{st}\Big|dsdt=ST\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\int_{D_p(S)}\int_{D_q(T)}\Big|\frac{f(s,t)}{st}\Big|dsdt$$

$$<4\varepsilon\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2^{p+q}}=16\varepsilon \quad \text{if} \quad S,T>\rho_2.$$

Since $\varepsilon>0$ is arbitrary, this proves (3.8). $\square$

LEMMA 5. *For every real number $u \neq 0$, we have*

$$0 \leqslant 1 - \frac{\sin u}{u} \leqslant 2|u|. \tag{3.10}$$

Proof. By the Taylor expansion of the function $\sin u$, we have

$$0 \leqslant 1 - \frac{\sin u}{u} \leqslant \frac{u^2}{3!} \quad \text{if} \quad |u| \leqslant 1.$$

Clearly, we also have that

$$0 \leqslant 1 - \frac{\sin u}{u} \leqslant 2|u| \quad \text{if} \quad |u| \geqslant 1.$$

Combining these two inequalities gives (3.10). □

4. Proof of Theorem 2

By conditions (2.5) and (2.6), we may apply Lemmas 2 and 3. As a result, it follows from (3.4) and (3.6) that

$$\frac{f(s,t)}{s} \in L^1(\{s \in \mathbb{R} : |s| > S\} \times \mathbb{R})$$

and

$$\frac{f(s,t)}{t} \in L^1(\mathbb{R} \times \{t \in \mathbb{R} : |t| > T\})$$

for large enough S and T, respectively. Since $f(s,t) \in L^1_{\text{loc}}(\mathbb{R}^2)$, it follows that the double integral in (2.4) exists in Lebesgue's sense.

Let $h > 0$ and $k > 0$ be arbitrary real numbers. Keeping notations in (2.2) and (2.4) in mind, the difference in (2.7) can be represented as follows

$$\begin{aligned}
&\frac{\Delta L(x,y;h,k)}{4hk} - I_{1/h,1/k}(x,y) \qquad (4.1)\\
&= \int\int_{\mathbb{R}^2} f(s,t)e^{i(sx+ty)} \frac{\sin sh}{sh} \frac{\sin tk}{tk} dsdt \\
&\quad - \int_{|s|<1/h}\int_{|t|<1/k} f(s,t)e^{i(sx+ty)} dsdt \\
&= \int_{|s|>1/h}\int_{|t|>1/k} f(s,t)e^{i(sx+ty)} \frac{\sin sh}{sh} \frac{\sin tk}{tk} dsdt \\
&\quad + \Big(\int_{\mathbb{R}}\int_{|t|<1/k} f(s,t)e^{i(sx+ty)} \frac{\sin sh}{sh} \frac{\sin tk}{tk} dsdt \\
&\quad - \int_{|s|<1/h}\int_{|t|<1/k} f(s,t)e^{i(sx+ty)} \frac{\sin tk}{tk} dsdt\Big)
\end{aligned}$$

$$+\Big(\int_{|s|<1/h}\int_{\mathbb{R}} f(s,t)e^{i(sx+ty)}\frac{\sin sh}{sh}\frac{\sin tk}{tk}dsdt$$

$$-\int_{|s|<1/h}\int_{|t|<1/k} f(s,t)e^{i(sx+ty)}\frac{\sin sh}{sh}dsdt\Big)$$

$$-\int_{|s|<1/h}\int_{|t|<1/k} f(s,t)e^{i(sx+ty)}\Big(\frac{\sin sh}{sh}-1\Big)\Big(\frac{\sin tk}{tk}-1\Big)dsdt$$

$$=: J^{(1)}_{1/h,1/k}(x,y)+J^{(2)}_{1/h,1/k}(x,y)+J^{(3)}_{1/h,1/k}(x,y)+J^{(4)}_{1/h,1/k}(x,y), \quad \text{say.}$$

First, by (2.5) (as well as by (2.6)), we may apply Lemma 4 to obtain that

$$\Big|J^{(1)}_{1/h,1/k}(x,y)\Big| \leqslant \frac{1}{hk}\int_{|s|>1/h}\int_{|t|>1/k}\Big|\frac{f(s,t)}{st}\Big|dsdt \to 0 \quad \text{as} \quad h,k\to 0. \tag{4.2}$$

Second, we rewrite $J^{(2)}_{1/h,1/k}(x,y)$ into the following equivalent form:

$$J^{(2)}_{1/h,1/k}(x,y) = \int_{|s|<1/h}\int_{|t|<1/k} f(s,t)e^{i(sx+ty)}\Big(\frac{\sin sh}{sh}-1\Big)\frac{\sin tk}{tk}dsdt$$

$$+\int_{|s|>1/h}\int_{|t|<1/k} f(s,t)e^{i(sx+ty)}\frac{\sin sh}{sh}\frac{\sin tk}{tk}dsdt.$$

By (2.5), we may apply Lemma 2 to obtain that

$$\Big|J^{(2)}_{1/h,1/k}(x,y)\Big| \leqslant 2h\int_{|s|<1/h}\int_{|t|<1/k}|sf(s,t)|dsdt \tag{4.3}$$

$$+\frac{1}{h}\int_{|s|>1/h}\int_{|t|<1/k}\Big|\frac{f(s,t)}{s}\Big|dsdt \to 0 \quad \text{as} \quad h,k\to 0,$$

where we also used Lemma 5.

Third, the symmetric counterpart of (4.3) can be proved in an analogous way:

$$\Big|J^{(3)}_{1/h,1/k}(x,y)\Big| \leqslant 2k\int_{|s|<1/h}\int_{|t|<1/k}|tf(s,t)|dtds \tag{4.4}$$

$$+\frac{1}{k}\int_{|s|<1/h}\int_{|t|>1/k}\Big|\frac{f(s,t)}{t}\Big|dtds \to 0 \quad \text{as} \quad h,k\to 0,$$

thanks to (2.6) and Lemmas 3 and 5.

Fourth, by either one of the conditions (2.5) or (2.6), we may apply Lemma 4 to obtain that

$$\Big|J^{(4)}_{1/h,1/k}(x,y)\Big| \leqslant 4hk\int_{|s|<1/h}\int_{|t|<1/k}|stf(s,t)|dsdt \to 0 \quad \text{as} \quad h,k\to 0. \tag{4.5}$$

Combining (4.1)–(4.5) yields (2.7) to be a proved. □

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