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ON AN INEQUALITY FOR THE RATIO OF GAMMA FUNCTIONS

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Abstract. The following inequality relating to the ratio of the gamma functions

$$\alpha \log x \leqslant x - \frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{x}\right)},$$

where α is a suitable constant, is established for every $x > 0$. This inequality gives a contribution to the recent results, proved by several authors, involving the functions $\Gamma(x)$ and $\Gamma\left(\frac{1}{x}\right)$. It also gives an alternative proof of a conjecture formulated by D. Kershaw and recently proved by G.J.O. Jameson and T.P. Jameson [5].

1. Introduction and main result

In 1974, W. Gautschi [3] proved the following inequality, conjectured by R. Uppuluri, involving the harmonic mean of $\Gamma(x)$ and $\Gamma\left(\frac{1}{x}\right)$

$$\frac{2}{1/\Gamma(x)+1/\Gamma\left(\frac{1}{x}\right)} \geqslant 1, \qquad \forall x > 0, \tag{1.1}$$

which can be written in equivalent form as follows

$$\frac{\Gamma(x)+\Gamma\left(\frac{1}{x}\right)}{2} \leqslant \Gamma(x)\Gamma\left(\frac{1}{x}\right), \qquad \forall x > 0, \tag{1.2}$$

where

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad z > 0, \tag{1.3}$$

denotes the classical Euler's gamma function. As a consequence of the well known inequalities between the harmonic, geometric and arithmetic means, inequality (1.1) implies

$$\Gamma(x)\Gamma\left(\frac{1}{x}\right) \geqslant 1, \qquad \forall x > 0. \tag{1.4}$$

An alternative proof of (1.4) was given by Kairies [6].

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A first extension of (1.4) was presented by Laforgia and Sismondi [7]. They established the inequality

$$\left[\frac{\Gamma(x+1)\Gamma\left(\frac{1}{x}+1\right)}{\Gamma(x+\lambda)\Gamma\left(\frac{1}{x}+\lambda\right)}\right]^{1/2} \geqslant \frac{1}{\Gamma(\lambda+1)},$$

for $x>0$ and $0<\lambda<1$. In the case $\lambda>1$ the inequality must be reversed. Moreover, Giordano and Laforgia [4] proved the following more accurate inequalities than (1.4) for the product of gamma functions

$$\frac{1}{2}\Gamma\left(1+x+\frac{1}{x}\right) \leqslant \Gamma(x)\Gamma\left(\frac{1}{x}\right) \leqslant \Gamma\left(1+x+\frac{1}{x}\right), \qquad \forall x>0. \tag{1.5}$$

An improvement of (1.1) was given by Alzer [1]. Denoting by $M_t(a,b)$ the power mean of order t of the positive real numbers a and b, that is,

$$M_t(a,b) = \left(\frac{a^t+b^t}{2}\right)^{1/t} \qquad (t\neq 0), \qquad M_0(a,b) = (ab)^{1/2},$$

he proved that the inequality

$$M_r\left(\Gamma(x),\Gamma\left(\frac{1}{x}\right)\right) \geqslant 1, \qquad x>0,$$

holds if and only if $r \geqslant \frac{1}{\gamma} - \frac{\pi^2}{6\gamma^2} = -3.20464\ldots$. Here and in the sequel, $\gamma = 0.57721\ldots$ denotes the Euler's constant.

Recently, Alzer [2] obtained the following inequality involving $\Gamma(x)$ and $\Gamma\left(\frac{1}{x}\right)$

$$\Gamma(x)+\Gamma\left(\frac{1}{x}\right) \leqslant b\Gamma\left(x+\frac{1}{x}\right), \qquad x>0, \tag{1.6}$$

where $b \approx 2.098$. In this direction, G.J.O Jameson and T.P. Jameson [5] proved the following inequality conjectured by Kershaw

$$\Gamma(x)+\Gamma\left(\frac{1}{x}\right) \leqslant \Gamma\left(1+x+\frac{1}{x}\right), \qquad x>0, \tag{1.7}$$

where the equality occurs only at $x=1$. Observing that $x+\frac{1}{x} \geqslant 2$, for all $x>0$, we see that (1.6) implies a version of (1.7) with an intervening factor $b/2$. However (1.6) fails to reproduce equality at $x=1$.

Note that the lower bound in (1.5), together with (1.7), implies (1.2).

Motivated by these results, we want to give, in this paper, a contribution about the research of the inequalities which involve the functions $\Gamma(x)$ and $\Gamma\left(\frac{1}{x}\right)$. In particular, our main result refers to the following inequality

THEOREM 1.1. *Let $\alpha = 1+\gamma = 1.57721\ldots$. Then, for every $x>0$, the inequality*

$$\alpha \log x \leqslant x - \frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{x}\right)}, \tag{1.8}$$

holds. Equality occurs only at $x=1$.

REMARK 1.2. It is easy to observe that the inequality (1.8) can be written in the following equivalent way

$$\frac{\Gamma\left(\frac{1}{x}\right)}{\Gamma\left(x+\frac{1}{x}\right)} - \frac{1}{x} \leqslant \alpha \log x, \qquad x>0. \tag{1.9}$$

Therefore, to prove (1.8) for $x>0$, it is sufficient to prove (1.8) and (1.9) only for $x \geqslant 1$.

REMARK 1.3. As an immediate consequence of (1.8) and (1.9) we obtain the inequality

$$\frac{\Gamma(x)+\Gamma\left(\frac{1}{x}\right)}{\Gamma\left(x+\frac{1}{x}\right)} \leqslant x+\frac{1}{x}, \qquad x>0,$$

from which, by the well known formula $\Gamma(z+1) = z\Gamma(z)$, inequality (1.7), conjectured by Kershaw, follows.

Important tools for our proofs are the following series representations

$$\psi(z) = -\gamma + \sum_{k=1}^{\infty} \frac{z-1}{k(z+k-1)} \tag{1.10}$$

$$= -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \frac{z}{k(z+k)}, \tag{1.11}$$

where

$$\psi(z) = \frac{d}{dz}\log\Gamma(z)$$

denotes the ψ or digamma function.

In order to prove Theorem 1.1 we need some preliminary results.

2. Preliminary results

LEMMA 2.1. *The inequalities*

$$-\frac{\alpha x^2 - x}{\alpha x \log x + 1} - \psi\left(\frac{1}{x}\right) \leqslant 2(1-\gamma)(x-1) \leqslant (x^2-1)\psi\left(x+\frac{1}{x}\right) \tag{2.1}$$

hold true $\forall x \in [1,3]$. Further the upper bound holds for every $x \geqslant 1$.

Proof. From the inequality

$$\psi\left(x+\frac{1}{x}\right) \geqslant \psi(2) = 1-\gamma, \qquad x>0, \tag{2.2}$$

it follows

$$(x+1)\psi\left(x+\frac{1}{x}\right) \geqslant 2(1-\gamma), \qquad x \geqslant 1,$$

from which the right-hand inequality in (2.1) follows.

Now, we consider the function

$$u(x) = \frac{\alpha x^2 - x}{\alpha x \log x + 1} + \psi\left(\frac{1}{x}\right) + 2(1-\gamma)(x-1).$$

We establish the lower bound in (2.1) by showing that $u(x)$ or, equivalently, $v(x) = (\alpha x \log x + 1)u(x)$ is non-negative in $[1,3]$.

By using the series representation (1.11), we have for every $x \geqslant 1$

$$\begin{aligned}
v(x) &= \alpha x^2 - x + (\alpha x \log x + 1)\left[\psi\left(\frac{1}{x}\right) + 2(1-\gamma)(x-1)\right] \\
&= \alpha x^2 - x + (\alpha x \log x + 1)\left[-\gamma - x + \sum_{k=1}^{\infty} \frac{1}{k(kx+1)} + 2(1-\gamma)(x-1)\right] \\
&\geqslant \alpha x^2 - x + (\alpha x \log x + 1)\left[-\gamma - x + \frac{1}{x}\sum_{k=1}^{\infty} \frac{1}{k(k+1)} + 2(1-\gamma)(x-1)\right] \\
&= \alpha x^2 - x + (\alpha \log x + \frac{1}{x})p(x) = \nu(x),
\end{aligned}$$

where $p(x) = (1-2\gamma)x^2 + (-2+\gamma)x + 1$. It is easy to observe that the polynomial $p(x)$ is negative when $x \geqslant 1$. Indeed the coefficient $(1-2\gamma) = -0.1542\ldots$ is negative and its bigger zero $\frac{2-\gamma-\sqrt{\gamma^2+4\gamma}}{2(1-2\gamma)} = 0.656\ldots$ is less than 1. As a consequence, the function $\nu(x)$, and then $v(x)$ too, is positive on $[1,3]$ if and only if the function

$$h(x) = \frac{\nu(x)}{p(x)} = \frac{\alpha x^2 - x}{p(x)} + \alpha \log x + \frac{1}{x}, \tag{2.3}$$

is negative on the same interval. Now $h(1) = 0$ and, after some laborious algebraic calculation, we obtain

$$h'(x) = \frac{q(x)}{x^2 p^2(x)},$$

where

$$\begin{aligned}
q(x) =& (4\gamma^3 - 3\gamma + 1)x^5 + (-4\gamma^3 + 3\gamma^2 + 7\gamma - 6)x^4 \\
&+ (\gamma^3 - 3\gamma^2 - 10\gamma + 12)x^3 + (\gamma^2 + 6\gamma - 11)x^2 - (\gamma - 5)x - 1.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
q'(x) =&5(4\gamma^3-3\gamma+1)x^4+4(-4\gamma^3+3\gamma^2+7\gamma-6)x^3\\
&+3(\gamma^3-3\gamma^2-10\gamma+12)x^2+2(\gamma^2+6\gamma-11)x-(\gamma-5),\\
q''(x) =&20(4\gamma^3-3\gamma+1)x^3+12(-4\gamma^3+3\gamma^2+7\gamma-6)x^2\\
&+6(\gamma^3-3\gamma^2-10\gamma+12)x+2(\gamma^2+6\gamma-11),\\
q'''(x) =&60(4\gamma^3-3\gamma+1)x^2+24(-4\gamma^3+3\gamma^2+7\gamma-6)x\\
&+6(\gamma^3-3\gamma^2-10\gamma+12).
\end{aligned}$$

Since $q'''(1)=6(-2-12\gamma+9\gamma^2+25\gamma^3)=-6.72059\ldots$, $q'''(3)=6(30-196\gamma+33\gamma^2+313\gamma^3)=-71.6679\ldots$ and the coefficient $(4\gamma^3-3\gamma+1)=0.0375\ldots$, it follows that $q'''(x)<0$ on $I=[1,3]$, then $q''(x)$ decreases on I. But $q''(1)=-2-24\gamma+20\gamma^2+38\gamma^3=-1.88163\ldots$. This means that $q''(x)<0$ on I and then also $q'(x)$ decreases on I. In similar way, since $q'(1)=\gamma(-6+5\gamma+7\gamma^2)=-0.451196\ldots$, we have $q'(x)<0$ on I. By this last result and since $q(1)=-\gamma+\gamma^2+\gamma^3=-0.0517222\ldots$ we can conclude that $q(x)$, and equivalently $h'(x)$, are negative on I.

Finally we have shown that $h(x)$ decreases on $[1,3]$ and, as a consequence, it is negative in the same interval. □

REMARK 2.2. The function $h(x)$, defined in (2.3), is not negative for every $x\geqslant 1$. Indeed $\lim_{x\to+\infty} h(x)=+\infty$.

LEMMA 2.3. *For every $x\geqslant 1$, the inequalities*

$$\alpha\log x\left[1-\psi\left(x+\frac{1}{x}\right)\right]\leqslant x-1\leqslant x^2-\gamma x-x\psi\left(x+\frac{1}{x}\right), \tag{2.4}$$

hold true.

Proof. From (2.2) we have

$$\alpha\log x\left[1-\psi\left(x+\frac{1}{x}\right)\right]\leqslant \alpha\gamma\log x,$$

then, since $\alpha\gamma\log x\leqslant x-1$ when $x\geqslant 1$, the left-hand inequality in (2.4) follows at once.

Let v be the function defined by

$$v(x)=x^2-\alpha x+1-x\psi\left(x+\frac{1}{x}\right). \tag{2.5}$$

Clearly, the right-hand inequality in (2.4) is equivalent to the inequality $v(x)\geqslant 0$, $\forall x\geqslant$

1. By using the series representation (1.10), we have

$$\begin{aligned}
v(x) &= x^2 - \alpha x + 1 - x\left\{-\gamma + \sum_{k=1}^{\infty} \frac{x^2 - x + 1}{k[x^2 + (k-1)x + 1]}\right\} \\
&= x^2 - x + 1 - \sum_{k=1}^{\infty} \frac{x(x^2 - x + 1)}{k[x^2 + (k-1)x + 1]} \\
&= (x^2 - x + 1)\left\{\sum_{k=1}^{\infty} \frac{1}{k(k+1)} - \sum_{k=1}^{\infty} \frac{x}{k[x^2 + (k-1)x + 1]}\right\} \\
&= (x^2 - x + 1)\sum_{k=1}^{\infty} \frac{(x-1)^2}{k(k+1)[x^2 + (k-1)x + 1]},
\end{aligned}$$

from which we obtain $v(x) \geqslant 0$, $\forall x \geqslant 1$. □

PROPOSITION 2.4. *For every $x \geqslant 1$, the inequality*

$$\frac{x^2 - \alpha x}{x - \alpha \log x} - \psi\left(x + \frac{1}{x}\right) \geqslant -1, \tag{2.6}$$

holds true.

Proof. Let f be the function defined by

$$f(x) = x^2 - \gamma x - x\psi\left(x + \frac{1}{x}\right) + \alpha \log x \left[\psi\left(x + \frac{1}{x}\right) - 1\right].$$

By means of the double inequalities (2.4) we have that $f(x) \geqslant 0$, $\forall x \geqslant 1$. From this last result, dividing f by the function $(x - \alpha \log x)$, which is positive on the interval $[1, +\infty)$, the inequality (2.6) follows. □

PROPOSITION 2.5. *For every $x \geqslant 1$, the inequality*

$$x^2\left[\psi\left(x + \frac{1}{x}\right) - \psi(x)\right] \geqslant 1, \tag{2.7}$$

holds true.

Proof. We will prove that the function

$$u(x) = \psi\left(x + \frac{1}{x}\right) - \psi(x) - \frac{1}{x^2},$$

is non-negative on the interval $[1, +\infty)$.

By means of the series representation (1.11) we have for every $x \geqslant 1$

$$\begin{aligned}
u(x) &= -\frac{x}{x^2+1} + \sum_{k=1}^{\infty} \frac{x^2+1}{k(x^2+kx+1)} + \frac{1}{x} - \sum_{k=1}^{\infty} \frac{x}{k(k+x)} - \frac{1}{x^2} \\
&= -\frac{x^2-x+1}{x^2(x^2+1)} + \sum_{k=1}^{\infty} \left[\frac{x^2+1}{k(x^2+kx+1)} - \frac{x}{k(k+x)} \right] \\
&= -\frac{x^2-x+1}{x^2(x^2+1)} + \sum_{k=1}^{\infty} \frac{1}{(k+x)(x^2+kx+1)} \\
&\geqslant -\frac{x^2-x+1}{x^2(x^2+1)} + \frac{1}{x} \sum_{k=1}^{\infty} \frac{1}{(k+x)(x+k+1)} \\
&= -\frac{x^2-x+1}{x^2(x^2+1)} + \frac{1}{x(x+1)} = \frac{x-1}{x^2(x^2+1)(x+1)} \geqslant 0.
\end{aligned}$$

This completes the proof of Proposition 2.5. □

3. Proof of main result

In this section we prove the Theorem 1.1.

Proof. of Theorem 1.1. We consider first the inequality (1.8). It can rewritten as follows

$$\frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{x}\right)} \leqslant x - \alpha \log x. \tag{3.1}$$

In order to prove the inequality (3.1) for $x \geqslant 1$, we will establish the following equivalent logarithmic inequality

$$\log\left(\frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{x}\right)}\right) \leqslant \log\left(x - \alpha \log x\right), \qquad x \geqslant 1. \tag{3.2}$$

To this purpose we consider the following function

$$f(x) = \log\left(x - \alpha \log x\right) + \log \Gamma\left(x + \frac{1}{x}\right) - \log \Gamma(x), \tag{3.3}$$

on the interval $[1,+\infty)$. Since $f(1) = \log \Gamma(2) - \log \Gamma(1) = 0$, if we show that f increases on $[1,+\infty)$ then $f(x) \geqslant 0 \ \forall x \geqslant 1$ and, as consequence, the inequality (3.2) will be proved.

We have

$$f'(x) = \frac{1}{x^2}\left\{ \frac{x^2-\alpha x}{x-\alpha \log x} - \psi\left(x+\frac{1}{x}\right) + x^2\left[\psi\left(x+\frac{1}{x}\right) - \psi(x)\right]\right\},$$

then, by inequalities (2.6) and (2.7), it follows that $f'(x) \geqslant 0$, $\forall x \geqslant 1$. This achieves the proof of inequality (3.1).

Now we consider, for $x \geqslant 1$, the inequality (1.9) which can be written in the following way

$$\frac{\Gamma\left(\frac{1}{x}\right)}{\Gamma\left(x+\frac{1}{x}\right)} \leqslant \frac{1}{x} + \alpha \log x. \tag{3.4}$$

As in the previous case, we achieve the proof of the inequality (3.4) when the inequality

$$\log\left(\frac{\Gamma\left(\frac{1}{x}\right)}{\Gamma\left(x+\frac{1}{x}\right)}\right) \leqslant \log\left(\frac{1}{x} + \alpha \log x\right), \qquad x \geqslant 1. \tag{3.5}$$

will be proved. Therefore we define the following function

$$g(x) = \log\left(\frac{1}{x} + \alpha \log x\right) + \log\Gamma\left(x+\frac{1}{x}\right) - \log\Gamma\left(\frac{1}{x}\right), \tag{3.6}$$

on the interval $[1,+\infty)$. Since also $g(1) = 0$, the inequality (3.5) will be achieved when it will be shown that g increases on $[1,+\infty)$.

We have

$$g'(x) = \frac{1}{x^2}\left\{\frac{\alpha x^2 - x}{\alpha x \log x + 1} + (x^2-1)\psi\left(x+\frac{1}{x}\right) + \psi\left(\frac{1}{x}\right)\right\}. \tag{3.7}$$

In order to prove that $g'(x) \geqslant 0$ for every $x \geqslant 1$, we will consider two different cases. In particular, in the first case, we will show that $g'(x) \geqslant 0$ when $1 \leqslant x \leqslant 3$, while, in the last one, we will establish that this inequality is also true for every $x \geqslant 3$.

Clearly, from (3.7), it results that $g'(x) \geqslant 0$ in $I = [1,3]$ if and only if

$$(x^2-1)\psi\left(x+\frac{1}{x}\right) \geqslant -\psi\left(\frac{1}{x}\right) - \frac{\alpha x^2 - x}{\alpha x \log x + 1} \tag{3.8}$$

in the same interval I. But this last relation immediately follows by the double inequalities (2.1).

Finally, we show that $g'(x)$ is non-negative also for every $x \geqslant 3$. Indeed, recalling that $\psi\left(\frac{1}{x}\right) = \psi\left(1+\frac{1}{x}\right) - x > -x - \gamma$ and that, for $x \geqslant 3$, $\psi\left(x+\frac{1}{x}\right) \geqslant 1$, it follows

$$(x^2-1)\psi\left(1+\frac{1}{x}\right) + \psi\left(\frac{1}{x}\right) \geqslant x^2 - 1 - x - \gamma = x(x-1) - 1 - \gamma > 0$$

when $x \geqslant 3$. This shows that $g'(x)$ is non-negative for every $x \geqslant 3$.

The proof of Theorem 1.1 is now complete. $\square$

REFERENCES

[1] H. ALZER, *Inequalities for the gamma functions*, Proc. Amer. Math. Soc., **128**, (1999), 3355–3362.

[2] H. ALZER, *Inequalities involving* $\Gamma(x)$ *and* $\Gamma(1/x)$ *for the gamma functions*, J. Comput. Appl. Math., **192**, (2006), 460–480.

[3] W. GAUTSCHI, *A harmonic mean inequality for the gamma function*, SIAM J. Math. Anal., **5**, (1974), 278–281.

[4] C. GIORDANO, A. LAFORGIA, *Inequalities and monotonicity properties for the gamma function*, J. Comput. Appl. Math., **133**, (2001), 387–396.

[5] G. J. O. JAMESON, T. P. JAMESON, *An inequality for the gamma function conjectured by D. Kershaw*, J. Math. Inequal., **6**, no. 2, (2012), 175–181.

[6] H. H. KAIRIES, *An inequality for Krull solutions of a certain difference equations*, in E. F. Beckenbach, W. Walter (Eds.), ISNM General Inequalities, Vol. 3, Birkhäuser, Basel, 1983, 277–280.

[7] A. LAFORGIA, S. SISMONDI, *A geometric mean inequality for the gamma function*, Boll. Un. Mat. Ital. A, **7** 3, (1989), 339–342.

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