Mathematical
Inequalities
& Applications
Volume 17, Number 1 (2014), 179–185 doi:10.7153/mia-17-13

ON THE STABILITY OF AN ALTERNATIVE QUADRATIC FUNCTIONAL EQUATION

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(*Communicated by Zs. Páles*)

Abstract. We will investigate the stability of the alternative quadratic functional equation

$$f(x+y)+f(x-y)+2f(x)+2f(y)\neq 0 \Rightarrow f(x+y)+f(x-y)=2f(x)+2f(y).$$

1. Introduction

In 1940, S. M. Ulam [8] posed a famous problem on the stability of an additive mapping before the Mathematics Club of the University of Wisconsin. In 1941, D. H. Hyers [5] considered the case of approximately additive mappings $f: E \longrightarrow E'$ where E and E' are Banach spaces and f satisfies $\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon$ for all $x,y \in E$. It was shown that the limit $l(x)=\lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that $l: E \longrightarrow E'$ is the unique additive mapping satisfying $\|f(x)-l(x)\| \leqslant \varepsilon$. In 1978, Th. M. Rassias [7] investigated the stability problem of an additive mapping where the approximation is controlled by powers of norms. The problem of stability has later been generalized and investigated by a number of authors (cf. [4]) for various functional equations including the quadratic functional equation (cf. [6])

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1}$$

In 1999, B. Batko and J. Tabor [3] investigated the Hyers-Ulam stability of the generalized alternative Cauchy equation $|f(x+y)|=|f(x)+f(y)|$ where f is a function from a commutative semigroup to a complete Archimedean Riesz Space, and $|\cdot|$ is defined from the order structure. They [2] also studied the Hyers-Ulam stability of an alternative Cauchy equation $|f(x+y)|=|f(x)+f(y)|$ on a restricted domain, where f is a real-valued function defined on a commutative semigroup. In 2005, B. Batko [1] treated the stability of an alternative Cauchy equation

$$f(x+y)+f(x)+f(y)\neq 0 \Rightarrow f(x+y)=f(x)+f(y)$$

Mathematics subject classification (2010): 39B52, 39B82.
Keywords and phrases: Alternative functional equation; Quadratic functional equation; Stability of functional equation.

and showed that for an abelian group, $(S,+)$, and a Banach space, $(X,\|\cdot\|)$, if a function $f: S \to X$ satisfies

$$\|f(x+y)+f(x)+f(y)\| > \delta \Rightarrow \|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon$$

for all $x,y \in S$ and for some $\delta,\varepsilon \geqslant 0$, then there exists a unique additive function $a: S \to X$ such that

$$\|f(x)-a(x)\| \leqslant \max\{\varepsilon,\delta\} \quad \text{for all } x \in S.$$

We will apply the concepts in the aforementioned work to the classical quadratic functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \tag{2}$$

by proposing an alternative quadratic functional equation

$$\|f(x+y)+f(x-y)\| = \|2f(x)+2f(y)\|,$$

where f is a function from a group to a normed space. Hence, in this paper, we aim to investigate the stability of the following alternative functional equation

$$f(x+y)+f(x-y)+2f(x)+2f(y) \neq 0 \Rightarrow f(x+y)+f(x-y) = 2f(x)+2f(y). \tag{3}$$

All quadratic functions satisfying (2) will definitely satisfy (3). But, to the best of the author's knowledge, it is not known whether there exist any other solutions.

2. Auxiliary lemmas

Let $(G,+)$ be a group and let $(E,\|\cdot\|)$ be a normed space. Given nonnegative real numbers δ and ε, we will give lemmas concerning a function $f: G \to E$ satisfying

$$\begin{aligned} &\|f(x+y)+f(x-y)+2f(x)+2f(y)\| > \delta \\ &\qquad \Rightarrow \|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leqslant \varepsilon \end{aligned} \tag{4}$$

for all $x,y \in G$.

LEMMA 1. *If a function $f: G \to E$ satisfies (4), then $\|f(0)\| \leqslant \frac{\delta}{6}$ or $\|f(0)\| \leqslant \frac{\varepsilon}{2}$.*

Proof. Putting $x=0$ and $y=0$ in (4) yields

$$\|6f(0)\| \leqslant \delta \quad \text{or} \quad \|-2f(0)\| \leqslant \varepsilon$$

which immediately gives us the desired result. □

LEMMA 2. *If a function $f: G \to E$ satisfies (4) and $\|f(0)\| > \frac{\varepsilon}{2}$, then $\|f(x)\| \leqslant \frac{\delta}{3}$ for all $x \in G$.*

Proof. Putting $y=0$ in (4), we get

$$\|4f(x)+2f(0)\| \leqslant \delta \quad \text{or} \quad \|-2f(0)\| \leqslant \varepsilon.$$

Since $\|f(0)\| > \frac{\varepsilon}{2}$, we conclude that $\|4f(x)+2f(0)\| \leqslant \delta$. In addition, by Lemma 1, $\|f(0)\| \leqslant \frac{\delta}{6}$. Therefore,

$$\|f(x)\| \leqslant \frac{\delta+\|2f(0)\|}{4} \leqslant \frac{\delta}{3} \quad \text{for all } x \in G. \qquad \square$$

LEMMA 3. *If a function $f: G \to E$ satisfies (4) and, for some $x \in G$,*

$$\|f(2x)+4f(x)+f(0)\| \leqslant \delta, \tag{5}$$

then

$$\|f(x)\| \leqslant \frac{1}{4}(5\varepsilon+19\delta).$$

Proof. Assume that the condition in the lemma holds. We fix $x \in G$ such that (5) is satisfied. If $\|f(0)\| > \frac{\varepsilon}{2}$, then, by Lemma 2, $\|f(x)\| \leqslant \frac{\delta}{3}$ which satisfies the bound given in the lemma.

We now assume that $\|f(0)\| \leqslant \frac{\varepsilon}{2}$. Putting $(x,y)=(2x,x)$ in (4), we get

$$\|f(3x)+2f(2x)+3f(x)\| \leqslant \delta \quad \text{or} \quad \|f(3x)-2f(2x)-f(x)\| \leqslant \varepsilon. \tag{6}$$

Combining (6) and (5), we have

$$\|f(3x)-5f(x)-2f(0)\| \leqslant 3\delta \quad \text{or} \quad \|f(3x)+7f(x)+2f(0)\| \leqslant \varepsilon+2\delta. \tag{7}$$

Putting $(x,y)=(2x,2x)$ in (4), we get

$$\|f(4x)+f(0)+4f(2x)\| \leqslant \delta \quad \text{or} \quad \|f(4x)+f(0)-4f(2x)\| \leqslant \varepsilon. \tag{8}$$

We will handle each inequality in (8) separately. First consider the case where $\|f(4x)+f(0)+4f(2x)\| \leqslant \delta$. Combining this inequality with (5), we have

$$\|f(4x)-16f(x)-3f(0)\| \leqslant 5\delta. \tag{9}$$

Putting $(x,y)=(4x,x)$ in (4), we get

$$\|f(5x)+f(3x)+2f(4x)+2f(x)\| \leqslant \delta \quad \text{or} \quad \|f(5x)+f(3x)-2f(4x)-2f(x)\| \leqslant \varepsilon. \tag{10}$$

Combining (10) and (9), we have

$$\begin{aligned} &\|f(5x)+f(3x)+34f(x)+6f(0)\| \leqslant 11\delta \quad \text{or} \\ &\|f(5x)+f(3x)-34f(x)-6f(0)\| \leqslant \varepsilon+10\delta. \end{aligned} \tag{11}$$

Putting $(x,y)=(3x,2x)$ in (4), we get

$$\|f(5x)+f(x)+2f(3x)+2f(2x)\| \leqslant \delta \quad \text{or} \quad \|f(5x)+f(x)-2f(3x)-2f(2x)\| \leqslant \varepsilon. \tag{12}$$

Combining (12) and (5), we have

$$\begin{aligned} &\|f(5x)+2f(3x)-7f(x)-2f(0)\| \leqslant 3\delta \quad \text{or} \\ &\|f(5x)-2f(3x)+9f(x)+2f(0)\| \leqslant \varepsilon+2\delta. \end{aligned} \tag{13}$$

Combining (11) and (13), we have

$$\begin{aligned} &\|f(3x)-41f(x)-8f(0)\| \leqslant 14\delta && \text{or } \|3f(3x)+25f(x)+4f(0)\| \leqslant \varepsilon+13\delta \\ \text{or } &\|f(3x)+27f(x)+4f(0)\| \leqslant \varepsilon+13\delta && \text{or } \|3f(3x)-43f(x)-8f(0)\| \leqslant 2\varepsilon+12\delta. \end{aligned} \tag{14}$$

Combining (7) and (14), we have

$$\begin{aligned} &\|36f(x)+6f(0)\| \leqslant 17\delta && \text{or } \|48f(x)+10f(0)\| \leqslant \varepsilon+16\delta \\ \text{or } &\|40f(x)+10f(0)\| \leqslant \varepsilon+22\delta && \text{or } \|4f(x)-2f(0)\| \leqslant 4\varepsilon+19\delta \\ \text{or } &\|32f(x)+6f(0)\| \leqslant \varepsilon+16\delta && \text{or } \|20f(x)+2f(0)\| \leqslant 2\varepsilon+15\delta \\ \text{or } &\|28f(x)+2f(0)\| \leqslant 2\varepsilon+21\delta && \text{or } \|64f(x)+14f(0)\| \leqslant 5\varepsilon+18\delta. \end{aligned} \tag{15}$$

Taking into account $\|f(0)\| \leqslant \frac{\varepsilon}{2}$, we conclude from (15) that

$$\|f(x)\| \leqslant \frac{5}{4}\varepsilon+\frac{19}{4}\delta. \tag{16}$$

Now turn to the case where $\|f(4x)+f(0)-4f(2x))\| \leqslant \varepsilon$ in (8). Combining this inequality with (5), we have

$$\|f(4x)+16f(x)+5f(0)\| \leqslant \varepsilon+4\delta. \tag{17}$$

Putting $(x,y)=(3x,x)$ in (4), we get

$$\|f(4x)+f(2x)+2f(3x)+2f(x)\| \leqslant \delta \quad \text{or} \quad \|f(4x)+f(2x)-2f(3x)-2f(x)\| \leqslant \varepsilon. \tag{18}$$

Combining (18), (17), and (5), we have

$$\|2f(3x)-18f(x)-6f(0)\| \leqslant \varepsilon+6\delta \quad \text{or} \quad \|2f(3x)+22f(x)+6f(0)\| \leqslant 2\varepsilon+5\delta. \tag{19}$$

Combining (19) and (7), we have

$$\begin{aligned} &\|8f(x)+2f(0)\| \leqslant \varepsilon+12\delta && \text{or } \|32f(x)+10f(0)\| \leqslant 2\varepsilon+11\delta \\ \text{or } &\|32f(x)+10f(0)\| \leqslant 3\varepsilon+10\delta && \text{or } \|8f(x)+2f(0)\| \leqslant 4\varepsilon+9\delta. \end{aligned} \tag{20}$$

Taking into account $\|f(0)\| \leqslant \frac{\varepsilon}{2}$, we conclude from (20) that

$$\|f(x)\| \leqslant \frac{5}{8}\varepsilon+\frac{3}{2}\delta. \tag{21}$$

From (16) and (21), we infer that $\|f(x)\| \leqslant \frac{1}{4}(5\varepsilon+19\delta)$ as desired. □

LEMMA 4. *If a function $f: G \to E$ satisfies (4), then*

$$\|4^{-1}f(2x)-f(x)\| \leqslant \frac{21}{8}\varepsilon+\frac{39}{4}\delta \quad \textit{for all } x \in G. \tag{22}$$

Proof. If $\|f(0)\| > \frac{\varepsilon}{2}$, then, by Lemma 2, $\|f(x)\| \leqslant \frac{\delta}{3}$ for all $x \in G$. Therefore,

$$\|f(2x) - 4f(x)\| \leqslant \frac{5}{3}\delta. \tag{23}$$

If $\|f(0)\| \leqslant \frac{\varepsilon}{2}$, then putting $y = x$ in (4) gives

$$\|f(2x) + f(0) + 4f(x)\| \leqslant \delta \quad \text{or} \quad \|f(2x) + f(0) - 4f(x)\| \leqslant \varepsilon.$$

For the case $\|f(2x) + f(0) + 4f(x)\| \leqslant \delta$, Lemma 3 gives $\|f(x)\| \leqslant \frac{1}{4}(5\varepsilon + 19\delta)$. Thus,

$$\|f(2x) - 4f(x)\| \leqslant \delta + 8\|f(x)\| + \|f(0)\| \leqslant \frac{21}{2}\varepsilon + 39\delta. \tag{24}$$

For the case $\|f(2x) + f(0) - 4f(x)\| \leqslant \varepsilon$,

$$\|f(2x) - 4f(x)\| \leqslant \varepsilon + \|f(0)\| \leqslant \frac{3}{2}\varepsilon. \tag{25}$$

From (23), (24), and (25), we infer that $\|4^{-1}f(2x) - f(x)\| \leqslant \frac{21}{8}\varepsilon + \frac{39}{4}\delta$. □

LEMMA 5. *If a function $f : G \to E$ satisfies (3), then, for all positive integers n,*

$$f(2^n x) = \pm 4^n f(x) \quad \textit{for all } x \in G.$$

Proof. Putting $x = y = 0$ in (3), we get

$$6f(0) = 0 \quad \text{or} \quad -2f(0) = 0.$$

Thus, $f(0) = 0$. Putting $y = x$ in (3), we get

$$f(2x) + 4f(x) = 0 \quad \text{or} \quad f(2x) - 4f(x) = 0 \quad \text{for all } x \in G.$$

Therefore, $f(2x) = \pm 4f(x)$ for all $x \in G$. We can now employ mathematical induction to prove that $f(2^n x) = \pm 4^n f(x)$ for all $x \in G$ for all positive integers n. □

3. Stability

We will now prove the stability of the alternative functional equation (3).

THEOREM 1. *Let $(G, +)$ be an abelian group and let $(E, \|\cdot\|)$ be a Banach space. If a function $f : G \to E$ satisfies (4) for some nonnegative real numbers δ and ε, then there exists a unique function $q : G \to E$ such that q satisfies (3) and*

$$\|f(x) - q(x)\| \leqslant \frac{7}{2}\varepsilon + 13\delta \quad \textit{for all } x \in G. \tag{26}$$

Proof. From Lemma 4,

$$\|4^{-1}f(2x)-f(x)\| \leqslant \frac{21}{8}\varepsilon+\frac{39}{4}\delta \quad \text{for all } x\in G. \tag{27}$$

For a positive integer n and for each $x\in G$,

$$\begin{aligned}\|4^{-n}f(2^nx)-f(x)\| &\leqslant \sum_{i=0}^{n-1}\|4^{-(i+1)}f(2^{i+1}x)-4^{-i}f(2^ix)\|\\ &\leqslant \sum_{i=0}^{n-1}4^{-i}\|4^{-1}f(2\cdot 2^ix)-f(2^ix)\|\\ &\leqslant \sum_{i=0}^{n-1}4^{-i}\left(\frac{21}{8}\varepsilon+\frac{39}{4}\delta\right).\end{aligned}$$

Consider the sequence $\left\{4^{-n}f(2^nx)\right\}$. For positive integers $0<m<n$,

$$\begin{aligned}\|4^{-n}f(2^nx)-4^{-m}f(2^mx)\| &\leqslant 4^{-m}\|4^{-(n-m)}f(2^{n-m}\cdot 2^mx)-f(2^mx)\|\\ &\leqslant 4^{-m}\sum_{i=0}^{n-m-1}4^{-i}\left(\frac{21}{8}\varepsilon+\frac{39}{4}\delta\right)\\ &\leqslant 4^{-m}\sum_{i=0}^{\infty}4^{-i}\left(\frac{21}{8}\varepsilon+\frac{39}{4}\delta\right).\end{aligned}$$

The right-hand side of the above inequality approaches 0 as $m\to\infty$; therefore, $\left\{4^{-n}f(2^nx)\right\}$ is a Cauchy sequence for all $x\in G$. Thus, we define

$$q(x)=\lim_{n\to\infty}4^{-n}f(2^nx) \quad \text{for all } x\in G. \tag{28}$$

It can be verified that

$$\|f(x)-q(x)\| \leqslant \sum_{i=0}^{\infty}4^{-i}\left(\frac{21}{8}\varepsilon+\frac{39}{4}\delta\right)=\frac{7}{2}\varepsilon+13\delta.$$

We will now prove that q satisfies (3). By the definition of q in (28), we have

$$\begin{aligned}&q(x+y)+q(x-y)+2q(x)+2q(y)\\ &\quad=\lim_{n\to\infty}4^{-n}\left(f(2^n(x+y))+f(2^n(x-y))+2f(2^nx)+2f(2^ny)\right)\end{aligned}$$

for all $x,y\in G$. Suppose that for some $x,y\in G$, $q(x+y)+q(x-y)+2q(x)+2q(y)\neq 0$. Then, for a sufficiently large n,

$$\|f(2^n(x+y))+f(2^n(x-y))+2f(2^nx)+2f(2^ny)\|>\delta.$$

We deduce from the condition (4) that

$$\|f(2^n(x+y))+f(2^n(x-y))-2f(2^nx)-2f(2^ny)\|\leqslant\varepsilon.$$

If we multiply the above inequality by 4^{-n} and take the limit as $n \to \infty$, then $q(x+y)+q(x-y)-2q(x)-2q(y)=0$.

To prove the uniqueness of q, we note the property of q from Lemma 5: $\|q(2^n x)\| = 4^n\|q(x)\|$ for all $x \in G$ and for all positive integers n. Suppose that there exists a function $q' : G \to E$ satisfying the alternative functional equation (3) and the inequality (26). Then, for each $x \in G$,

$$\begin{aligned}\|q'(x)-q(x)\| &= 4^{-n}\|q'(2^n x)-q(2^n x)\| \\ &\leqslant 4^{-n}\big(\|f(2^n x)-q(2^n x)\|+\|f(2^n x)-q'(2^n x)\|\big) \\ &\leqslant 4^{-n}(7\varepsilon+26\delta) \underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Hence, $q(x)=q'(x)$ for all $x \in G$. $\square$

Acknowledgement. The author is indebted to the referees for all valuable comments which improve the quality of the paper.

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(Received March 9, 2012)

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