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NEW CONVERSES OF THE JESSEN AND LAH–RIBARIČ INEQUALITIES

ROZARIJA JAKŠIĆ AND JOSIP PEČARIĆ

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Abstract. New converses of the Jessen and Lah-Ribarič inequalities for continuous convex functions with applications to means, the Hölder inequality, the Hadamard inequality, and the inequalities of Giaccardi and Petrović are given.

1. Introduction

Let E be a nonempty set and L be a linear class of real-valued functions $f\colon E\to\mathbb{R}$ having the properties:

L1: $f,g\in L\Rightarrow (af+bg)\in L$ for all $a,b\in\mathbb{R}$;

L2: $1\in L$, i.e., if $f(t)=1$ for every $t\in E$, then $f\in L$.

We also consider positive linear functionals $A\colon L\to\mathbb{R}$. That is, we assume that:

A1: $A(af+bg)=aA(f)+bA(g)$ for $f,g\in L$ and $a,b\in\mathbb{R}$;

A2: $f\in L$, $f(t)\geqslant 0$ for every $t\in E\Rightarrow A(f)\geqslant 0$ (A is positive).

Jessen [5] gave the following generalization of Jensen's inequality for convex functions (see also [7, p. 47]):

THEOREM 1.1. ([5]) *Let L satisfy properties L1, L2 on a nonempty set E, and assume that ϕ is a continous convex function on an interval $I\subset\mathbb{R}$. If A is a positive linear functional with $A(1)=1$, then for all $f\in L$ such that $\phi(f)\in L$ we have $A(f)\in I$ and*

$$\phi(A(f))\leqslant A(\phi(f)). \tag{1.1}$$

The following result is proved in [1] by Beesack and Pečarić (see also [7, p. 98]):

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THEOREM 1.2. ([1]) *Let ϕ be convex on $I=[m,M]$ $(-\infty<m<M<\infty)$. Let L satisfy conditions $L1, L2$ on E and let A be any positive linear functional on L with $A(1)=1$. Then for every $f\in L$ such that $\phi(f)\in L$ (so that $m\leqslant f(t)\leqslant M$ for all $t\in E$), we have*

$$A(\phi(f))\leqslant\frac{(M-A(f))\phi(m)+(A(f)-m)\phi(M)}{M-m}. \tag{1.2}$$

Let $(\Omega,\mathscr{A},\mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathscr{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathscr{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w\colon\Omega\to\mathbb{R}$, with $w(x)\geqslant 0$ for μ-a.e. (almost every) $x\in\Omega$, consider the Lebesgue space

$$L_w(\Omega,\mu):=\{f\colon\Omega\to\mathbb{R}, f \text{ is } \mu-\text{measurable and } \textstyle\int_\Omega w(x)|f(x)|\mathrm{d}\mu(x)<\infty\}.$$

S.S.Dragomir [3] gave the following converse of Jensen's inequality:

THEOREM 1.3. ([3]) *Let $\phi\colon I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\overset{\circ}{I}$, where $\overset{\circ}{I}$ is interior of I. Let $w>0$ such that $\int w\mathrm{d}\mu=1$. If $f\colon\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty<m\leqslant f(t)\leqslant M<\infty \text{ for } \mu\text{-a.e. } t\in\Omega$$

and such that $f,\phi\circ f\in L_w(\Omega,\mu)$, then

$$\begin{aligned}0&\leqslant\int_\Omega w(t)\phi(f(t))\mathrm{d}\mu(t)-\phi(\bar{f}_{\Omega,w})\\&\leqslant(M-\bar{f}_{\Omega,w})(\bar{f}_{\Omega,w}-m)\frac{\phi'_-(M)-\phi'_+(m)}{M-m}\\&\leqslant\frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).\end{aligned} \tag{1.3}$$

In this paper we shall give new converses of Lah-Ribarič's and Jessen's inequality for positive linear functionals. Also, we shall give applications of these results to generalized means, power means, Hölder's inequality, Hadamard's inequality and to inequalities of Giaccardi and Petrović.

2. Results

The results in this section are converses of Lah-Ribarič's and Jessen's inequality for positive linear functionals.

THEOREM 2.1. *Let ϕ be a continuous convex function on an interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I. Let L satisfy conditions $L1, L2$ on E and let A be any positive linear functional on L with $A(1)=1$. If $f\in L$ satisfies the bounds*

$$-\infty<m\leqslant f(t)\leqslant M<\infty \text{ for every } t\in E$$

and $\phi \circ f \in L$, *then*

$$\begin{aligned} 0 &\leqslant A(\phi(f)) - \phi(A(f)) \\ &\leqslant (M - A(f))(A(f) - m)\frac{\phi'_-(M) - \phi'_+(m)}{M - m} \\ &\leqslant \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)). \end{aligned} \tag{2.1}$$

If ϕ *is concave on* I, *then the inequalities in (2.1) are reversed.*

Proof. First we assume that ϕ is convex.

The first inequality follows directly from Theorem 1.1. By Theorem 1.2, we have

$$A(\phi(f)) - \phi(A(f)) \leqslant \frac{M - A(f)}{M - m}\phi(m) + \frac{A(f) - m}{M - m}\phi(M) - \phi(A(f)) =: B.$$

Then, by the convexity of ϕ we have the gradient inequality:

$$\phi(t) - \phi(M) \geqslant \phi'_-(M)(t - M)$$

for any $t \in [m, M]$. If we multiply this inequality with $t - m \geqslant 0$, we deduce

$$(t - m)\phi(t) - (t - m)\phi(M) \geqslant \phi'_-(M)(t - M)(t - m), \quad t \in [m, M]. \tag{2.2}$$

Similarly, we get

$$(M - t)\phi(t) - (M - t)\phi(m) \geqslant \phi'_+(m)(t - m)(M - t), \quad t \in [m, M]. \tag{2.3}$$

Adding (2.2) to (2.3) and dividing by $m - M$, we deduce that for any $t \in [m, M]$

$$\frac{(t - m)\phi(M) + (M - t)\phi(m)}{M - m} - \phi(t) \leqslant \frac{(M - t)(t - m)}{M - m}(\phi'_-(M) - \phi'_+(m)). \tag{2.4}$$

Substituting t with $A(f)$ in (2.4), we obtain

$$B \leqslant \frac{(M - A(f))(A(f) - m)}{M - m}(\phi'_-(M) - \phi'_+(m))$$

which is the second inequality in (2.1).

To prove the third inequality in (2.1), we notice that for every $t \in [m, M]$, the inequality $\frac{(M - t)(t - m)}{M - m} \leqslant \frac{1}{4}(M - m)$ is valid, and the proof is completed.

If ϕ is concave, then $-\phi$ is convex, so we can apply (2.1) to function $-\phi$ and obtain reversed inequalities for ϕ. □

THEOREM 2.2. *Let us suppose that the assumptions from Theorem 2.1 hold. If* $f \in L$ *satisfies the bounds*

$$-\infty < m \leqslant f(t) \leqslant M < \infty \text{ for every } t \in E$$

and $\phi \circ f \in L$*, then*

$$\begin{aligned} 0 &\leqslant \frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\phi(f)) \\ &\leqslant \frac{\phi'_-(M)-\phi'_+(m)}{M-m}A([M-f][f-m]) \\ &\leqslant \frac{\phi'_-(M)-\phi'_+(m)}{M-m}(M-A(f))(A(f)-m) \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)). \end{aligned} \tag{2.5}$$

If ϕ *is concave, the inequalities in (2.5) are reversed.*

Proof. Let us assume that ϕ is convex.

The first inequality in (2.5) is obtained from (1.2) by subtracting $\phi(A(f))$ from both sides of the inequality. If we replace t with $f(t)$ in (2.4), we obtain

$$\begin{aligned} &\frac{M-f(t)}{M-m}\phi(m)+\frac{f(t)-m}{M-m}\phi(M)-\phi(f(t)) \\ &\quad\leqslant \frac{(M-f(t))(f(t)-m)}{M-m}(\phi'_-(M)-\phi'_+(m)). \end{aligned}$$

Due to the linearity of A, when we apply it to the previous inequality, the second inequality in (2.5) follows:

$$\begin{aligned} &\frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\phi(f)) \\ &\quad\leqslant \frac{(\phi'_-(m)-\phi'_+(m))}{M-m}A([M-f][f-m]). \end{aligned}$$

Since the function $g(t)=(M-t)(t-m)$ is concave, from Jessen's inequality it follows that $A([M-f][f-m]) \leqslant (M-A(f))(A(f)-m)$, which gives us the third inequality in (2.5):

$$\begin{aligned} &\frac{(\phi'_-(m)-\phi'_+(m))}{M-m}A([M-f][f-m]) \\ &\quad\leqslant \frac{(\phi'_-(m)-\phi'_+(m))}{M-m}(M-A(f))(A(f)-m). \end{aligned}$$

To prove the last inequality in (2.5), we notice that for every $t \in [m,M]$, the inequality $\dfrac{(M-t)(t-m)}{M-m} \leqslant \dfrac{1}{4}(M-m)$ is valid, and thus the proof is completed.

If ϕ is concave, then $-\phi$ is convex, so we can apply (2.5) to to function $-\phi$ and obtain reversed inequalities for ϕ. □

3. Applications

3.1. Generalized means

DEFINITION 3.1.1. Let $I = \langle a,b \rangle$, $-\infty \leqslant a < b \leqslant \infty$, and let $\psi \colon I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that L and A satisfy the conditions $L1,L2$ and $A1,A2$ with $A(1) = 1$ on a non-empty set E, and that $\psi(f) \in L$ for some $f \in L$. Generalized mean with respect to the operator A and ψ for $f \in L$ is defined by

$$M_\psi(f,A) = \psi^{-1}A(\psi(f)). \tag{3.1.1}$$

We need the following result (generalization to functionals of the general means inequality found in [4]):

THEOREM 3.1.1. ([4, p. 75, Theorem 92]) *Let $I = \langle a,b \rangle$, $-\infty \leqslant a < b \leqslant \infty$, and let $\psi,\chi \colon I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that L and A satisfy the conditions $L1,L2$ and $A1,A2$ with $A(1) = 1$ on a non-empty set E, and let $f \in L$ be such that $\psi(f),\chi(f) \in L$. Then the following inequality is valid*

$$M_\psi(f,A) \leqslant M_\chi(f,A), \tag{3.1.2}$$

provided either χ is increasing and $\phi = \chi \circ \psi^{-1}$ is convex, or χ is decreasing and $\phi = \chi \circ \psi^{-1}$ is concave.

THEOREM 3.1.2. ([7, p. 108, Theorem 4.3]) *Let L, A, ψ and χ be as in Theorem 3.1.1, but with $I = [m,M]$, $-\infty < m < M < \infty$. Then for every $f \in L$ such that $m \leqslant f(t) \leqslant M$, $t \in E$ we have*

$$(\psi(M) - \psi(m))A(\chi(f)) - (\chi(M) - \chi(m))A(\psi(f)) \leqslant \psi(M)\chi(m) - \chi(M)\psi(m), \tag{3.1.3}$$

provided that $\phi = \chi \circ \psi^{-1}$ is convex. The inequality in (3.1.3) is reversed if ϕ is concave.

The following results are converses of the inequality for generalized means:

THEOREM 3.1.3. *Let L,A,ψ,χ satisfy conditions of the Theorem 3.1.1. Let $I \supset [m,M]$, $-\infty < m < M < \infty$, and let us assume that the function $\phi = \chi \circ \psi^{-1}$ is convex. Then for every $f \in L$ such that $m \leqslant f(t) \leqslant M$ for $t \in [m,M]$ and $\psi(f),\chi(f) \in L$ we have*

$$\begin{aligned} 0 &\leqslant \chi(M_\chi(f,A)) - \chi(M_\psi(f,A)) \\ &\leqslant (M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi)\frac{[\chi \circ \psi^{-1}]'_-(M_\psi) - [\chi \circ \psi^{-1}]'_+(m_\psi)}{M_\psi - m_\psi} \\ &\leqslant \frac{1}{4}(M_\psi - m_\psi)([\chi \circ \psi^{-1}]'_-(M_\psi) - [\chi \circ \psi^{-1}]'_+(m_\psi)) \end{aligned} \tag{3.1.4}$$

where $[m_\psi,M_\psi] = \psi([m,M])$. If ϕ is concave, then the inequalities in (3.1.4) are reversed.

Proof. Function $\phi = \chi \circ \psi^{-1}$ is obviously continuous. Let us assume that ϕ is convex.

Since $m \leqslant f(t) \leqslant M$ for $t \in [m,M]$, we have $m_\psi \leqslant \psi(f(t)) \leqslant M_\psi$ for every $t \in [m,M]$ (if ψ is increasing, then $m_\psi = \psi(m)$ and $M_\psi = \psi(M)$; if ψ is decreasing, then $m_\psi = \psi(M)$ and $M_\psi = \psi(m)$). Conditions of Theorem 2.1 are satisfied, so we can obtain (3.1.4) by substituting $m \leftrightarrow m_\psi$, $M \leftrightarrow M_\psi$, $\phi \leftrightarrow \chi \circ \psi^{-1}$ and $f \leftrightarrow \psi \circ f$ in (2.1).

Now let us assume that $\phi = \chi \circ \psi^{-1}$ is concave. Then the function $-\phi = -\chi \circ \psi^{-1}$ is convex, so we can obtain reversed inequalities by replacing ϕ with $-\phi$ in (3.1.4). $\square$

THEOREM 3.1.4. *Under the same assumptions as in the previous theorem, if the function $\phi = \chi \circ \psi^{-1}$ is convex, the following inequalities are valid:*

$$\begin{aligned}
0 &\leqslant \frac{\psi(M) - A(\psi(f))}{\psi(M) - \psi(m)}\chi(m) + \frac{A(\psi(f)) - \psi(m)}{\psi(M) - \psi(m)}\chi(M) - \chi(M_\chi(f,A)) \\
&\leqslant \frac{[\chi \circ \psi^{-1}]'_-(M_\psi) - [\chi \circ \psi^{-1}]'_+(m_\psi)}{M_\psi - m_\psi} A([M_\psi - \psi(f)][\psi(f) - m_\psi]) \\
&\leqslant \frac{[\chi \circ \psi^{-1}]'_-(M_\psi) - [\chi \circ \psi^{-1}]'_+(m_\psi)}{M_\psi - m_\psi} (M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi) \\
&\leqslant \frac{1}{4}(M_\psi - m_\psi)([\chi \circ \psi^{-1}]'_-(M_\psi) - [\chi \circ \psi^{-1}]'_+(m_\psi))
\end{aligned} \tag{3.1.5}$$

where $[m_\psi, M_\psi] = \psi([m,M])$. If ϕ is concave, the inequalities in (3.1.5) are reversed.

Proof. Function $\phi = \chi \circ \psi^{-1}$ is obviously continuous. Let us assume that ϕ is convex.

Since $m \leqslant f(t) \leqslant M$ for $t \in [m,M]$, we have $m_\psi \leqslant \psi(f(t)) \leqslant M_\psi$ for every $t \in [m,M]$ (if ψ is increasing, then $m_\psi = \psi(m)$ and $M_\psi = \psi(M)$; if ψ is decreasing, then $m_\psi = \psi(M)$ and $M_\psi = \psi(m)$). Conditions of Theorem 2.2 are satisfied, so we can obtain (3.1.5) by substituting $m \leftrightarrow m_\psi$, $M \leftrightarrow M_\psi$, $\phi \leftrightarrow \chi \circ \psi^{-1}$ and $f \leftrightarrow \psi \circ f$ in (2.5).

Now let us assume that $\phi = \chi \circ \psi^{-1}$ is concave. Then the function $-\phi = -\chi \circ \psi^{-1}$ is convex, so we can obtain reversed inequalities by replacing ϕ with $-\phi$ in (3.1.5). $\square$

3.2. Power means

DEFINITION 3.2.1. Suppose that L and A satisfy the conditions $L1, L2$ and $A1, A2$ with $A(1) = 1$, on a non-empty set E. For $f \in L$, the power mean $M^{[r]}(f,A)$ is defined for $r \in \mathbb{R}$ with:

$$M^{[r]}(f,A) = \begin{cases} (A(f^r))^{1/r} & : r \neq 0 \\ \exp(A(\log f)) & : r = 0 \end{cases} \tag{3.2.1}$$

where $f(t) > 0$ for $t \in E$, $f^r \in L$ for $r \in \mathbb{R}$ and $\log f \in L$.

From Theorem 3.1.1 ([4, p. 75, Theorem 92]) it follows as a special case:

THEOREM 3.2.1. *Let $-\infty < r \leqslant s < \infty$ and let us assume that the assumptions from Definition 3.2.1 are valid. Then*

$$M^{[r]}(f,A) \leqslant M^{[s]}(f,A). \tag{3.2.2}$$

We can also obtain Goldman's inequality for positive linear functionals from (3.1.3) as a special case (see [2, p. 203]):

$$(M^r - m^r)(M^{[s]}(f,A))^s - (M^s - m^s)(M^{[r]}(f,A))^r \leqslant M^r m^r - M^s m^s \tag{3.2.3}$$

for $0 < r < s$ or $r < 0 < s$, and the inequality is reversed for $r < s < 0$.

Similarly, for $r = 0$ and $s \in \mathbb{R}$ we obtain

$$(M^{[s]}(f,A))^s \log\frac{M}{m} - (M^s - m^s)\log(M^{[0]}(f,A)) \leqslant m^s \log M - M^s \log m. \tag{3.2.4}$$

When we apply theorems 2.1 and 2.2 to the power means, we obtain the following results:

THEOREM 3.2.2. *Suppose that L and A satisfy the conditions $L1, L2$ and $A1, A2$ with $A(1) = 1$, on a non-empty set E. Let $0 < m \leqslant f(t) \leqslant M < \infty$ for $t \in E$, $f^r, f^s \in L$ for $r, s \in \mathbb{R}$, $r < s$ and $\log f \in L$.*

If $0 < r < s$ or $r < 0 < s$ then:

$$\begin{aligned} 0 &\leqslant (M^{[s]}(f,A))^s - (M^{[r]}(f,A))^s \\ &\leqslant \frac{s}{r}(M^r - A(f^r))(A(f^r) - m^r)\frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\leqslant \frac{s}{4r}(M^r - m^r)(M^{s-r} - m^{s-r}). \end{aligned} \tag{3.2.5}$$

If $r < s < 0$ then:

$$\begin{aligned} 0 &\geqslant (M^{[s]}(f,A))^s - (M^{[r]}(f,A))^s \\ &\geqslant \frac{s}{r}(M^r - A(f^r))(A(f^r) - m^r)\frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\geqslant \frac{s}{4r}(M^r - m^r)(M^{s-r} - m^{s-r}). \end{aligned} \tag{3.2.6}$$

If $s = 0$ and $r < 0$, then:

$$\begin{aligned} 0 &\leqslant \log(M^{[0]}(f,A)) - \log(M^{[r]}(f,A)) \\ &\leqslant -\frac{1}{r}\frac{(M^r - A(f^r))(A(f^r) - m^r)}{M^r m^r} \\ &\leqslant -\frac{1}{4r}\frac{(M^r - m^r)^2}{M^r m^r}. \end{aligned} \tag{3.2.7}$$

If $r=0$ and $s>0$, then:

$$\begin{aligned} 0 &\leqslant (M^{[s]}(f,A))^s - (M^{[0]}(f,A))^s \\ &\leqslant (\log M - A(\log f))(A(\log f) - \log m)\frac{s(e^{sM}-e^{sm})}{\log M - \log m} \\ &\leqslant \frac{s}{4}(e^{sM}-e^{sm})\log\frac{M}{m}. \end{aligned} \tag{3.2.8}$$

Proof. If we put $\chi(t)=t^s$ and $\psi(t)=t^r$, we have $\phi(t)=\chi(\psi^{-1}(t))=t^{s/r}$, which is continuous, and convex for $0<r<s$ and $r<0<s$. Function ψ is strictly increasing for $r>0$, and the conditions of Theorem 2.1 are satisfied, so we can obtain (3.2.5) by replacing $m\leftrightarrow\psi(m)=m^r$, $M\leftrightarrow\psi(M)=M^r$, $\phi(t)\leftrightarrow\chi\circ\psi^{-1}(t)=t^{s/r}$ and $f\leftrightarrow\psi\circ f=f^r$ in (2.1). Function ψ is strictly decreasing for $r<0$, so we can obtain (3.2.5) by replacing $M\leftrightarrow\psi(m)=m^r$, $m\leftrightarrow\psi(M)=M^r$, $\phi(t)\leftrightarrow\chi\circ\psi^{-1}(t)=t^{s/r}$ and $f\leftrightarrow\psi\circ f=f^r$ in (2.1).

In case $r<s<0$, function $\psi(t)=t^r$ is strictly decreasing and $\phi(t)=\chi(\psi^{-1}(t))=t^{s/r}$ is concave, so we obtain (3.2.6) by making substitutions $M\leftrightarrow\psi(m)=m^r$, $m\leftrightarrow\psi(M)=M^r$, $\phi(t)\leftrightarrow-\chi\circ\psi^{-1}(t)=-t^{s/r}$ and $f\leftrightarrow\psi\circ f=f^r$ in (2.1).

In case $r<0$ and $s=0$ we put $\chi(t)=\log t$ and $\psi(t)=t^r$. Then $\phi(t)=\chi(\psi^{-1}(t))=\frac{1}{r}\log t$ is continuous and convex, and ψ is strictly decreasing for $r<0$, so the conditions of Theorem 2.1 are satisfied and we can obtain (3.2.7) by making substitutions $M\leftrightarrow\psi(m)=m^r$, $m\leftrightarrow\psi(M)=M^r$, $\phi(t)\leftrightarrow\chi\circ\psi^{-1}(t)=\frac{1}{r}\log t$ and $f\leftrightarrow\psi\circ f=f^r$ in (2.1).

In case $r=0$, $s>0$, we put $\chi(t)=t^s$ and $\psi(t)=\log t$. Then $\phi(t)=\chi(\psi^{-1}(t))=e^{st}$ is continuous and convex, and ψ is strictly increasing. The inequalities (3.2.8) are now obtained by replacing $m\leftrightarrow\psi(m)=\log m$, $M\leftrightarrow\psi(M)=\log M$, $\phi(t)\leftrightarrow\chi\circ\psi^{-1}(t)=e^{st}$ and $f\leftrightarrow\psi\circ f=\log f$ in (2.1). □

THEOREM 3.2.3. *Under the same hypothesis as in the previous theorem, if $0<r<s$ or $r<0<s$, then:*

$$\begin{aligned} 0 &\leqslant \frac{M^r-A(f^r)}{M^r-m^r}m^s + \frac{A(f^r)-m^r}{M^r-m^r}M^s - (M^{[s]}(f,A))^s \\ &\leqslant \frac{s}{r}\frac{M^{s-r}-m^{s-r}}{M^r-m^r}A([M^r-f^r][f^r-m^r]) \\ &\leqslant \frac{s}{r}\frac{M^{s-r}-m^{s-r}}{M^r-m^r}(M^r-A(f^r))(A(f^r)-m^r) \\ &\leqslant \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r}). \end{aligned} \tag{3.2.9}$$

If $r < s < 0$, then:

$$\begin{aligned} 0 &\geqslant \frac{M^r - A(f^r)}{M^r - m^r} m^s + \frac{A(f^r) - m^r}{M^r - m^r} M^s - (M^{[s]}(f,A))^s \\ &\geqslant \frac{s}{r} \frac{M^{s-r} - m^{s-r}}{M^r - m^r} A([M^r - f^r][f^r - m^r]) \\ &\geqslant \frac{s}{r} \frac{M^{s-r} - m^{s-r}}{M^r - m^r} (M^r - A(f^r))(A(f^r) - m^r) \\ &\geqslant \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r}). \end{aligned} \tag{3.2.10}$$

If $s = 0$ and $r < 0$, then:

$$\begin{aligned} 0 &\leqslant \frac{M^r - A(f^r)}{M^r - m^r} \log m + \frac{s}{r} \frac{A(f^r) - m^r}{M^r - m^r} \log M - \log(M^{[0]}(f,A)) \\ &\leqslant -\frac{1}{r} \frac{A([M^r - f^r][f^r - m^r])}{M^r m^r} \\ &\leqslant -\frac{1}{r} \frac{(M^r - A(f^r))(A(f^r) - m^r)}{M^r m^r} \\ &\leqslant \frac{1}{4r} (M^r - m^r) \left(\frac{1}{M^r} - \frac{1}{m^r} \right). \end{aligned} \tag{3.2.11}$$

If $r = 0$ and $s > 0$, then:

$$\begin{aligned} 0 &\leqslant \frac{\log M - A(\log f)}{\log M - \log m} m^s + \frac{A(\log f) - \log m}{\log M - \log m} M^s - (M^{[s]}(f,A))^s \\ &\leqslant s \frac{e^{sM} - e^{sm}}{\log M - \log m} A([\log M - \log(f)][\log(f) - \log m]) \\ &\leqslant s \frac{e^{sM} - e^{sm}}{\log M - \log m} (\log M - A(\log(f))][A(\log(f)) - \log m]) \\ &\leqslant \frac{s}{4} (e^{sM} - e^{sm}) \log \frac{M}{m}. \end{aligned} \tag{3.2.12}$$

Proof. All the inequalities can be obtained directly from (2.5) by making the same supstitutions as in the proof of the previous theorem. □

THEOREM 3.2.4. *Suppose that L and A satisfy the conditions $L1, L2$ and $A1, A2$ with $A(1) = 1$, on a non-empty set E. Let $f(t) > 0$, $0 < m \leqslant f(t) \leqslant M < \infty$ for $t \in E$, $f^r, f^s \in L$ for $r, s \in \mathbb{R}$, $r < s$ and $\log f \in L$.*

If $r < 0 < s$ or $r < s < 0$, then:

$$\begin{aligned} 0 &\leqslant (M^{[r]}(f,A))^r - (M^{[s]}(f,A))^r \\ &\leqslant \frac{r}{s} (M^s - A(f^s))(A(f^s) - m^s) \frac{M^{r-s} - m^{r-s}}{M^s - m^s} \\ &\leqslant \frac{r}{4s} (M^s - m^s)(M^{r-s} - m^{r-s}). \end{aligned} \tag{3.2.13}$$

If $0<r<s$*, then:*

$$\begin{aligned}0 &\geqslant (M^{[r]}(f,A))^r-(M^{[s]}(f,A))^r\\ &\geqslant \frac{r}{s}(M^s-A(f^s))(A(f^s)-m^s)\frac{M^{r-s}-m^{r-s}}{M^s-m^s}\\ &\geqslant \frac{r}{4s}(M^s-m^s)(M^{r-s}-m^{r-s}).\end{aligned} \tag{3.2.14}$$

If $s=0$ *and* $r<0$*, then:*

$$\begin{aligned}0 &\leqslant (M^{[r]}(f,A))^r-(M^{[0]}(f,A))^r\\ &\leqslant (\log M-A(\log f))(A(\log f)-\log m)\frac{r(M^r-m^r)}{\log M-\log m}\\ &\leqslant \frac{r}{4}(M^r-m^r)\log\frac{M}{m}.\end{aligned} \tag{3.2.15}$$

If $r=0$ *and* $s>0$*, then:*

$$\begin{aligned}0 &\geqslant \log(M^{[0]}(f,A))-\log(M^{[s]}(f,A))\\ &\geqslant -\frac{1}{s}(M^s-A(f^s))(A(f^s)-m^s)\frac{1}{M^sm^s}\\ &\geqslant \frac{1}{4s}(M^s-m^s)\left(\frac{1}{M^s}-\frac{1}{m^s}\right).\end{aligned} \tag{3.2.16}$$

Proof. If we put $\chi(t)=t^r$ and $\psi(t)=t^s$, we have $\phi(t)=\chi(\psi^{-1}(t))=t^{r/s}$, which is a continuous function, convex for $r<0<s$ and $r<s<0$. Function ψ is strictly increasing for $s>0$ and the conditions of Theorem 2.1 are satisfied, so we can obtain (3.2.13) by replacing $m\leftrightarrow\psi(m)=m^s$, $M\leftrightarrow\psi(M)=M^s$, $\phi(t)\leftrightarrow\chi\circ\psi^{-1}(t)=t^{r/s}$ and $f\leftrightarrow\psi\circ f=f^s$ in (2.1). If $r<s<0$, function $\psi(t)=t^s$ is strictly decreasing, so we obtain (3.2.13) by making substitutions $M\leftrightarrow\psi(m)=m^s$, $m\leftrightarrow\psi(M)=M^s$, $\phi(t)\leftrightarrow\chi\circ\psi^{-1}(t)=t^{r/s}$ and $f\leftrightarrow\psi\circ f=f^s$ in (2.1)

In case $0<r<s$ function $\phi(t)=\chi(\psi^{-1}(t))=t^{r/s}$ is concave and $\psi(t)=t^s$ is strictly increasing, so we obtain (3.2.14) by making substitutions $m\leftrightarrow\psi(m)=m^s$, $M\leftrightarrow\psi(M)=M^s$, $\phi(t)\leftrightarrow-\chi\circ\psi^{-1}(t)=-t^{r/s}$ and $f\leftrightarrow\psi\circ f=f^s$ in (2.1).

In case $r<0$ and $s=0$ we put $\chi(t)=t^r$ and $\psi(t)=\log t$. Then $\phi(t)=\chi(\psi^{-1}(t))=e^{rt}$ is a continuous, convex function and ψ is strictly increasing, so the conditions of Theorem 2.1 are satisfied and we can obtain (3.2.15) by making substitutions $m\leftrightarrow\psi(m)=\log m$, $M\leftrightarrow\psi(M)=\log M$, $\phi(t)\leftrightarrow\chi\circ\psi^{-1}(t)=e^{rt}$ and $f\leftrightarrow\psi\circ f=\log f$ in (2.1).

In case $r=0$, $s>0$, we put $\chi(t)=\log t$ and $\psi(t)=t^s$. Then $\phi(t)=\chi(\psi^{-1}(t))=\frac{1}{s}\log t$ is a continuous, concave function and ψ is strictly increasing. The inequalities (3.2.16) are now obtained by replacing $m\leftrightarrow\psi(m)=m^s$, $M\leftrightarrow\psi(M)=M^s$, $\phi(t)\leftrightarrow-\chi\circ\psi^{-1}(t)=-\frac{1}{s}\log t$ and $f\leftrightarrow\psi\circ f=f^s$ in (2.1). $\square$

THEOREM 3.2.5. *Under the same hypothesis as in the previous theorem, if $r < s < 0$ or $r < 0 < s$, then:*

$$\begin{aligned} 0 &\leqslant \frac{M^s - A(f^s)}{M^s - m^s} m^r + \frac{A(f^s) - m^s}{M^s - m^s} M^r - (M^{[r]}(f,A))^r \\ &\leqslant \frac{r}{s} \frac{M^{r-s} - m^{r-s}}{M^s - m^s} A([M^s - f^s][f^s - m^s]) \\ &\leqslant \frac{r}{s} \frac{M^{r-s} - m^{r-s}}{M^s - m^s} (M^s - A(f^s))(A(f^s) - m^s) \\ &\leqslant \frac{r}{4s} (M^s - m^s)(M^{r-s} - m^{r-s}). \end{aligned} \tag{3.2.17}$$

If $0 < r < s$, then:

$$\begin{aligned} 0 &\geqslant \frac{M^s - A(f^s)}{M^s - m^s} m^r + \frac{A(f^s) - m^s}{M^s - m^s} M^r - (M^{[r]}(f,A))^r \\ &\geqslant \frac{r}{s} \frac{M^{r-s} - m^{r-s}}{M^s - m^s} A([M^s - f^s][f^s - m^s]) \\ &\geqslant \frac{r}{s} \frac{M^{r-s} - m^{r-s}}{M^s - m^s} (M^s - A(f^s))(A(f^s) - m^s) \\ &\geqslant \frac{r}{4s} (M^s - m^s)(M^{r-s} - m^{r-s}). \end{aligned} \tag{3.2.18}$$

If $s = 0$ and $r < 0$, then:

$$\begin{aligned} 0 &\leqslant \frac{\log M - A(\log f)}{\log M - \log m} m^r + \frac{A(\log f) - \log m}{\log M - \log m} M^r - (M^{[r]}(f,A))^r \\ &\leqslant \frac{r(M^r - m^r)}{\log M - \log m} A([\log M - \log(f)][\log(f) - \log m]) \\ &\leqslant \frac{r(M^r - m^r)}{\log M - \log m} (\log M - A(\log f))(A(\log f) - \log m) \\ &\leqslant \frac{r}{4} (M^r - m^r) \log \frac{M}{m}. \end{aligned} \tag{3.2.19}$$

If $r = 0$ and $s > 0$, then:

$$\begin{aligned} 0 &\geqslant \frac{M^s - A(f^s)}{M^s - m^s} \log m + \frac{A(f^s) - m^s}{M^s - m^s} \log M - \log(M^{[0]}(f,A)) \\ &\geqslant -\frac{1}{s} \frac{1}{M^s m^s} A([M^s - (f^s)][(f^s) - m^s]) \\ &\geqslant -\frac{1}{s} \frac{1}{M^s m^s} (M^s - A(f^s))[A(f^s) - m^s]) \\ &\geqslant \frac{1}{s} (M^s - m^s) \left(\frac{1}{M^s} - \frac{1}{m^s} \right). \end{aligned} \tag{3.2.20}$$

Proof. All the inequalities can be obtained directly from (2.5) by making the same supstitutions as in the proof of the previous theorem. □

REMARK 3.2.1. It is easy to see that $M^{[r]}(f,A) = (M^{[-r]}(f^{-1},A))^{-1}$ holds for every $f \in L$ and $r \in \mathbb{R}$. Using that result, we can obtain Theorem 3.2.4 from Theorem 3.2.2, and Theorem 3.2.5 from Theorem 3.2.3 by replacing $f \leftrightarrow f^{-1}$, $-r \leftrightarrow s$ and $-s \leftrightarrow r$.

3.3. The Hölder inequality

THEOREM 3.3.1. [7, p. 113] (Hölder's inequality for positive functionals) *Let L satisfy conditions $L1,L2$, and A satisfy conditions $A1,A2$ on a non-empty set E. Let $p > 1$ and $q = p/(p-1)$. If $w, f, g \geqslant 0$ on E and $wf^p, wg^q, wfg \in L$, then we have*

$$A(wfg) \leqslant A^{1/p}(wf^p)A^{1/q}(wg^q) \tag{3.3.1}$$

In case $0 < p < 1$ and $A(wg^q) > 0$ (or $p < 0$ and $A(wf^p) > 0$) the inequality in (3.3.1) is reversed.

THEOREM 3.3.2. [7, p. 114, Theorem 4.14] *Let L and A satisfy conditions $L1,L2$, and $A1,A2$ on a non-empty set E. Let $p > 1$ and $q = p/(p-1)$, and $w, f, g \geqslant 0$ on E with $wf^p, wg^q, wfg \in L$. If $0 < m \leqslant f(t)g^{-q/p}(t) \leqslant M$ for $t \in E$, then*

$$(M-m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \leqslant (M^p - m^p)A(wfg). \tag{3.3.2}$$

If $p < 0$, then (3.3.2) also holds provided either $A(wf^p) > 0$ or $A(wg^q) > 0$. If $0 < p < 1$, then the reversed inequality in (3.3.2) holds provided either $A(wf^p) > 0$ or $A(wg^q) > 0$.

The following results are converses of Hölder's inequality:

THEOREM 3.3.3. *Let L satisfy conditions $L1,L2$, and A satisfy conditions $A1,A2$ on a non-empty set E. Let $p > 1$ and $q = p/(p-1)$. If $w, f, g \geqslant 0$ on E and $wf^p, wg^q, wfg \in L$, $A(wg^q) > 0$, then we have*

$$\begin{aligned} 0 &\leqslant A(wf^p)A^{p/q}(wg^q) - A^p(wfg) \\ &\leqslant (MA(wg^q) - A(wfg))(A(wfg) - mA(wg^q))\frac{p(M^{p-1} - m^{p-1})}{M-m}A^{p-2}(wg^q) \\ &\leqslant \frac{p}{4}(M-m)(M^{p-1} - m^{p-1})A^p(wg^q) \end{aligned} \tag{3.3.3}$$

where $m \leqslant f(t)g^{-q/p}(t) \leqslant M$ for $t \in E$. If $A(wfg) > 0$, then the inequalities (3.3.3) also hold for $p < 0$. In case $0 < p < 1$ the inequalities in (3.3.3) are reversed.

Proof. Function $\phi(t) = t^p$ is continuous, and convex for $p > 1$ and $p < 0$. We define functional $B(f) = \dfrac{A(wf)}{A(w)}$ for $w \in L$ such that $w \geqslant 0$ and $A(w) > 0$. $B(1) = \dfrac{A(w)}{A(w)} = 1$, so B satisfies the conditions of Theorem 2.1. Now we can obtain the inequalities (3.3.3) from (2.1) by replacing $A \leftrightarrow B$, $w \leftrightarrow wg^q$ and $f \leftrightarrow fg^{-q/p}$.

For $0 < p < 1$, $\phi(t) = t^p$ is concave, so we obtain the reversed inequalities in the same way as above. □

THEOREM 3.3.4. *With the assumptions in Theorem 3.3.3, if $p > 1$ or $p < 0$ the following inequalities are valid*

$$\begin{aligned} 0 &\leqslant \frac{MA(wg^q) - A(wfg)}{M-m} m^p + \frac{A(wfg) - mA(wg^q)}{M-m} M^p - A(wf^p) \\ &\leqslant p \frac{M^{p-1} - m^{p-1}}{M-m} A(wg^q[M - fg^{-q/p}][fg^{-q/p} - m]) \\ &\leqslant \frac{p}{A(wg^q)} \frac{M^{p-1} - m^{p-1}}{M-m} (MA(wg^q) - A(wfg))(A(wfg) - mA(wg^q)) \\ &\leqslant \frac{p}{4}(M-m)(M^{p-1} - m^{p-1}) A(wg^q) \end{aligned} \tag{3.3.4}$$

where $m \leqslant f(t)g^{-q/p}(t) \leqslant M$ for $t \in E$. If $0 < p < 1$, the inequalities in (3.3.4) are reversed.

Proof. Function $\phi(t) = t^p$ is continuous, and convex for $p > 1$ and $p < 0$. We define functional $B(f) = \frac{A(wf)}{A(w)}$ for $w \in L$ such that $w \geqslant 0$ and $A(w) > 0$. $B(1) = \frac{A(w)}{A(w)} = 1$, so B satisfies the conditions of Theorem 2.1. We can obtain the inequalities (3.3.4) from (2.5) by replacing $A \leftrightarrow B$, $w \leftrightarrow wg^q$ and $f \leftrightarrow fg^{-q/p}$.

For $0 < p < 1$, $\phi(t) = t^p$ is concave, so we obtain the reversed inequalities in the same way as above. □

THEOREM 3.3.5. *Let L satisfy conditions $L1, L2$, and A satisfy conditions $A1, A2$ on a non-empty set E. Let $0 < p < 1$ and $q = p/(p-1)$. If $f, g \geqslant 0$ on E and $f^p, g^q, fg \in L$, $A(g^q) > 0$, then we have*

$$\begin{aligned} 0 &\leqslant A(fg) - A^{1/p}(f^p) A^{1/q}(g^q) \\ &\leqslant \frac{1}{pA(g^q)} (MA(g^q) - A(f^p))(A(f^p) - mA(g^q)) \frac{M^{-1/q} - m^{-1/q}}{M-m} \\ &\leqslant \frac{1}{4p}(M-m)(M^{-1/q} - m^{-1/q}) A(g^q) \end{aligned} \tag{3.3.5}$$

where $m \leqslant f^p(t)g^{-q}(t) \leqslant M$ for $t \in E$. If $A(f^p) > 0$, inequalities (3.3.5) hold for $p < 0$. In case $p > 1$ the inequalities in (3.3.5) are reversed.

Proof. We define functional $B(f) = \frac{A(wf)}{A(w)}$ for $w \in L$ such that $w \geqslant 0$ and $A(w) > 0$. $B(1) = \frac{A(w)}{A(w)} = 1$, so B satisfies the conditions of Theorem 2.1. Function $\phi(t)=t^{1/p}$ is continuous, and for $p<1$ convex, so we can obtain the inequalities (3.3.5) from (2.1) by replacing $A \leftrightarrow B$, $w \leftrightarrow \frac{g^q}{A(g^q)}$ and $f \leftrightarrow \frac{f^p}{g^q}$.

For $p>1$, the function $\phi(t)=t^{1/p}$ is concave, so we obtain the reversed inequalities in the same way as above. □

THEOREM 3.3.6. *With the assumptions in Theorem 3.3.5, if $p<1$ the following inequalities are valid*

$$\begin{aligned}
0 &\leqslant \frac{MA(g^q)-A(f^p)}{M-m}m^{1/p}+\frac{A(f^p)-mA(g^q)}{M-m}M^{1/p}-A(fg)\\
&\leqslant \frac{1}{p}\frac{M^{-1/q}-m^{-1/q}}{M-m}A(g^{-q}[Mg^q-f^p][f^p-mg^q])\\
&\leqslant \frac{1}{pA(g^q)}\frac{M^{-1/q}-m^{-1/q}}{M-m}(MA(g^q)-A(f^p))(A(f^p)-mA(g^q))\\
&\leqslant \frac{1}{4p}(M-m)(M^{-1/q}-m^{-1/q})A(g^q) \qquad (3.3.6)
\end{aligned}$$

where $m \leqslant f^p(t)g^{-q}(t) \leqslant M$ for $t \in E$. If $p>1$, the inequalities in (3.3.6) are reversed.

Proof. We define functional $B(f)=\frac{A(wf)}{A(w)}$ for $w \in L$ such that $w \geqslant 0$ and $A(w)>0$. $B(1)=\frac{A(w)}{A(w)}=1$, so B satisfies the conditions of Theorem 2.1. Function $\phi(t)=t^{1/p}$ is continuous, and convex for $p<1$, so we can obtain the inequalities (3.3.6) from (2.5) by replacing $A \leftrightarrow B$, $w \leftrightarrow \frac{g^q}{A(g^q)}$ and $f \leftrightarrow \frac{f^p}{g^p}$.

For $p>1$, $\phi(t)=t^{1/p}$ is concave, so we obtain the reversed inequalities by applying (3.3.6) to $-\phi$. □

THEOREM 3.3.7. *Let L satisfy conditions $L1,L2$, and A satisfy conditions $A1,A2$ on a non-empty set E. Let $p>1$ or $p<0$ and $q=p/(p-1)$. If $f,g \geqslant 0$ on E and $g^q, fg \in L$, $A(g^q)>0$, then we have*

$$\begin{aligned}
0 &\leqslant A(f^p)A^{p/q}(g^q)-A^p(fg)\\
&\leqslant p(MA(g^q)-A(fg))(A(fg)-mA(g^q))\frac{M^{p-1}-m^{p-1}}{M-m}A^{p-2}(g^q)\\
&\leqslant \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A^p(g^q) \qquad (3.3.7)
\end{aligned}$$

where $m \leqslant f(t)g^{1-q}(t) \leqslant M$ for $t \in E$. In case $0<p<1$ the inequalities in (3.3.7) are reversed.

Proof. Function $\phi(t) = t^p$ is continuous, and convex for $p > 1$ and $p < 0$. We define functional $B(f) = \dfrac{A(wf)}{A(w)}$ for $w \in L$ such that $w \geqslant 0$ and $A(w) > 0$. $B(1) = \dfrac{A(w)}{A(w)} = 1$, so B satisfies the conditions of Theorem 2.1. Now we can obtain the inequalities (3.3.7) from (2.1) by replacing $A \leftrightarrow B$, $w \leftrightarrow g^q$ and $f \leftrightarrow fg^{1-q}$.

For $0 < p < 1$, the function $\phi(t) = t^p$ is concave, so we obtain the reversed inequalities in the same way as above. □

THEOREM 3.3.8. *With the assumptions in Theorem 3.3.7, if $p > 1$ or $p < 0$ the following inequalities are valid*

$$\begin{aligned} 0 &\leqslant \frac{MA(g^q) - A(fg)}{M-m} m^p + \frac{A(fg) - mA(g^q)}{M-m} M^p - A(f^p) \\ &\leqslant p \frac{M^{p-1} - m^{p-1}}{M-m} A(g^{-q}[Mg^q - fg][fg - mg^q]) \\ &\leqslant \frac{p}{A(g^q)} \frac{M^{p-1} - m^{p-1}}{M-m} (MA(g^q) - A(fg))(A(fg) - mA(g^q)) \\ &\leqslant \frac{p}{4}(M-m)(M^{p-1} - m^{p-1}) A(g^q) \end{aligned} \tag{3.3.8}$$

where $m \leqslant f(t)g^{1-q}(t) \leqslant M$ for $t \in E$. If $0 < p < 1$, the inequalities in (3.3.8) are reversed.

Proof. We define functional $B(f) = \dfrac{A(wf)}{A(w)}$ for $w \in L$ such that $w \geqslant 0$ and $A(w) > 0$. $B(1) = \dfrac{A(w)}{A(w)} = 1$, so B satisfies the conditions of Theorem 2.1. Function $\phi(t) = t^p$ is continuous, and convex for $p > 1$ and $p < 0$, so we can obtain the inequalities (3.3.8) from (2.5) by replacing $A \leftrightarrow B$, $w \leftrightarrow g^q$ and $f \leftrightarrow fg^{1-q}$.

For $0 < p < 1$, $\phi(t) = t^p$ is concave, so we obtain the reversed inequalities by applying (3.3.8) to $-\phi$. □

REMARK 3.3.1.

(i) Under the assumptions of theorem 3.3.6, for $p < 1$ from the first inequality in (3.3.6) we can obtain the inequality

$$(M-m)A(fg) \leqslant A(f^p)(M^{1/p} - m^{1/p}) - A(g^q)(mM^{1/p} - Mm^{1/p}). \tag{3.3.9}$$

If $p > 1$, the inequality (3.3.9) is reversed.

(ii) Analogously, from theorem 3.3.8 for $p > 1$ or $p < 0$, we can obtain the following inequality

$$(M-m)A(f^p) + A(g^q)(mM^p - Mm^p) \leqslant A(fg)(M^p - m^p). \tag{3.3.10}$$

If $0 < p < 1$, the inequality (3.3.10) is reversed.

This inequality can also be obtained from (3.3.2) for $w = 1$.

3.4. Hadamard's inequality and generalizations

THEOREM 3.4.1. ([6]) (Hermite – Hadamard's inequality) *Let* $-\infty < a < b < \infty$ *and* $f\colon [a,b] \to \mathbb{R}$. *If f is convex, then*

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leqslant \frac{f(a)+f(b)}{2} \tag{3.4.1}$$

If f *is concave, the inequalities in (3.4.1) are reversed.*

The following two results are obtained by applying Theorem 2.1 and Theorem 2.2 to Hamadard's inequality.

THEOREM 3.4.2. *Let* $a < b$ *and let us assume that f is a continuous convex function on an open interval of real numbers* $I \supset [a,b]$. *Then*

$$\begin{aligned} 0 &\leqslant \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\Big(\frac{a+b}{2}\Big) \\ &\leqslant \frac{1}{4}(b-a)(f'_-(b) - f'_+(a)). \end{aligned} \tag{3.4.2}$$

If f *is concave, the inequalities in (3.4.2) are reversed.*

Proof. Inequalities (3.4.2) are obtained from (2.1) by replacing

$$A(f) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t, \quad f(t) \leftrightarrow t \quad \text{and} \quad \phi \leftrightarrow f.$$

If f is concave, the reversed inequalities follow from the convexity of $-f$. □

THEOREM 3.4.3. *Let* $a < b$ *and let us assume that f is a continuous convex function on an open interval of real numbers* $I \supset [a,b]$. *Then*

$$\begin{aligned} 0 &\leqslant \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\ &\leqslant \frac{1}{6}(b-a)(f'_-(b) - f'_+(a)). \end{aligned} \tag{3.4.3}$$

If f *is concave, the inequalities in (3.4.3) are reversed.*

Proof. Inequalities (3.4.3) are obtained from (2.5) by replacing

$$A(f) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t, \quad f(t) \leftrightarrow t \quad \text{and} \quad \phi \leftrightarrow f.$$

If f is concave, the reversed inequalities follow from the convexity of $-f$. □

REMARK 3.4.1. Let $a < b$ and let us assume that f is continuous convex function on an open interval of real numbers $I \supset [a,b]$. By combining the above results, we obtain

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{1}{6}(b-a)(f'_-(b) - f'_+(a)) \leqslant \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\ &\leqslant f\Big(\frac{a+b}{2}\Big) + \frac{1}{4}(b-a)(f'_-(b) - f'_+(a)).\end{aligned} \tag{3.4.4}$$

If f is concave, the inequalities in (3.4.4) are reversed.

The following result is a generalization of Hadamard's inequality for positive linear functionals given in [8]:

THEOREM 3.4.4. ([8]) *Let ϕ be a continuous convex function on an interval $I \supset [m,M]$, where $-\infty < m < M < \infty$. Suppose that $f\colon E \to \mathbb{R}$ satisfies $m \leqslant f(t) \leqslant M$ for every $t \in E$, $f \in L$ and $\phi(f) \in L$. Let $A\colon L \to \mathbb{R}$ be a positive linear functional with $A(1) = 1$, and let $p = p_f$, $q = q_f$ be nonnegative real numbers (with $p+q > 0$) for which*

$$A(f) = \frac{pm+qM}{p+q}. \tag{3.4.5}$$

Then

$$\phi\Big(\frac{pm+qM}{p+q}\Big) \leqslant A(\phi(f)) \leqslant \frac{p\phi(m)+q\phi(M)}{p+q}. \tag{3.4.6}$$

Applying Theorem 2.1 and Theorem 2.2 to the previous theorem, we obtain:

THEOREM 3.4.5. *Let ϕ be a continuous convex function on an open interval of real numbers $I \supset [m,M]$, where $-\infty < m < M < \infty$. Suppose that $f\colon E \to \mathbb{R}$ satisfies $m \leqslant f(t) \leqslant M$ for every $t \in E$, $f \in L$ and $\phi(f) \in L$. Let $A\colon L \to \mathbb{R}$ be a positive linear functional with $A(1) = 1$, and let $p = p_f$, $q = q_f$ be nonnegative real numbers (with $p+q > 0$) for which*

$$A(f) = \frac{pm+qM}{p+q}. \tag{3.4.7}$$

Then

$$\begin{aligned}0 &\leqslant A(\phi(f)) - \phi\Big(\frac{pm+qM}{p+q}\Big) \\ &\leqslant \frac{pq}{(p+q)^2}(M-m)(\phi'_-(M) - \phi'_+(m)) \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)).\end{aligned} \tag{3.4.8}$$

If ϕ is concave, the inequalities in (3.4.8) are reversed.

Proof. We first need to observe that since $m \leqslant A(f) \leqslant M$, there always exist $p \geqslant 0$ and $q \geqslant 0$, $p+q > 0$ satisfying (3.4.7).

Inequalities (3.4.8) are obtained from (2.1) by replacing $A(f) \leftrightarrow \dfrac{pm+qM}{p+q}$. If ϕ is concave, the reversed inequalities follow from the convexity of $-\phi$. □

THEOREM 3.4.6. *Under the same assumptions as in the previous theorem, we have*

$$\begin{aligned} 0 &\leqslant \frac{p\phi(m)+q\phi(M)}{p+q} - A(\phi(f)) \\ &\leqslant \frac{\phi'_-(M)-\phi'_+(m)}{M-m} A([M-f][f-m]) \\ &\leqslant \frac{pq}{(p+q)^2}(M-m)(\phi'_-(M)-\phi'_+(m)) \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)). \end{aligned} \tag{3.4.9}$$

If ϕ is concave, the inequalities in (3.4.9) are reversed.

Proof. Inequalities (3.4.9) are obtained from (2.5) by replacing $A(f) \leftrightarrow \dfrac{pm+qM}{p+q}$. If ϕ is concave, the reversed inequalities follow from the convexity of $-\phi$. □

REMARK 3.4.2. Under the same assumptions as in last two theorems, we have

$$\begin{aligned} &\frac{p\phi(m)+q\phi(M)}{p+q} - \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) \leqslant A(\phi(f)) \\ &\leqslant \phi\Big(\frac{pm+qM}{p+q}\Big) + \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)). \end{aligned} \tag{3.4.10}$$

If ϕ is concave, the inequalities in (3.4.10) are reversed.

3.5. The inequalities of Giaccardi and Petrović

THEOREM 3.5.1. (Giaccardi, [10]) *Let* $\boldsymbol{p}$ *be a nonnegative n-tuple and* $\boldsymbol{x}$ *be a real n-tuple such that*

$$(x_i - x_0)\Big(\sum_{j=1}^{n} p_j x_j - x_i\Big) \geqslant 0, \; i = 1, \ldots, n; \tag{3.5.1}$$

$$x_0, \sum_{i=1}^{n} p_i x_i \in [a,b]$$

$$\sum_{k=1}^{n} p_k x_k \neq x_0.$$

If $f\colon [a,b] \to \mathbb{R}$ is a convex function, then

$$\sum_{i=1}^{n} p_i f(x_i) \leqslant A f\Big(\sum_{i=1}^{n} p_i x_i\Big) + B\Big(\sum_{i=1}^{n} p_i - 1\Big) f(x_0) \tag{3.5.2}$$

where

$$A = \frac{\sum_{i=1}^{n} p_i (x_i - x_0)}{\sum_{i=1}^{n} p_i x_i - x_0}, \quad B = \frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i x_i - x_0}. \tag{3.5.3}$$

The succeeding results is a converse of Giaccardi's inequality obtained directly from Theorem 2.2:

THEOREM 3.5.2. *Let $[a,b]$ be a closed interval and let $I \supset [a,b]$ be an open interval of real numbers. Let* $\mathbf{p}$ *be a nonnegative n-tuple and* $\mathbf{x}$ *be a real n-tuple such that*

$$(x_i - x_0)(\sum_{j=1}^{n} p_j x_j - x_i) \geqslant 0, \quad i = 1,\dots,n; \quad \sum_{k=1}^{n} p_k x_k \neq x_0; \quad x_0, \sum_{i=1}^{n} p_i x_i \in [a,b]. \tag{3.5.4}$$

If $f\colon I \to \mathbb{R}$ is a continuous convex function, then

$$\begin{aligned} 0 &\leqslant Af\Big(\sum_{i=1}^{n} p_i x_i\Big) + B\Big(\sum_{i=1}^{n} p_i - 1\Big) f(x_0) - \sum_{i=1}^{n} p_i f(x_i) \\ &\leqslant \frac{f'_-(M) - f'_+(m)}{M-m} \sum_{i=1}^{n} p_i (M - x_i)(x_i - m) \\ &\leqslant \Big(M - \frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\Big)\Big(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i} - m\Big) \frac{f'_-(M) - f'_+(m)}{M-m} \sum_{i=1}^{n} p_i \\ &\leqslant \frac{1}{4}(M-m)(f'_-(M) - f'_+(m)) \sum_{i=1}^{n} p_i \end{aligned} \tag{3.5.5}$$

where $m = \min\{x_0, \sum_{i=1}^{n} p_i x_i\}$, $M = \max\{x_0, \sum_{i=1}^{n} p_i x_i\}$, and A, B are defined in (3.5.3). If f is concave, the inequalities in (3.5.5) are reversed.

Proof. Let f be a convex function. The inequalities (3.5.5) are obtained from (2.5) by substituting $A(\mathbf{x}) = \frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}$ and $\phi \leftrightarrow f$. If f is concave, then the reversed inequalities follow from (3.5.5) by substituting $f \leftrightarrow -f$ which is convex. □

The well-known Petrović's inequality [9] for convex function $f\colon [0,a] \to \mathbb{R}$ is given by

$$\sum_{i=1}^{n} f(x_i) \leqslant f\Big(\sum_{i=1}^{n} x_i\Big) + (n-1) f(0) \tag{3.5.6}$$

where $x_i, i = 1,\dots,n$ are nonnegative numbers such that $x_1,\dots,x_n, \sum_{i=1}^{n} x_i \in [0,a]$.

The following result follows directly by applying Theorem 2.2 to Petrović's inequality, but can also be obtained as a special case of Theorem 3.5.2 for $p_1 = \dots = p_n = 1$ and $x_0 = 0$.

THEOREM 3.5.3. *Let f be a continuous convex function on an open interval of real numbers $I \supset [0,a]$ If $x_1,\dots,x_n \in [0,a]$ are real numbers such that $\sum_{i=1}^{n} x_i \in \langle 0,a]$,* *then*

$$\begin{aligned}0 &\leqslant f\Big(\sum_{i=1}^{n} x_i\Big) + (n-1)f(0) - \sum_{i=1}^{n} f(x_i)\\ &\leqslant \frac{f'_-(\sum_{i=1}^{n} x_i) - f'_+(0)}{\sum_{i=1}^{n} x_i} \sum_{i=1}^{n} x_i(\sum_{j=1}^{n} x_j - x_i)\\ &\leqslant (f'_-(\sum_{i=1}^{n} x_i) - f'_+(0))(\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n} x_i)\\ &\leqslant \frac{n}{4}(f'_-(\sum_{i=1}^{n} x_i) - f'_+(0))\sum_{i=1}^{n} x_i\end{aligned} \tag{3.5.7}$$

If f is concave, the inequalities in (3.5.7) are reversed.

Proof. Let f be a convex function. The inequalities (3.5.7) are obtained from (2.5) by substituting $A(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\phi \leftrightarrow f$. If f is concave, then the reversed inequalities follow from (3.5.7) by substituting $f \leftrightarrow -f$ which is convex. □

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Rozarija Jakšić
Faculty of textile technology, University of Zagreb
Prilaz baruna Filipovića 28A
10000 Zagreb, Croatia
e-mail: rozarija.jaksic@ttf.hr

Josip Pečarić
Faculty of textile technology, University of Zagreb
Prilaz baruna Filipovića 28A
10000 Zagreb, Croatia
e-mail: pecaric@hazu.hr