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ON OPIAL–TYPE INTEGRAL INEQUALITIES AND APPLICATIONS

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Abstract. In the present paper we establish some new Opial-type inequalities involving higher order partial derivatives. Our results in special cases yield some of the recent results on Opial's inequality. As application, we prove the uniqueness of the solution of initial value problem involving higher order partial differential equation.

1. Introduction and statement of results

In 1960, Opial [17] established the following integral inequality:

THEOREM A. *Suppose $x \in C^1[0,a]$ satisfies $x(0) = x(a) = 0$ and $x(t) > 0$ for all $t \in (0,a)$. Then the integral inequality holds*

$$\int_0^a \left|x(t)x'(t)\right| dt \leqslant \frac{a}{4}\int_0^a (x'(t))^2 dt, \tag{1.1}$$

where this constant $\frac{a}{4}$ is best possible.

The first natural extension of Opial's inequality (1.1) involving the higher order derivatives $x^{(n)}(s)(n \geqslant 1)$ instead of $x'(s)$ is embodied in the following:

THEOREM B. [1] *Let $x(t) \in C^{(n)}[0,a]$ be such that $x^{(i)}(0) = 0,\ 0 \leqslant i \leqslant n-1$ $(n \geqslant 1)$. Then the following inequality holds*

$$\int_0^a \left|x(t)x^{(n)}(t)\right| dt \leqslant \frac{1}{2}a^n \int_0^a \left|x^{(n)}(t)\right|^2 dt. \tag{1.2}$$

A sharp version of (1.2) is the following:

THEOREM C. [11] *Let $x(t) \in C^{(n-1)}[0,a]$ be such that $x^{(i)}(0) = 0,\ 0 \leqslant i \leqslant n-1$ $(n \geqslant 1)$. Further, let $x^{(n-1)}(t)$ be absolutely continuous, and $\int_0^a \left|(x^{(n)}(t)\right|^2 dt < \infty$. Then the following inequality holds*

$$\int_0^a \left|x(t)x^{(n)}(t)\right| dt \leqslant \frac{1}{2n!}\left(\frac{n}{2n-1}\right)^{1/2} a^n \int_0^a \left|x^{(n)}(t)\right|^2 dt. \tag{1.3}$$

A more general version of (1.3) is following:

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THEOREM D. [18] *Let* $x(t) \in C^{(n-1)}[0,a]$ *be such that* $x^{(i)}(0) = 0,\ 0 \leqslant i \leqslant n-1$ $(n \geqslant 1)$. *Further, let* $x^{(n-1)}(t)$ *be absolutely continuous, and* $\int_0^a \left|(x^{(n)}(t)\right|^2 dt < \infty$. *Then the following inequality holds for* $0 \leqslant \kappa \leqslant n-1$

$$\int_0^a \left|\prod_{\kappa=0}^{n} x^{(\kappa)}(t)\right| dt \leqslant K(n) a^{(n^2+1)/2} \left(\int_0^a \left|x^{(n)}(t)\right|^2 dt\right)^{(n+1)/2}, \tag{1.4}$$

where

$$K(n) = \frac{1}{(n^2+1)(n+1)\prod_{\kappa=0}^{n-1}(n-\kappa-1)!} \left(\frac{(n^2+1)(n+1)}{\prod_{\kappa=0}^{n-1}(2n-2\kappa-1)}\right)^{1/2}.$$

Opial's inequality and its generalizations, extensions and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [1, 2, 4, 14, 15]. The inequality (1.1) has received considerable attention and a large number of papers dealing with new proofs, extensions, generalizations, variants and discrete analogues of Opial's inequality have appeared in the literature [3, 5–13, 16–26]. For an extensive survey on these inequalities, see [1, 15].

The first aim of the present paper is to establish a new Opial-type inequalities involving higher order partial derivatives which is a generalization of inequality (1.4).

THEOREM 1.1. *Let* $x(s,t) \in C^{(n-1,m-1)}([0,a]\times[0,b])$ *be such that* $\frac{\partial^\kappa}{\partial s^\kappa} x(s,t)|_{s=0} = 0,\ \ 0 \leqslant \kappa \leqslant n-1$ *and* $\frac{\partial^\lambda}{\partial t^\lambda} x(s,t)|_{t=0} = 0,\ \ 0 \leqslant \lambda \leqslant m-1.$ *Further, let* $\frac{\partial^n}{\partial s^n}\left(\frac{\partial^{m-1}}{\partial t^{m-1}} x(s,t)\right)$ *and* $\frac{\partial^{n-1}}{\partial s^{n-1}}\left(\frac{\partial^m}{\partial t^m} x(s,t)\right)$ *are absolutely continuous on* $[0,a]\times[0,b]$, *and* $\int_0^a \int_0^b \left|\frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t)\right|^2 dsdt$, *exist. Then*

$$\int_0^a \int_0^b \left|\frac{\prod_{\kappa=0}^{n}\prod_{\lambda=0}^{m} \frac{\partial^{\kappa+\lambda}}{\partial s^\kappa \partial t^\lambda} x(s,t)}{\prod_{\kappa=0}^{n-1} \frac{\partial^{\kappa+m}}{\partial s^\kappa \partial t^m} x(s,t) \cdot \prod_{\lambda=0}^{m-1} \frac{\partial^{n+\lambda}}{\partial s^n \partial t^\lambda} x(s,t)}\right| dsdt \leqslant M(n,m) a^{(n^2+1)/2} b^{(m^2+1)/2}$$

$$\times \left(\int_0^a \int_0^b \left|\frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t)\right|^2 dsdt\right)^{(mn+1)/2}, \tag{1.5}$$

where

$$M(n,m) = \frac{1}{(n^2+1)(m^2+1)(mn+1)\prod_{\kappa=0}^{n-1}\prod_{\lambda=0}^{m-1}(n-\kappa-1)!(m-\lambda-1)!}$$

$$\times \left(\frac{(n^2+1)(m^2+1)(mn+1)}{\prod_{\kappa=0}^{n-1}\prod_{\lambda=0}^{m-1}\left[(2n-2\kappa-1)(2m-2\lambda-1)\right]}\right)^{1/2}.$$

REMARK 1.1. Let $x(s,t)$ reduce to $x(t)$ and with suitable modifications, then (1.5) changes to (1.4).

A result involving two functions and their higher order derivatives is embodied in the following:

THEOREM E. [19] *For $j=1,2$ let $x_j(t)\in C^{n-1}[0,a]$ be such that $x_j^{(i)}(0)=0$, $0\leqslant i\leqslant n-1$. Further, let $x_j^{(n-1)}$ be absolutely continuous, and $\int_0^a |x_j^{(n)}(t)|^2dt<\infty$. Then*

$$\int_0^a \left\{\left|x_2(t)x_1^{(n)}(t)\right|+\left|x_1(t)x_2^{(n)}(t)\right|\right\}dt$$
$$\leqslant \frac{1}{2n!}\left(\frac{n}{2n-1}\right)^{1/2} a^n\left[\int_0^a\left(\left|x_1^{(n)}(t)\right|^2+\left|x_2^{(n)}(t)\right|^2\right)dt\right], \tag{1.6}$$

The second aim of the present paper is to establish a new Opial-type inequalities involving higher order partial derivatives which is a generalization of inequality (1.6).

THEOREM 1.2. *For $i=1,2$, let $x_i(s,t)\in C^{(n-1,m-1)}([0,a]\times[0,b])$ be such that $\frac{\partial^\kappa}{\partial s^\kappa}x_i(s,t)|_{s=0}=0$, $0\leqslant\kappa\leqslant n-1$ and $\frac{\partial^\lambda}{\partial t^\lambda}x_i(s,t)|_{t=0}=0$, $0\leqslant\lambda\leqslant m-1$. Further, let $\frac{\partial^n}{\partial s^n}\left(\frac{\partial^{m-1}}{\partial t^{m-1}}x_i(s,t)\right)$ and $\frac{\partial^{n-1}}{\partial s^{n-1}}\left(\frac{\partial^m}{\partial t^m}x_i(s,t)\right)$ are absolutely continuous on $[0,a]\times[0,b]$, and $\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_i(s,t)\right|^2dsdt$, exist. Then*

$$\int_0^a\int_0^b\left(\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x_1(s,t)\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_2(s,t)\right|+\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x_2(s,t)\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_1(s,t)\right|\right)dsdt$$
$$\leqslant\sqrt{2}D(n,m)a^{n-\kappa}b^{m-\lambda}\int_0^a\int_0^b\left[\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_1(s,t)\right|^2+\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_2(s,t)\right|^2\right]dsdt, \tag{1.7}$$

where

$$D(n,m)=\frac{1}{4(n-\kappa)!(m-\lambda)!}\left(\frac{(n-\kappa)(m-\lambda)}{(2n-2\kappa-1)(2m-2\lambda-1)}\right)^{1/2}.$$

REMARK 1.2. For $i=1,2$, let $x_i(s,t)$ reduce to $x_i(t)$, respectively and with suitable modifications, then (1.7) changes to (1.6).

As application, we prove the uniqueness of the solution of initial value problem involving higher order partial differential equation.

THEOREM 1.3. *For the partial differential equation*

$$\frac{\partial^{n+m}\phi}{\partial s^n\partial t^m}=f(s,t,\langle\phi\rangle) \tag{1.8}$$

together with the initial conditions

$$\frac{\partial^\kappa\phi(0,t)}{\partial s^\kappa}=\alpha_\kappa(t),\quad 0\leqslant\kappa\leqslant n-1, \tag{1.9}$$

$$\frac{\partial^{\lambda}\phi(s,0)}{\partial t^{\lambda}}=\beta_{\lambda}(s),\quad 0\leqslant\lambda\leqslant m-1,\tag{1.10}$$

where

$$\langle\phi\rangle=(\phi_{0,0},\phi_{1,0},\ldots,\phi_{n-1,0},\phi_{0,1},\ldots,\phi_{n-1,1},\phi_{0,2},\ldots,\phi_{n-1,m-1})$$

and

$$\phi_{i,j}=\frac{\partial^{i+j}\phi}{\partial s^{i}\partial t^{j}},$$

we assume that $f:([0,s]\times[0,t])\times\mathbb{R}^{nm}\to\mathbb{R}$ *are continuous, and* $\alpha_{\kappa}:[0,s]\to\mathbb{R}$, $\beta_{\lambda}:[0,t]\to\mathbb{R}$ *are m and n times differentiable respectively, and* $\alpha_{\kappa}^{(\lambda)}(0)=\beta_{\lambda}^{(\kappa)}(0)$, $0\leqslant\kappa\leqslant n-1$, $0\leqslant\lambda\leqslant m-1$. *Further, the function* f *satisfies the following condition, i.e., for all* $(s,t,\langle\phi\rangle),(s,t,\langle\bar{\phi}\rangle)\in([0,s]\times[0,t])\times\mathbb{R}^{nm}$,

$$|f(s,t,\langle\phi\rangle)-f(s,t,\langle\bar{\phi}\rangle)|\leqslant\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|\phi_{i,j}-\bar{\phi}_{i,j}|.$$

Then the problem (1.8)–(1.10) has at most one solution on $[0,s]\times[0,t]$.

2. Proofs of results

Proof of Theorem 1.1. From the hypotheses of the Theorem 1.1, we have for $0\leqslant\kappa\leqslant n-1$, $0\leqslant\lambda\leqslant m-1$

$$\left|\frac{\partial^{\kappa+\lambda}}{\partial s^{\kappa}\partial t^{\lambda}}x(s,t)\right|\leqslant\frac{1}{(n-\kappa-1)!(m-\lambda-1)!}$$

$$\times\int_{0}^{s}\int_{0}^{t}(s-\sigma)^{n-\kappa-1}(t-\tau)^{m-\lambda-1}\left|\frac{\partial^{n+m}}{\partial\sigma^{n}\partial\tau^{m}}x(\sigma,\tau)\right|d\sigma d\tau.\tag{2.1}$$

By using Cauchy-Schwarz inequality, we obtain

$$\left|\frac{\partial^{\kappa+\lambda}}{\partial s^{\kappa}\partial t^{\lambda}}x(s,t)\right|\leqslant\frac{1}{(n-\kappa-1)!(m-\lambda-1)!}\left(\int_{0}^{s}\int_{0}^{t}(s-\sigma)^{2(n-\kappa-1)}(t-\tau)^{2(m-\lambda-1)}d\sigma d\tau\right)^{1/2}$$

$$\times\left(\int_{0}^{s}\int_{0}^{t}\left|\frac{\partial^{n+m}}{\partial\sigma^{n}\partial\tau^{m}}x(\sigma,\tau)\right|^{2}d\sigma d\tau\right)^{1/2}$$

$$=\frac{1}{(n-\kappa-1)!(m-\lambda-1)!\Big[(2n-2k-1)(2m-2\lambda-1)\Big]^{1/2}}s^{n-k-1/2}t^{m-\lambda-1/2}$$

$$\times\left(\int_{0}^{s}\int_{0}^{t}\left|\frac{\partial^{n+m}}{\partial\sigma^{n}\partial\tau^{m}}x(\sigma,\tau)\right|^{2}d\sigma d\tau\right)^{1/2}.\tag{2.2}$$

Multiplying (2.2) by $\left|\frac{\partial^{n+m}}{\partial s^n \partial t^m}x(s,t)\right|$, we obtain for $0 \leqslant \kappa \leqslant n-1$, $0 \leqslant \lambda \leqslant m-1$

$$\left|\frac{\prod_{\kappa=0}^{n}\prod_{\lambda=0}^{m}\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa \partial t^\lambda}x(s,t)}{\prod_{\kappa=0}^{n-1}\frac{\partial^{\kappa+m}}{\partial s^\kappa \partial t^m}x(s,t)\cdot\prod_{\lambda=0}^{m-1}\frac{\partial^{n+\lambda}}{\partial s^n \partial t^\lambda}x(s,t)}\right|$$

$$\leqslant \frac{s^{n^2/2}t^{m^2/2}}{\prod_{\kappa=0}^{n-1}\prod_{\lambda=0}^{m-1}(n-\kappa-1)!(m-\lambda-1)!\prod_{\kappa=0}^{n-1}\prod_{\lambda=0}^{m-1}\Big[(2n-2k-1)(2m-2\lambda-1)\Big]^{1/2}}$$

$$\times\left|\frac{\partial^{n+m}}{\partial s^n \partial t^m}x(s,t)\right|\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial \sigma^n \partial \tau^m}x(\sigma,\tau)\right|^2 d\sigma d\tau\right)^{mn/2}. \qquad (2.3)$$

Integrating the two sides of (2.3) over t from 0 to b first and then integrating the resulting inequality over s from 0 to a and then applying the Cauchy-Schwarz inequality to the right side again, we observe

$$\int_0^a\int_0^b\left|\frac{\prod_{\kappa=0}^{n}\prod_{\lambda=0}^{m}\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa \partial t^\lambda}x(s,t)}{\prod_{\kappa=0}^{n-1}\frac{\partial^{\kappa+m}}{\partial s^\kappa \partial t^m}x(s,t)\cdot\prod_{\lambda=0}^{m-1}\frac{\partial^{n+\lambda}}{\partial s^n \partial t^\lambda}x(s,t)}\right|dsdt$$

$$\leqslant \frac{1}{\prod_{\kappa=0}^{n-1}\prod_{\lambda=0}^{m-1}(n-\kappa-1)!(m-\lambda-1)!\prod_{\kappa=0}^{n-1}\prod_{\lambda=0}^{m-1}\Big[(2n-2k-1)(2m-2\lambda-1)\Big]^{1/2}}$$

$$\times\left(\int_0^a\int_0^b s^{n^2}t^{m^2}dsdt\right)^{1/2}$$

$$\times\left(\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n \partial t^m}x(s,t)\right|^2\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial \sigma^n \partial \tau^m}x(\sigma,\tau)\right|^2 d\sigma d\tau\right)^{mn} dsdt\right)^{1/2}. \qquad (2.4)$$

On the other hand, from the hypotheses of Theorem 1.1 and in view of the following facts

$$\frac{\partial^2}{\partial s \partial t}\left[\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial \sigma^n \partial \tau^m}x(\sigma,\tau)\right|^2 d\sigma d\tau\right)^{mn+1}\right]$$

$$=(mn+1)\left|\frac{\partial^{n+m}}{\partial s^n \partial t^m}x(s,t)\right|^2\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial \sigma^n \partial \tau^m}x(\sigma,\tau)\right|^2 d\sigma d\tau\right)^{mn} \qquad (2.5)$$

and

$$\left(\int_0^a\int_0^b s^{n^2}t^{m^2}dsdt\right)^{1/2}=\frac{a^{(n^2+1)/2}b^{(m^2+1)/2}}{[(n^2+1)(m^2+1)]^{1/2}}. \qquad (2.6)$$

From (2.4), (2.5) and (2.6), we have

$$\int_0^a \int_0^b \left| \frac{\prod_{\kappa=0}^{n} \prod_{\lambda=0}^{m} \frac{\partial^{\kappa+\lambda}}{\partial s^\kappa \partial t^\lambda} x(s,t)}{\prod_{\kappa=0}^{n-1} \frac{\partial^{\kappa+m}}{\partial s^\kappa \partial t^m} x(s,t) \cdot \prod_{\lambda=0}^{m-1} \frac{\partial^{n+\lambda}}{\partial s^n \partial t^\lambda} x(s,t)} \right| dsdt$$

$$\leqslant M(n,m) a^{(n^2+1)/2} b^{(m^2+1)/2} \left(\int_0^a \int_0^b \left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right|^2 dsdt \right)^{(mn+1)/2},$$

where

$$M(n,m) = \frac{1}{(n^2+1)(m^2+1)(mn+1) \prod_{\kappa=0}^{n-1} \prod_{\lambda=0}^{m-1} (n-\kappa-1)!(m-\lambda-1)!}$$

$$\times \left(\frac{(n^2+1)(m^2+1)(mn+1)}{\prod_{\kappa=0}^{n-1} \prod_{\lambda=0}^{m-1} \left[(2n-2\kappa-1)(2m-2\lambda-1) \right]} \right)^{1/2}. \quad \square$$

Proof of Theorem 1.2. From the hypotheses of the Theorem 1.2, we have for $0 \leqslant \kappa \leqslant n-1$, $0 \leqslant \lambda \leqslant m-1$

$$\left| \frac{\partial^{\kappa+\lambda}}{\partial s^\kappa \partial t^\lambda} x_1(s,t) \right| \leqslant \frac{1}{(n-\kappa-1)!(m-\lambda-1)!}$$

$$\times \int_0^s \int_0^t (s-\sigma)^{n-\kappa-1} (t-\tau)^{m-\lambda-1} \left| \frac{\partial^{n+m}}{\partial \sigma^n \partial \tau^m} x_1(\sigma,\tau) \right| d\sigma d\tau. \tag{2.7}$$

Multiplying (2.7) by $\left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x_2(s,t) \right|$, and using Cauchy-Schwarz inequality, we obtain

$$\left| \frac{\partial^{\kappa+\lambda}}{\partial s^\kappa \partial t^\lambda} x_1(s,t) \frac{\partial^{n+m}}{\partial s^n \partial t^m} x_2(s,t) \right| \leqslant \frac{1}{(n-\kappa-1)!(m-\lambda-1)!} \left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x_2(s,t) \right|$$

$$\times \left(\int_0^s \int_0^t (s-\sigma)^{2(n-\kappa-1)} (t-\tau)^{2(m-\lambda-1)} d\sigma d\tau \right)^{1/2} \left(\int_0^s \int_0^t \left| \frac{\partial^{n+m}}{\partial \sigma^n \partial \tau^m} x_1(\sigma,\tau) \right|^2 d\sigma d\tau \right)^{1/2}$$

$$= \frac{s^{n-k-1/2} t^{m-\lambda-1/2}}{(n-\kappa-1)!(m-\lambda-1)! \left[(2n-2k-1)(2m-2\lambda-1) \right]^{1/2}} \left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x_2(s,t) \right|$$

$$\times \left(\int_0^s \int_0^t \left| \frac{\partial^{n+m}}{\partial \sigma^n \partial \tau^m} x_1(\sigma,\tau) \right|^2 d\sigma d\tau \right)^{1/2}. \tag{2.8}$$

Integrating the two sides of (2.8) over t from 0 to b first and then integrating the resulting inequality over s from 0 to a and then applying the Cauchy-Schwarz inequality to the right side again, we observe

$$\int_0^a\int_0^b\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x_1(s,t)\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_2(s,t)\right|dsdt$$

$$\leqslant\frac{1}{(n-\kappa-1)!(m-\lambda-1)!\Big[(2n-2k-1)(2m-2\lambda-1)\Big]^{1/2}}\left(\int_0^a\int_0^b s^{2n-2k-1}t^{2m-2\lambda-1}dsdt\right)^{1/2}$$

$$\times\left(\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_2(s,t)\right|^2\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x_1(\sigma,\tau)\right|^2d\sigma d\tau\right)dsdt\right)^{1/2}. \quad (2.9)$$

Similarly

$$\int_0^a\int_0^b\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x_2(s,t)\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_1(s,t)\right|dsdt$$

$$\leqslant\frac{1}{(n-\kappa-1)!(m-\lambda-1)!\Big[(2n-2k-1)(2m-2\lambda-1)\Big]^{1/2}}\left(\int_0^a\int_0^b s^{2n-2k-1}t^{2m-2\lambda-1}dsdt\right)^{1/2}$$

$$\times\left(\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_1(s,t)\right|^2\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x_2(\sigma,\tau)\right|^2d\sigma d\tau\right)dsdt\right)^{1/2}. \quad (2.10)$$

Thus, in view of the elementary inequality $\alpha^{1/2}+\beta^{1/2}\leqslant[2(\alpha+\beta)]^{1/2}$, $\alpha\geqslant 0$, $\beta\geqslant 0$, we have

$$\int_0^a\int_0^b\left(\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x_1(s,t)\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_2(s,t)\right|+\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x_2(s,t)\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_1(s,t)\right|\right)dsdt$$

$$\leqslant\frac{1}{(n-\kappa-1)!(m-\lambda-1)!\Big[(2n-2k-1)(2m-2\lambda-1)\Big]^{1/2}}\left(\int_0^a\int_0^b s^{2n-2k-1}t^{2m-2\lambda-1}dsdt\right)^{1/2}$$

$$\times\left\{2\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_2(s,t)\right|^2\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x_1(\sigma,\tau)\right|^2d\sigma d\tau\right)dsdt\right.$$

$$\left.+2\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_1(s,t)\right|^2\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x_2(\sigma,\tau)\right|^2d\sigma d\tau\right)dsdt\right\}^{1/2}. \quad (2.11)$$

On the other hand, in view of the following facts

$$\frac{\partial^2}{\partial s\partial t}\left[\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x_1(\sigma,\tau)\right|^2 d\sigma d\tau\right)\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x_2(\sigma,\tau)\right|^2 d\sigma d\tau\right)\right]$$

$$=\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_1(s,t)\right|^2\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x_2(\sigma,\tau)\right|^2 d\sigma d\tau\right)+\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_2(s,t)\right|^2$$

$$\times\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x_1(\sigma,\tau)\right|^2 d\sigma d\tau\right) \tag{2.12}$$

and

$$\left(\int_0^a\int_0^b s^{2n-2k-1}t^{2m-2\lambda-1}dsdt\right)^{1/2}=\frac{a^{n-\kappa}b^{m-\lambda}}{2\Big[(n-\kappa)(m-\lambda)\Big]^{1/2}}. \tag{2.13}$$

From (2.11), (2.12), (2.13) and in view the elementary inequality $(\alpha\beta)^{1/2}\leqslant\frac{1}{2}(\alpha+\beta)$, $\alpha\geqslant 0$, $\beta\geqslant 0$, we have

$$\int_0^a\int_0^b\left(\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x_1(s,t)\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_2(s,t)\right|+\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x_2(s,t)\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_1(s,t)\right|\right)dsdt$$

$$\leqslant 2D(n,m)a^{n-\kappa}b^{m-\lambda}\left[2\left(\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_1(s,t)\right|^2 dsdt\right)\right.$$

$$\left.\times\left(\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_2(s,t)\right|^2 dsdt\right)\right]^{1/2}$$

$$\leqslant\sqrt{2}D(n,m)a^{n-\kappa}b^{m-\lambda}\int_0^a\int_0^b\left[\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_1(s,t)\right|^2+\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x_2(s,t)\right|^2\right]d\sigma d\tau,$$

where

$$D(n,m)=\frac{1}{4(n-\kappa)!(m-\lambda)!}\left(\frac{(n-\kappa)(m-\lambda)}{(2n-2\kappa-1)(2m-2\lambda-1)}\right)^{1/2}. \quad\square$$

Proof of Theorem 1.3. If the problem (1.8)–(1.10) has two solutions $\phi(s,t)$, $\bar{\phi}(s,t)$ then for the function $u(s,t)=\phi(s,t)-\bar{\phi}(s,t)$ the following inequality holds

$$\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}u\right|^2 d\sigma d\tau\leqslant\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\int_0^s\int_0^t\left|\frac{\partial^{i+j}u}{\partial s^i\partial t^j}\right|\left|\frac{\partial^{n+m}u}{\partial\sigma^n\partial\tau^m}\right|d\sigma d\tau. \tag{4.4}$$

For each term on the right side of (4.4), we apply the inequality (2.3) in Theorem 2.2 with $r_k = r_n = 1$, $r = 2$ and $x_1(s,t) = x_2(s,t) = u(s,t)$, to obtain

$$\int_0^s \int_0^t \left| \frac{\partial^{n+m} u}{\partial \sigma^n \partial \tau^m} \right|^2 d\sigma d\tau \leqslant \left(\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} Q_{i,j}(s,t) \right) \int_0^s \int_0^t \left| \frac{\partial^{n+m} u}{\partial \sigma^n \partial \tau^m} \right|^2 d\sigma d\tau, \qquad (4.5)$$

where

$$Q_{i,j}(s,t) = \frac{\sqrt{2}}{4(n-i)!(m-j)!} \left(\frac{(n-i)(m-j)}{(2n-2i-1)(2m-2j-1)} \right)^{1/2} s^{n-i} t^{n-j}.$$

Here, $Q_{i,j}(s,t)$ are continuous functions with the property $Q_{i,j}(0,0) = 0$. Thus, (4.5) implies that $u(s,t) = 0$, i.e. $\phi(s,t) = \bar{\phi}(s,t)$. $\square$

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