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ON THE SUPER–STABILITY OF TRIGONOMETRIC HILBERT–VALUED FUNCTIONAL EQUATIONS

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Abstract. We generalize the super-stability result for cosine functional equation mappings with values in the field of complex numbers to the case of an arbitrary Hilbert space with the Hadamard product. On the other hand, by an example we prove that this generalization is not true for sine functional equation.

1. Introduction

A stability problem in functional equation theory have been first raised by S. M. Ulam in [21] and D. H. Hyers [11] who proved the stability of a linear functional equation. Generalizations of this result appeared then in many papers. Hyers's theorem was generalized by Aoki [1] for additive mappings and by Th.M.Rassias [17, 18, 19] for linear mappings by considering an unbounded Cauchy difference. J. A. Baker, J. Lawrence and F. Zorzitto [4] have proved the super-stability of the exponential functional equation: If $(G,.)$ is group and a function $f: G \to \mathbb{R}$ is approximately exponential function, i.e., there exists a nonnegative number α such that

$$|f(x+y) - f(x)f(y)| \leqslant \alpha$$

for $x, y \in G$, then f is either bounded or exponential. The super-stability of the cosine functional equation (also called the d'Alembert equation):

$$f(x+y) + f(x-y) = 2f(x)f(y); \tag{1.1}$$

is investigated by Baker [3], and again is proved under a simple new technique by Gavruta [6, 7].

THEOREM 1.1. *Let G, be a uniquely 2 divisible Abelian group, $\mathbb{C}$ the field of complex numbers and let $\alpha > 0$ be given. If a function $f: G \to \mathbb{C}$ satisfies the inequality*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leqslant \alpha \tag{1.2}$$

for all $x; y \in G$, then either

$$|f(x)| \leqslant \frac{1+\sqrt{1+2\alpha}}{2}$$

for all $x \in G$ or f is a solution of (1.1).

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R. Badora and R. Ger [2] generalized this result, by replacing δ with functions $\delta(x)$ or $\delta(y)$ in (1.2). In the other way, the cosine functional equation is generalized by replacing $f(x)f(y)$ with $f(x)g(y)$, $g(x)f(y)$ or $g(x)g(y)$ in the equations (1.1) and (1.2). The above cosine type equations have been investigated by Badora, Ger, Kannappan, ([3], [2], [10]) and others. The super-stability bounded by constant for the sine functional equation

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2$$

is investigated by Cholewa [8], and is improved in Badora and Ger [2], and Kim [15].

Let H be a Hilbert space with a countable orthonormal basis $\{e_n : n \in \mathbb{N}\}$. For two vectors $x, y \in H$, we have the Hadamard product (named after French mathematician Jacques Hadamard), also known as the entrywise product on Hilbert space H as the following:

$$x * y = \sum_{n=1}^{+\infty} \langle x, e_n \rangle \langle y, e_n \rangle e_n$$

for every $x, y \in H$. The Cauchy-Schwartz inequality together with the Parseval identity insure that Hadamard multiplication is well defined. In fact,

$$\|x * y\| \leqslant (\sum_{n=1}^{+\infty} |\langle x, e_n \rangle|^2)^{1/2} (\sum_{n=1}^{+\infty} |\langle y, e_n \rangle|^2)^{1/2} = \|x\| \|y\|.$$

The super-stability of Hilbert-value functions with the Hadamard product has been started in [20]. In fact, it has been shown in [20] that if G is a semigroup and a function $f : G \to H$ satisfies the inequality

$$\|f(x.y) - f(x) * f(y)\|_H \leqslant \alpha$$

for some $\delta > 0$ and for every $x; y \in G$, then either there exist an integer $k \geqslant 1$ such that $|\langle f(x), e_k \rangle| \leqslant 2^k (1 + \sqrt{1+\alpha})$ for all $x \in G$ or $f(x.y) = f(x) * f(y)$ for all $x; y \in G$.

In the present paper we state a super-stability result for the approximately cosine Hilbert-valued functional equation by Hadamard product, see Theorem 2.4 below. As a consequence, we prove if a surjective function $f : H \to H$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) * f(y)\|_H \leqslant \alpha$$

for some $\alpha \geqslant 0$ and for all $x; y \in H$, then it must satisfy the following cosine functional equation with this product:

$$f(x+y) + f(x-y) = 2f(x) * f(y).$$

For the Hilbert-value sine functional equation, the situation is different. For every separable Hilbert space H and $\delta > 0$, we give some unbounded Hilbert-value function $f : \mathbb{C} \to H$ satisfying the inequality

$$\|f(x+y) * f(x-y) - f(x) * f(x) + f(y) * f(y)\| < \delta$$

which dose not satisfy the following sine functional equation:

$$f(x+y) * f(x-y) = f(x) * f(x) - f(y) * f(y).$$

2. Main results

The addition rule $\cos(x+y)+\cos(x-y)=2\cos x\cos y$ for the cosine function may be symbolized by the functional equation

$$f(x+y)+f(x-y)=2f(x)f(y). \tag{2.1}$$

This equation is called the cosine functional equation or d'Alembert equation. In 1968, Pl. Kannappan [10] determined the general solution of the cosine functional equation (2.1): Every nontrivial solution $f:\mathbb{R}\to\mathbb{C}$ of the functional equation (2.1) is given by

$$f(x)=\frac{1}{2}(m(x)+m(-x))$$

where $m:\mathbb{R}\to\mathbb{C}\backslash\{0\}$ is an exponential function.

DEFINITION 2.1. For a Hilbert space H and a semi-group $(G,.)$, we say a function $F:G\to H$ satisfies the cosine functional equation when

$$F(x+y)+F(x-y)=2F(x)*F(y); \tag{2.2}$$

for every $x,y\in G$.

The following proposition characterizes the Hilbert-valued function satisfying the cosine functional equation:

PROPOSITION 2.2. *Let H be a separable complex Hilbert space and the mapping $F:\mathbb{R}\to H$ satisfy the cosine functional equation* (2.2) *then either $F\equiv 0$ or there exist a positive integer N such that*

$$F(x)=\frac{1}{2}\sum_{n=1}^{N}(m_n(x)+m_n(-x))e_n$$

for all $x\in H$ where $m_n:\mathbb{R}\to\mathbb{C}\backslash\{0\}$ is an exponential function for $n=1,2,\ldots,N$.

Proof. Let $\{e_n:n\in\mathbb{N}\}$ be an orthonormal basis for H. For every integer $n\geqslant 1$, consider the function $e_n\otimes F:\mathbb{R}\to\mathbb{C}$ by

$$(e_n\otimes F)(h)=\langle F(h),e_n\rangle$$

for every $h\in H$. Since F satisfies the cosine equation (2.1), $e_n\otimes F$ dose so for every integer $n\geqslant 1$. Indeed, for $n\geqslant 1$ and $x,y\in H$ we see that

$$
\begin{aligned}
&\sum_{n=1}^{+\infty}(e_n\otimes F)(x+y)+(e_n\otimes F)(x-y)\\
&=\sum_{n=1}^{+\infty}\langle F(x+y)+F(x-y),e_n\rangle e_n\\
&=\sum_{n=1}^{+\infty}\langle 2F(x)*F(y),e_n\rangle e_n\\
&=\sum_{n=1}^{+\infty}\langle F(x),e_n\rangle\langle F(y),e_n\rangle e_n\\
&=\sum_{n=1}^{+\infty}(e_n\otimes F)(x)(e_n\otimes F)(y)e_n.
\end{aligned}
$$

This yields that

$$(e_n\otimes F)(x+y)+(e_n\otimes F)(x-y)=2(e_n\otimes F)(x)(e_n\otimes F)(y)$$

for every $n\geqslant 1$. Hence, either

$$(e_n\otimes F)(x)=\frac{1}{2}(m_n(x)+m_n(-x)) \tag{2.3}$$

for all $x\in\mathbb{R}$ or $(e_n\otimes F)(x)=0$ for all $x\in\mathbb{R}$; Here $m_n:\mathbb{R}\to\mathbb{C}\backslash\{0\}$ is an exponential function. The continuation of proof depend on the dimension of H. In fact, if H is infinite dimensional, since

$$(e_n\otimes F)(x)=\langle F(x),e_n\rangle\to 0$$

for every $x\in H$ as $n\to+\infty$ the equation (2.3) is not possible for infinitely many positive integer n and hence there exists some positive integer N such that $e_n\otimes F=0$ for every integer $n>N$. Thus, F can be represented as

$$
\begin{aligned}
F(x)=\sum_{n=1}^{+\infty}\langle F(x),e_n\rangle e_n &= \sum_{n=1}^{N}\langle F(x),e_n\rangle e_n\\
&=\frac{1}{2}\sum_{n=1}^{N}(m_n(x)+m_n(-x))e_n
\end{aligned}
$$

In the case that, H is of finite dimensional type the proof is clear. □

In the following theorem we generalize the well-known Baker and Gavruta superstability result for trigonometric mappings with values in the field of complex numbers to the case of an arbitrary Hilbert space with the Hadamard product.

For simplicity and avoiding the complicated formulas and expressions in the proof of theorems of this paper, the following notations is used frequently: Let $\alpha>0$ and A,B be two complex numbers. We use the notation $A\overset{\alpha}{\simeq}B$ for indicating the relation $|A-B|<\alpha$.

LEMMA 2.3. *For every two complex numbers A,B and $\alpha > 0$ the following statements hold:*

(i) If $A \overset{\alpha}{\simeq} B$ then $B \overset{\alpha}{\simeq} A$,

(ii) If $A_1 \overset{\alpha_1}{\simeq} B_1$ and $A_2 \overset{\alpha_2}{\simeq} B_2$ then $A_1 + A_2 \overset{\alpha_1+\alpha_2}{\simeq} B_1 + B_2$

(iii) If $A_1 \overset{\alpha_1}{\simeq} B$ and $A_2 \overset{\alpha_2}{\simeq} B$ then $A_1 \overset{\alpha_1+\alpha_2}{\simeq} A_2$.

THEOREM 2.4. *Let G be a semigroup and let $\alpha > 0$ be given. If a function $f : G \to H$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) * f(y)\|_H \leqslant \alpha \tag{2.4}$$

for all $x;y \in G$, then either there exists an integer $k \geqslant 1$ such that

$$|\langle f(x), e_k\rangle| \leqslant 2^{k-1}(1+\sqrt{1+2\alpha}) \tag{2.5}$$

for all $x \in G$ or

$$f(x+y) + f(x-y) = 2f(x) * f(y)$$

for all $x;y \in G$.

Proof. Assume that (2.5) is not true. Hence, for every integer $k \geqslant 1$ there exists a $a_k \in G$ such that

$$|\langle f(a_k), e_k\rangle| > 2^{k-1}(1+\sqrt{1+2\alpha}).$$

Let $\beta := 2^{-1}(1+\sqrt{1+2\alpha})$, $f_k(x) = \langle f(x), e_k\rangle$, and $g_k = 2^{-k} f_k$. Then $2\beta^2 - 2\beta = \alpha$, $\beta > 1$ and $|f_k(a_k)| > 2^k \beta$ whence $|g_k(a_k)| > \beta$. By applying the Parseval identity and definition of Hadamard product with relation (2.4), we find that each scalar-valued function f_k is approximately cosine, i.e.,

$$|f_k(x+y) + f_k(x-y) - 2f_k(x)f_k(y)| < \alpha \tag{2.6}$$

for every $x,y \in G$. Applying the notation of the preceding lemma, we can write

$$2f_k(x)f_k(y) \overset{\alpha}{\simeq} f_k(x+y) + f_k(x-y) \tag{2.7}$$

for every $x,y \in G$. If we put $y = x$ in (2.6), we get

$$|f_k(2x) + f_k(0) - 2f_k(x)^2| \leqslant \alpha,$$

and hence

$$\begin{aligned} |g_k(2x)| &\geqslant |2^{k+1} g_k(x)^2 - g_k(0)| - |g_k(2x) + g_k(0) - 2^{k+1} g_k(x)^2| \\ &\geqslant 2^{k+1}|g_k(x)^2| - |g_k(0)| - 2^{-k}|f_k(2x) + f_k(0) - 2f_k(x)^2| \\ &> 2^{k+1}|g_k(x)|^2 - |g_k(0)| - \alpha. \end{aligned}$$

Since $|g_k(0)| \leqslant \beta$, we have

$$|g_k(2x)| \geqslant 4|g_k(x)|^2 - (\beta + \alpha) \tag{2.8}$$

Now for $x = a_k$ we have $|g_k(a_k)| > \beta$ and so

$$\begin{aligned}|g_k(2a_k)| \geqslant 4|g_k(a_k)|^2 - (\beta+\alpha) &\geqslant 4|g_k(a_k)|^2 - \beta + 2\beta - \beta^2 \\ &\geqslant 2|g_k(a_k)|^2 + 2(|g_k(a_k)|^2 - \beta^2) + \beta > 2\beta.\end{aligned}$$

Assume that

$$|g_k(2^n a_k)| \geqslant 2^n \beta \tag{2.9}$$

for some integer $n \geqslant 1$. Then, if we replace x in (2.8) with $2^n a_k$ simultaneously, it then follows from (2.9) that

$$\begin{aligned}|g_k(2^{n+1}a_k)| &\geqslant 4|g_k(2^n a_k)|^2 - (\beta+\alpha) \\ &\geqslant 4(4^n\beta^2) - (\beta + 2\beta^2 - 2\beta) > 2^{n+1}\beta.\end{aligned}$$

Hence, (2.10) holds true for all $n \in \mathbb{N}$. Therefore,

$$|\langle f(2^n a_k), e_k\rangle| > 2^{k+n}\beta. \tag{2.10}$$

Fix $k \in \mathbb{N}$ and put $t_n = 2^n a_k$ for all $n \in \mathbb{N}$. Let

$$A_{n,k}(x) := \frac{1}{2f_k(t_n)}\big(f_k(x+t_n) + f_k(x-t_n)\big)$$

for every $x \in G$. Let x be any element of G, then by (2.6) we conclude that

$$|2f_k(x)f_k(t_n) - f_k(x+t_n) - f_k(x-t_n)| < \alpha. \tag{2.11}$$

In the other word by (2.7),

$$2f_k(x)f_k(t_n) \overset{\alpha}{\simeq} f_k(x+t_n) + f_k(x-t_n). \tag{2.12}$$

By the relation (2.10), $|f_k(t_n)| \to +\infty$ as $n \to +\infty$, hence

$$\lim_n A_{n,k}(x) = f_k(x).$$

Let $x, y \in G$ and $n \in \mathbb{N}$ be given. Then by using frequently the relation (2.12), we obtain that

$$\begin{aligned}2f_k(t_n)^2 A_{n,k}(x+y) &= 2f(x+y+t_n)f_k(t_n) + 2f(x+y-t_n)f_k(t_n) \\ &\overset{\alpha}{\simeq} f_k(x+y+2t_n) + 2f_k(x+y) + f_k(x+y-2t_n).\end{aligned}$$

Similarly,

$$\begin{aligned}2f_k(t_n)^2 A_{n,k}(x-y) &= 2f(x-y+t_n)f_k(t_n) + 2f(x-y-t_n)f_k(t_n) \\ &\overset{\alpha}{\simeq} f_k(x-y+2t_n) + 2f_k(x-y) + f_k(x-y-2t_n).\end{aligned}$$

Hence, by the preceding lemma,

$$\begin{aligned} &2f_k(t_n)^2(A_{n,k}(x+y)+A_{n,k}(x-y)) \\ \overset{\alpha}{\simeq}\ & f_k(x+y+2t_n)+2f_k(x+y) \\ +\ & f_k(x+y-2t_n)+f_k(x-y+2t_n) \\ +\ & 2f_k(x-y)+f_k(x-y-2t_n). \end{aligned} \tag{2.13}$$

On the other hand,

$$\begin{aligned} 4f_k(t_n)^2A_{n,k}(x)A_{n,k}(y) &= 2(f_k(x+t_n)+f(x-t_n))(f_k(y+t_n)+f(y-t_n)) \\ &= 2f_k(x+t_n)f_k(y+t_n)+2f_k(x+t_n)f(y-t_n) \\ &+ 2f_k(x-t_n)f_k(y+t_n)+2f_k(x-t_n)f(y-t_n). \end{aligned}$$

Again by (2.7) and the preceding lemma we have

$$\begin{aligned} 2f_k(x+t_n)f_k(y+t_n) &\overset{\alpha}{\simeq} f_k(x+y+2t_n)+f(x-y) \\ 2f_k(x+t_n)f_k(y-t_n) &\overset{\alpha}{\simeq} f_k(x+y)+f(x+y+2t_n) \\ 2f_k(x-t_n)f_k(y+t_n) &\overset{\alpha}{\simeq} f_k(x+y)+f(x-y-2t_n) \\ 2f_k(x-t_n)f_k(y-t_n) &\overset{\alpha}{\simeq} f_k(x+y-2t_n)+f(x-y). \end{aligned}$$

Comparing the two last relations we see that

$$\begin{aligned} 4f_k(t_n)^2A_{n,k}(x)A_{n,k}(y) \overset{\alpha}{\simeq}\ & f_k(x+y+2t_n)+2f_k(x+y) \\ +\ & f_k(x+y-2t_n)+f_k(x-y+2t_n) \\ +\ & 2f_k(x-y)+f_k(x-y-2t_n). \end{aligned} \tag{2.14}$$

Now by relations (2.13), (2.14) and part (iii) of the preceding lemma we conclude that

$$4f_k(t_n)^2A_{n,k}(x)A_{n,k}(y) \overset{2\alpha}{\simeq} 2f_k(t_n)^2(A_{n,k}(x+y)+A_{n,k}(x-y)).$$

Hence,

$$|2f_k(t_n)^2(A_{n,k}(x+y)+A_{n,k}(x-y))-4f_k(t_n)^2A_{n,k}(x)A_{n,k}(y)| < 2\alpha$$

and consequently,

$$|A_{n,k}(x+y)+A_{n,k}(x-y)-2A_{n,k}(x)A_{n,k}(y)| < \frac{\alpha}{2|f_k(t_n)|^2}.$$

Since, $|f_k(t_n)| \to +\infty$ and $\lim_n A_{n,k}(x) = f_k(x)$ for every $x \in G$, we conclude from the above relation that

$$f_k(x+y)+f_k(x-y) = 2f_k(x)f_k(y).$$

Therefore,

$$f(x+y)+f(x-y) = 2f(x)*f(y). \quad \square$$

Notice that if $f: H \to H$ is a surjection function, then every component function $e_n \otimes f$ is unbounded. In fact, for every positive integer n, there exists an $x_n \in H$ such that $f(x_n) = ne_n$, and so $(e_n \otimes f)(x_n) = n$. This led to the following corollary:

COROLLARY 2.5. *If a surjective function $f : H \to H$ satisfies the inequality*

$$\|f(x+y)+f(x-y)-2f(x)*f(y)\|_H \leqslant \alpha$$

for some $\alpha \geqslant 0$ and for all $x;y \in G$, then

$$f(x+y)+f(x-y)=2f(x)*f(y)$$

for all $x;y \in H$.

3. remarks on Hilbert-value sine functional equation

Recall that if $(G,+)$ be a semigroup then a function $f : G \to \mathbb{C}$ is said to be satisfy in the sine functional equation when

$$f(x+y)f(x-y)=f(x)^2-f(y)^2 \tag{3.1}$$

for all $x,y \in G$. P. W. Cholewa [8] observed the super-stability phenomenon of the sine functional equation (3.1).

THEOREM 3.1. *Let $(G,+)$ be an abelian group in which division by 2 is uniquely performable. Every unbounded function $f : G \to \mathbb{C}$ satisfying the inequality*

$$|f(x+y)f(x-y)-f(x)^2+f(y)^2|<\delta \tag{3.2}$$

for some $\delta > 0$ and for all $x,y \in G$ is a solution of the sine functional equation (3.1).

Unfortunately, there is no generalization of the preceding theorem for Hilbert-value functions:

THEOREM 3.2. *For every separable Hilbert space H and every number $\delta > 0$ there exists a function $f : \mathbb{C} \to H$ satisfying the inequality*

$$\|f(x+y)*f(x-y)-f(x)*f(x)+f(y)*f(y)\|<\delta$$

which dose not satisfy the following Hilbert-value sine functional equation:

$$f(x+y)*f(x-y)=f(x)*f(x)-f(y)*f(y) \tag{3.3}$$

Proof. Let $\{e_n\}$ be an orthonormal basis for H. Let $\varepsilon=\sqrt{\delta}$ and pick a nonzero element $a \in G$. Define the functions $f_k : \mathbb{C} \to \mathbb{C}$ by $f_k(z)=2^{-k}x$ for every $k \geqslant 1$ and $f_0=\varepsilon$. Consider the Hilbert-value mapping $f : \mathbb{C} \to H$ by

$$f(x)=(f_0(x),f_1(x),f_2(x),\dots f_k(x),\dots)=\sum_{k=0}^{+\infty} f_k(x)e_k.$$

Notice that f is well defined and unbounded. In fact,

$$\|F(x)\|^2=\varepsilon^2+|x|^2$$

for every $x \in \mathbb{C}$. On the other hand, every $f_k, k \geqslant 1$ satisfies (3.1) hence we observe that

$$\begin{aligned} &\|f(x+y)*f(x-y)-f(x)*f(x)+f(y)*f(y)\|^2 \\ &= \sum_{k=0}^{+\infty} |f_k(x+y)f_k(x-y)-f_k(x)^2+f_k(y)^2|^2 \\ &= |f_0(x+y)f_0(x-y)-f_0(x)^2+f_0(y)^2|^2 < \varepsilon^4. \end{aligned}$$

and so

$$\|f(x+y)*f(x-y)-f(x)*f(x)+f(y)*f(y)\| < \varepsilon^2 = \delta.$$

Finally, since f_0 dose not satisfy (3.1), F also dose not (3.3) and the proof is completed. □

At the end of this paper let us consider the other type multiplication in a Hilbert space. In fact, for a separable Hilbert space H and two elements $x = \sum_{n=1}^{+\infty} x_n e_n$ and $y = \sum_{n=1}^{+\infty} y_n e_n$ of H one can define the convolution product by

$$x \bullet y = \left(\sum_{n=1}^{+\infty} \hat{x}(n)e_n\right) \bullet \left(\sum_{n=1}^{+\infty} \hat{y}(n)e_n\right) = \sum_{n=1}^{+\infty} \hat{z}(n)e_n$$

where the numbers $\hat{z}(n)$ can be obtained by discrete convolution:

$$\hat{z}(n) = \sum_{k=1}^{n} \hat{x}(k)\hat{y}(n-k).$$

Hence, it is interesting to study and to phrase the super-stability phenomenon for functions with values in $(H,\bullet)$. For instance, it is desirable to have a sufficient condition for approximately cosine mappings with values in $(H,\bullet)$ to satisfy the cosine functional equation with the convolution product.

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