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LOCAL GRADIENT ESTIMATES FOR THE $p(x)$–LAPLACIAN ELLIPTIC EQUATIONS

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Abstract. In this paper we give a new and direct proof of local L^q estimates for the non-homogeneous $p(x)$-Laplacian elliptic equation under some proper conditions on $p(x) > 1$. We prove that

$$|\mathbf{f}|^{p(x)} \in L^q_{loc} \Longrightarrow |\nabla u|^{p(x)} \in L^q_{loc} \quad \text{for any } q \geqslant 1$$

for weak solutions of

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p(x)-2}\mathbf{f}\right) \quad \text{in } \Omega.$$

1. Introduction

In this paper we discuss the non-homogeneous $p(x)$-Laplacian elliptic equation of the following form

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p(x)-2}\mathbf{f}\right) \quad \text{in} \quad \Omega, \tag{1.1}$$

where $p \in W^{1,s}(\Omega)$ for some $s > n$ satisfies

$$1 < p_1 = \inf_{\overline{\Omega}} p(x) \leqslant p(x) \leqslant \sup_{\overline{\Omega}} p(x) = p_2 < \infty. \tag{1.2}$$

We denote by $L^{p(x)}(\Omega)$ the variable exponent Lebesgue-Sobolev spaces

$$L^{p(x)}(\Omega) = \left\{ g : \Omega \to \mathbb{R} \mid g \text{ is measurable and } \int_\Omega |g|^{p(x)}\,dx < \infty \right\} \tag{1.3}$$

with the Luxemburg type norm

$$\|g\|_{L^{p(x)}(\Omega)} = \inf\left\{ \lambda > 0 \; : \; \int_\Omega \left|\frac{g}{\lambda}\right|^{p(x)} dx \leqslant 1 \right\}. \tag{1.4}$$

Furthermore, we define

$$W^{1,p(x)}(\Omega) = \left\{ g \in L^{p(x)}(\Omega) \; : \; |\nabla g| \in L^{p(x)}(\Omega) \right\} \tag{1.5}$$

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with the norm

$$\|g\|_{W^{1,p(x)}(\Omega)} = \|g\|_{L^{p(x)}(\Omega)} + \|\nabla g\|_{L^{p(x)}(\Omega)}. \tag{1.6}$$

By $W_0^{1,p(x)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Actually, the $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ spaces are Banach spaces. There have been many investigations (see for example [9, 11, 12, 13]) on properties of such variable exponent Sobolev spaces.

Now we state the definition of local weak solutions for (1.1).

DEFINITION 1.1. Assume that $\mathbf{f} \in L_{loc}^{p(x)}(\Omega)$. A function $u \in W_{loc}^{1,p(x)}(\Omega)$ is a local weak solution of (1.1) in Ω if for any $\varphi \in W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi \, dx = \int_\Omega |\mathbf{f}|^{p(x)-2}\mathbf{f} \cdot \nabla\varphi \, dx.$$

When $p(x)$ is a constant, L^q, $q \geqslant p$, gradient estimates for weak solutions of quasilinear elliptic equations of p-Laplacian type have been studied by many authors [4, 8, 15, 14]. When $p(x)$ is not a constant, such elliptic problems (1.1) appear in mathematical models of various physical phenomena, such as the electro-rheological fluids (see, e.g., [1, 17, 18]). There have been many investigations [7, 10, 16] on Hölder estimates for the $p(x)$-Laplacian elliptic equation (1.1) and the more general case. Moreover, Acerbi and Mingione [2] have proved that

$$|\mathbf{f}|^{p(x)} \in L_{loc}^q(\Omega) \quad \Longrightarrow \quad |\nabla u|^{p(x)} \in L_{loc}^q(\Omega) \ \text{ for any } \ q > 1$$

for weak solutions of (1.1) under the following assumptions

$$p(x) \in C(\Omega), \quad 1 < \gamma_1 \leqslant p(x) \leqslant \gamma_2 < \infty, \quad |p(x) - p(y)| \leqslant w(|x-y|), \tag{1.7}$$

where

$$\lim_{R\to 0} w(R) \ln\left(\frac{1}{R}\right) = 0.$$

We assume that $p(x) \in W^{1,s}(\Omega)$ for some $s > n$. Therefore, it follows from Sobolev embedding theorem that $p(x)$ is Hölder continuous with the exponent $\alpha = 1 - \frac{n}{s}$. Especially, the hypothesis (1.7) is satisfied. Now let us state the main result of this work.

THEOREM 1.2. *Let $u \in W_{loc}^{1,p(x)}(\Omega)$ be a local weak solution of (1.1) in Ω with $B_2 \subset \Omega$ under the assumptions (1.2). Then for all $\mathbf{f}$ with $|\mathbf{f}|^{p(x)} \in L_{loc}^q(\Omega)$ for any $q \geqslant 1$, we have $|\nabla u|^{p(x)} \in L_{loc}^q(\Omega)$ with the estimate*

$$\int_{B_1} \left(|\nabla u|^{p(x)}\right)^q dx \leqslant C\left[\left(\int_{B_2} |u|^{p(x)} dx\right)^q + \int_{B_2} \left(|\mathbf{f}|^{p(x)}\right)^q dx\right],$$

where the constant C only depends on n, p_1, p_2.

1.1. Preliminary tools

We use the Hardy-Littlewood maximal function which controls the local behavior of a function.

DEFINITION 1.3. Let v be a locally integrable function. The Hardy-Littlewood maximal function $\mathscr{M}v(x)$ is defined as

$$\mathscr{M}v(x) = \sup_{r>0} \fint_{B_r(x)} |v(y)|dy.$$

If v is not defined outside Ω, then we define

$$\mathscr{M}_\Omega v(x) = \mathscr{M}(v \cdot \mathscr{X}_\Omega)(x),$$

where the indicator function $\mathscr{X}_\Omega$ of Ω satisfies

$$\mathscr{X}_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega. \end{cases}$$

For maximal functions, they satisfy strong p-p estimate and weak 1-1 estimate.

LEMMA 1.4. ([19]) *(i) If $v \in L^p(\mathbb{R}^n)$ for $p > 1$, then $\mathscr{M}v \in L^p(\mathbb{R}^n)$ with the estimate*

$$\|\mathscr{M}v\|_{L^p(\mathbb{R}^n)} \leqslant C\|v\|_{L^p(\mathbb{R}^n)}.$$

(ii) If $v \in L^1(\mathbb{R}^n)$, then

$$|\{x \in \mathbb{R}^n : \mathscr{M}v(x) > \lambda\}| \leqslant \frac{C}{\lambda}\|v\|_{L^1(\mathbb{R}^n)}.$$

Moreover, in this paper we need the following version of the modified Vitali covering lemma.

LEMMA 1.5. ([3, 4, 20]) *Assume that $R > 0$ and $0 < \varepsilon < 1$. Let $C \subset D \subset B_R$ be two measurable sets. We suppose further that $|C| < \varepsilon|B_R|$ and for all $x \in B_R$ and for all $r \in (0,R]$ with $|C \cap B_r(x)| \geqslant \varepsilon|B_r|$, $B_r(x) \cap B_R \subset D$. Then $|C| \leqslant 10^n\varepsilon|D|$.*

We end this subsection by introducing the following standard measure theory.

LEMMA 1.6. ([3, 4, 5]) *Assume that g is a nonnegative and measurable function in Ω and $0 < p < \infty$. Let $\theta > 0$ and $m > 1$ be two constants. Then we have*

$$g \in L^p(\Omega) \qquad \text{iff} \quad S := \sum_{i \geqslant 1} m^{ip}|\{x \in \Omega : g(x) > \theta m^i\}| < \infty$$

and

$$\frac{1}{C}S \leqslant \|g\|^p_{L^p(\Omega)} \leqslant C(|\Omega| + S).$$

Here $C > 0$ does not depend on g.

1.2. Final proof

In this section we shall finish the proof of the main result, Theorem 1.2. When $q = 1$, the proof of Theorem 1.2 is trivial. We first present the corresponding proof of this result.

LEMMA 1.7. *Let $u \in W^{1,p(x)}_{loc}(\Omega)$ be a local weak solution of (1.1) in Ω with $B_{2R} \subset \Omega$. Then we have*

$$\int_{B_R} |\nabla u|^{p(x)} dx \leqslant C\left(\int_{B_{2R}} |u|^{p(x)} dx + \int_{B_{2R}} |\mathbf{f}|^{p(x)} dx\right),$$

where C only depend on n, p_1, p_2, R.

Proof. Without loss of generality we may assume that $R = 1$. We may as well select the test function $\varphi = \zeta^{p_2} u \in W^{1,p(x)}_0(B_2)$, where $\zeta \in C^\infty_0(\mathbb{R}^n)$ is a cut-off function satisfying

$$0 \leqslant \zeta \leqslant 1, \quad \zeta \equiv 1 \text{ in } B_1, \quad \zeta \equiv 0 \text{ in } \mathbb{R}^n \setminus B_2.$$

Then by Definition 1.1, we have

$$\int_{B_2} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla(\zeta^{p_2} u) dx = \int_{B_2} |\mathbf{f}|^{p(x)-2} \mathbf{f} \cdot \nabla(\zeta^{p_2} u) dx$$

and write the resulting expression as

$$I_1 = I_2 + I_3 + I_4,$$

where

$$I_1 = \textstyle\int_{B_2} \zeta^{p_2} |\nabla u|^{p(x)} \, dx, \qquad I_2 = -\displaystyle\int_{B_2} p_2 \zeta^{p_2-1} u |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \zeta \, dx,$$

$$I_3 = \textstyle\int_{B_2} \zeta^{p_2} |\mathbf{f}|^{p(x)-2} \mathbf{f} \cdot \nabla u \, dx, \quad I_4 = \displaystyle\int_{B_2} p_2 \zeta^{p_2-1} u |\mathbf{f}|^{p(x)-2} \mathbf{f} \cdot \nabla \zeta \, dx.$$

Estimate of I_2. From Young's inequality with τ we have

$$\begin{aligned} I_2 &\leqslant C \int_{B_2} \zeta^{p(x)-1} |\nabla u|^{p(x)-1} |u| \, dx \\ &\leqslant \tau \int_{B_2} \zeta^{p_2} |\nabla u|^{p(x)} \, dx + C(\tau) \int_{B_2} |u|^{p(x)} \, dx. \end{aligned}$$

Estimate of I_3. From Young's inequality with τ we have

$$I_3 \leqslant \tau \int_{B_2} \zeta^{p_2} |\nabla u|^{p(x)} \, dx + C(\tau) \int_{B_2} |\mathbf{f}|^{p(x)} \, dx.$$

Estimate of I_4. From Young's inequality we have

$$I_4 \leqslant C\left(\int_{B_2} |u|^{p(x)} \, dx + \int_{B_2} |\mathbf{f}|^{p(x)} \, dx\right).$$

Combining all the estimates of I_i $(1 \leqslant i \leqslant 4)$ and selecting $\tau = 1/3$, we deduce that

$$\int_{B_2} \zeta^{p_2} |\nabla u|^{p(x)}\, dx \leqslant C\left(\int_{B_2} |u|^{p(x)}\, dx + \int_{B_2} |\mathbf{f}|^{p(x)}\, dx\right),$$

which completes the proof. $\square$

Now we recall the following L^{∞}_{loc} estimates (see [6], Theorem 2.1).

LEMMA 1.8. *Let v be a local weak solution of*

$$div\left(|\nabla v|^{p(x)-2}\nabla v\right) = 0 \quad in \quad \Omega, \tag{1.8}$$

under the assumptions (1.2). Then there exists a positive constant R_0, depending on n, p_1, p_2, s, Ω, and $\int_\Omega |\nabla v|^{p(x)}\, dx$, such that for any $R \leqslant 2R_0$ and $\rho \in (0, R_0]$ we have

$$\sup_{B_\rho} |\nabla v|^{p(x)} \leqslant C\left[\fint_{B_{2R}} |\nabla v|^{p(x)}\, dx + \left(\fint_{B_{8R}} \left(1 + |\nabla v|^{p(x)}\right)\, dx\right)^{1+\delta_0}\right], \tag{1.9}$$

where δ_0 depends on s, n, and C depends on $n, p_1, p_2, s, \|p\|_{W^{1,s}}$, and $\int_\Omega |\nabla v|^{p(x)}\, dx$.

REMARK 1.9. From the corresponding proof in [6] we may as well assume that $\int_\Omega |\nabla v|^{p(x)} dx \leqslant 1$, and then R_0, C are no longer dependent on $\int_\Omega |\nabla v|^{p(x)}\, dx$. In fact, we can easily prove that

$$\sup_{B_\rho} \frac{|\nabla v|^{p(x)}}{\lambda} \leqslant C\left[\fint_{B_{2R}} \frac{|\nabla v|^{p(x)}}{\lambda}\, dx + \left(\fint_{B_{8R}} \left(1 + \frac{|\nabla v|^{p(x)}}{\lambda}\right)\, dx\right)^{1+\delta_0}\right], \tag{1.10}$$

if we consider

$$\frac{1}{\lambda}\mathrm{div}\left(|\nabla v|^{p(x)-2}\nabla v\right) = 0.$$

Thus, we get the desired result by choosing $\lambda = \int_\Omega |\nabla v|^{p(x)}\, dx$.

Next, we give two lemmas which are very important to obtain the main result, Theorem 1.2.

LEMMA 1.10. *Assume that $0 < r \leqslant 16R_0$, where R_0 is defined in Lemma 1.8. For any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ such that if u is a local weak solution of (1.1) in Ω,*

$$\fint_{B_r} |\nabla u|^{p(x)} dx \leqslant 1 \quad and \quad \fint_{B_r} |\mathbf{f}|^{p(x)} dx \leqslant \delta, \tag{1.11}$$

then there exists $N_0 > 1$ such that

$$\sup_{B_{r/16}} |\nabla v|^{p(x)} \leqslant N_0 \tag{1.12}$$

and

$$\fint_{B_r} |\nabla(u-v)|^{p(x)} dx \leqslant \varepsilon, \tag{1.13}$$

where v is the weak solution of (1.8) in B_r with $v = u$ on ∂B_r.

Proof. The conclusion (1.12) follows from Lemma 1.8, (1.11) and (1.13), since u and v are the weak solutions of (1.1) in Ω and (1.8) in B_r, respectively.

We may as well choose the test function $\varphi = v - u \in W_0^{1,p(x)}(B_r)$ for u and v, and then a direct calculation shows the resulting expression as

$$\begin{aligned} I_1 &=: \int_{B_r} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\cdot \nabla(u-v)\,dx \\ &= \int_{B_r} |\mathbf{f}|^{p(x)-2}\mathbf{f}\cdot\nabla(u-v)\,dx =: I_2, \end{aligned}$$

Estimate of I_1. We divide into two cases.

Case 1. $p(x) \geqslant 2$. Using the elementary inequality

$$C|\xi-\eta|^p \leqslant \left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi-\eta),$$

for every $\xi,\eta \in \mathbb{R}^n$ with $C = C(p,n)$, we have

$$I_1 \geqslant C\int_{B_r} |\nabla(u-v)|^{p(x)}dx.$$

Case 2. $1 < p(x) < 2$. Using the elementary inequality

$$|\xi-\eta|^p \leqslant C\tau^{(p-2)/p}\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi-\eta) + \tau|\eta|^p$$

for every $\xi,\eta \in \mathbb{R}^n$ and every $\tau \in (0,1)$ with $C = C(p,n)$, we have

$$I_1 + \tau\int_{B_r} |\nabla u|^{p(x)}dx \geqslant C(\tau)\int_{B_r} |\nabla(u-v)|^{p(x)}dx.$$

Estimate of I_2. Using Young's inequality and (1.11), we have

$$I_2 \leqslant \tau\int_{B_r} |\nabla(u-v)|^{p(x)}dx + C(\tau)\delta\,|B_r|.$$

Combing the estimates above, (1.11) and selecting τ,δ small enough, we can finish the proof. □

Next, we shall prove an important result.

LEMMA 1.11. *Let $R_1 = R_0/4$ and $B_{18R_0} \subset \Omega$. There is a constant $N_1 > 1$ so that for any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ and if u is a local weak solution of (1.1) in Ω, with*

$$\left\{\mathscr{M}\left(|\mathbf{f}|^{p(x)}\right) \leqslant \delta\right\} \cap \left\{\mathscr{M}\left(|\nabla u|^{p(x)}\right) \leqslant 1\right\} \cap B_{R_1} \neq \emptyset, \tag{1.14}$$

then

$$\left|\left\{\mathscr{M}\left(|\nabla u|^{p(x)}\right) > N_1\right\} \cap B_{R_1}\right| < \varepsilon\,|B_{R_1}|.$$

Proof. From (1.14) there exists $x_0 \in B_{R_1}$ such that

$$\mathscr{M}(|\mathbf{f}|^{p(x)})(x_0) \leqslant \delta \text{ and } \mathscr{M}(|\nabla u|^{p(x)})(x_0) \leqslant 1, \tag{1.15}$$

which implies that

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)\cap\Omega} |\nabla u|^{p(x)} dx \leqslant 1 \text{ and } \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)\cap\Omega} |\mathbf{f}|^{p(x)} dx \leqslant \delta \tag{1.16}$$

for any $r > 0$. Since $x_0 \in B_{R_1}$, $R_1 = R_0/4$ and $B_{64R_1} = B_{16R_0} \subset B_{17R_0}(x_0) \subset B_{18R_0} \subset \Omega$, from the inequality above we have

$$\frac{1}{|B_{64R_1}|}\int_{B_{64R_1}} |\nabla u|^{p(x)}\, dx \leqslant \frac{C}{|B_{17R_0}|}\int_{B_{17R_0}(x_0)\cap\Omega} |\nabla u|^{p(x)}\, dx \leqslant C.$$

Similarly, we deduce that

$$\frac{1}{|B_{64R_1}|}\int_{B_{64R_1}} |\mathbf{f}|^{p(x)} dx \leqslant C\delta.$$

Using Lemma 1.4 (ii) and Lemma 1.10, we have

$$\sup_{B_{4R_1}} |\nabla v|^{p(x)} \leqslant N_0 \tag{1.17}$$

and

$$\left|\left\{x \in B_{R_1} : \mathscr{M}\left(|\nabla(u-v)|^{p(x)}\right) > N_0\right\}\right| \leqslant \frac{C}{N_0}\int_{B_{R_1}} |\nabla(u-v)|^{p(x)} dx \leqslant \varepsilon |B_{R_1}|$$

by taking N_0 large enough. So, it is sufficient to prove the following formula to finish the proof

$$\left\{x \in B_{R_1} : \mathscr{M}(|\nabla u|^{p(x)}) > N_1\right\} \subset \left\{x \in B_{R_1} : \mathscr{M}(|\nabla(u-v)|^{p(x)}) > N_0\right\}, \tag{1.18}$$

where $N_1 = \max\left\{2^{p_2+1}N_0, 2^n\right\}$. To do this, fix any

$$x_1 \in \left\{x \in B_{R_1} : \mathscr{M}(|\nabla(u-v)|^{p(x)}) \leqslant N_0\right\}. \tag{1.19}$$

Then we divide into two cases.

Case 1. $0 < r \leqslant 2R_1$. Then $B_r(x_1) \subset B_{3R_1}$. Therefore, from (1.17) and (1.19) we have

$$\begin{aligned} \frac{1}{|B_r|}\int_{B_r(x_1)\cap\Omega} |\nabla u|^{p(x)} dx &\leqslant \frac{2^{p_2}}{|B_r|}\int_{B_r(x_1)\cap\Omega} \left(|\nabla v|^{p(x)} + |\nabla(u-v)|^{p(x)}\right) dx \\ &\leqslant 2^{p_2}\left(\mathscr{M}(|\nabla(u-v)|^{p(x)})(x_1) + N_0\right) \leqslant 2^{p_2+1}N_0. \end{aligned}$$

Case 2. $r > 2R_1$. Then $x_0 \in B_r(x_1) \subset B_{2r}(x_0)$. Thus, from the first inequality of (1.16) we have

$$\frac{1}{|B_r|}\int_{B_r(x_1)\cap\Omega}|\nabla u|^{p(x)}dx \leqslant \frac{2^n}{|B_{2r}|}\int_{B_{2r}(x_0)\cap\Omega}|\nabla u|^{p(x)}dx \leqslant 2^n.$$

Therefore, Case 1 and Case 2 imply that

$$\mathscr{M}(|\nabla u|^{p(x)})(x_1) \leqslant N_1,$$

and then (1.18) is true. This completes our proof. □

REMARK 1.12. As mentioned in Remark 1.9, the results in Lemma 1.10 and 1.11 still hold when we replace $|\nabla u|^{p(x)}, |\mathbf{f}|^{p(x)}, |\nabla(u-v)|^{p(x)}$ by $\frac{|\nabla u|^{p(x)}}{\lambda}, \frac{|\mathbf{f}|^{p(x)}}{\lambda}, \frac{|\nabla(u-v)|^{p(x)}}{\lambda}$ for any positive constant λ.

Furthermore, we can get the following result.

COROLLARY 1.13. *Let $R_1 = R_0/4$ and $B_{18R_0} \subset \Omega$. If u is a local weak solution of (1.1) in Ω with*

$$\left|\left\{\mathscr{M}\left(|\nabla u|^{p(x)}\right) > N_1\right\}\cap B_{R_1}\right| < \varepsilon|B_{R_1}|, \tag{1.20}$$

then for $\varepsilon_1 = 10^n\varepsilon$ we have

$$\begin{aligned}&\left|B_{R_1}\cap\left\{\mathscr{M}\left(|\nabla u|^{p(x)}\right) > N_1^k\right\}\right|\\ &\leqslant \sum_{i=1}^{k}\varepsilon_1^i\left|B_{R_1}\cap\left\{\mathscr{M}\left(|\mathbf{f}|^{p(x)}\right) > \delta N_1^{k-i}\right\}\right| + \varepsilon_1^k\left|B_{R_1}\cap\left\{\mathscr{M}\left(|\nabla u|^{p(x)}\right) > 1\right\}\right|.\end{aligned}$$

Proof.
Case 1. $k = 1$. The result above is an immediate consequence of Lemma 1.5 and Lemma 1.11 by selecting

$$\begin{aligned}C &= B_{R_1}\cap\left\{\mathscr{M}\left(|\nabla u|^{p(x)}\right) > N_1\right\},\\ D &= \left(B_{R_1}\cap\left\{\mathscr{M}\left(|\nabla u|^{p(x)}\right) > 1\right\}\right)\cup\left(B_{R_1}\cap\left\{\mathscr{M}\left(|\mathbf{f}|^{p(x)}\right) > \delta\right\}\right).\end{aligned}$$

Case 2. $k \geqslant 2$. Recalling Remark 1.12, we complete the proof after an iteration by choosing $\lambda = N_1^i, 1 \leqslant i \leqslant k-1$. □

Now we are ready to prove the main result, Theorem 1.2.

Proof. Without loss of generality, from Lemma 1.4, Lemma 1.6, Lemma 1.7 and Remark 1.9 we may assume that

$$\left|\left\{\mathscr{M}\left(\frac{|\nabla u|^{p(x)}}{\lambda}\right) > N_1\right\}\cap B_{R_1}\right| < \varepsilon|B_{R_1}|$$

and

$$\sum_{k=1}^{\infty} N_1^{qk} \left| \left\{ x \in B_{R_1} : \mathscr{M}\left(\frac{|\mathbf{f}|^{p(x)}}{\lambda} \right) > \delta N_1^k \right\} \right| \leqslant 1. \tag{1.21}$$

by choosing

$$\lambda = C \left\{ \int_{B_{18R_0}} |u|^{p(x)} dx + \left(\int_{B_{18R_0}} \left(|\mathbf{f}|^{p(x)} \right)^q dx \right)^{1/q} \right\}$$

where C is large enough. Then according to Corollary 1.13 and (1.21), we can find that

$$\begin{aligned}
&\sum_{k=1}^{\infty} N_1^{qk} \left| \left\{ x \in B_{R_1} : \mathscr{M}\left(\frac{|\nabla u|^{p(x)}}{\lambda} \right) > N_1^k \right\} \right| \\
&\leqslant \sum_{k=1}^{\infty} N_1^{qk} \sum_{i=1}^{k} \varepsilon_1^i \left| \left\{ x \in B_{R_1} : \mathscr{M}\left(\frac{|\mathbf{f}|^{p(x)}}{\lambda} \right) > \delta N_1^{k-i} \right\} \right| \\
&\quad + \sum_{k=1}^{\infty} N_1^{qk} \varepsilon_1^k \left| \left\{ x \in B_{R_1} : \mathscr{M}\left(\frac{|\nabla u|^{p(x)}}{\lambda} \right) > 1 \right\} \right| \\
&\leqslant \sum_{i=1}^{\infty} \left(N_1^q \varepsilon_1 \right)^i \sum_{k=i}^{\infty} N_1^{q(k-i)} \left| \left\{ x \in B_{R_1} : \mathscr{M}\left(\frac{|\mathbf{f}|^{p(x)}}{\lambda} \right) > \delta N_1^{k-i} \right\} \right| \\
&\quad + C \sum_{k=1}^{\infty} \left(N_1^q \varepsilon_1 \right)^k \leqslant C \sum_{k=1}^{\infty} \left(N_1^q \varepsilon_1 \right)^k \leqslant C,
\end{aligned}$$

by choosing ε_1 small enough such that $N_1^q \varepsilon_1 < 1$. Then from Lemma 1.6 we have

$$\mathscr{M}\left(\frac{|\nabla u|^{p(x)}}{\lambda} \right) \in L^q(B_{R_1}), \qquad \text{and then} \qquad \frac{|\nabla u|^{p(x)}}{\lambda} \in L^q(B_{R_1}),$$

with the estimate

$$\int_{B_{R_1}} \left(|\nabla u|^{p(x)} \right)^q dx \leqslant C \left\{ \int_{B_{18R_0}} \left(|\mathbf{f}|^{p(x)} \right)^q dx + \left(\int_{B_{18R_0}} |u|^{p(x)} dx \right)^q \right\}.$$

Without loss of generality we assume that $R_1 = R_0/4 \leqslant 1/36$. Thus we can complete the proof by the finite covering lemma. □

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