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NEW BOUNDS FOR THE SPREAD OF THE SIGNLESS LAPLACIAN SPECTRUM

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Abstract. The spread of the singless Laplacian of a simple graph G is defined as $SQ(G) = \mu_1(G) - \mu_n(G)$, where $\mu_1(G)$ and $\mu_n(G)$ are the maximum and minimum eigenvalues of the signless Laplacian matrix of G, respectively. In this paper, we will present some new lower and upper bounds for $SQ(G)$ in terms of clique and independence numbers. In the final section, as an application of the theory obtained in here, we will also show some new upper bounds for the spread of the singless Laplacian of tensor products of any two simple graphs.

1. Introductory material

For an $n \times n$ complex matrix $M = (m_{i,j})$ whose eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_n$, the *spread* $S(M)$ is defined as the diameter of its spectrum:

$$S(M) = \max_{i,j} |\lambda_i - \lambda_j|,$$

where the maximum is taken over all pairs of eigenvalues of M. With regard to the spread of an arbitrary matrix, we refer to [16, 26, 27, 34]. Among these works, in [26], Mirsky obtained one of the fundamental results for the spread of M as in the following.

PROPOSITION 1.1. *There exists the inequality*

$$S(M) \leqslant \left[2\sum_{i,j} |m_{i,j}|^2 - \frac{2}{n} |\sum_{i} m_{i,i}| \right]^{\frac{1}{2}}.$$

In here, the equality holds if and only if M is a normal matrix with $n-2$ of its eigenvalues all equal to the average of the remaining two.

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Now let us suppose that M is a Hermitian matrix. In that case, by [27], the eigenvalues $\lambda_i = \lambda_i(M)$ of M are all real numbers, and they may always be assumed to be in decreasing order

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n .$$

Then $S(M)$ will be equal to the $\lambda_1 - \lambda_n$, i.e. the distance between the maximum and minimum eigenvalues λ_1 and λ_n. For unit vectors $x, y \in \mathbb{C}^n$, we further have

$$\lambda_1 \geqslant x^* M x \quad \text{and} \quad \lambda_n \leqslant y^* M y . \tag{1}$$

In (1), the equality holds if and only if x and y are unit eigenvectors associated with λ_1 and λ_n, respectively. Therefore,

$$S(M) = \max_{x,y}(x^* M x - y^* M y) = \max_{x,y} \sum_{i,j} m_{i,j}(\bar{x}_i x_j - \bar{y}_i y_j) , \tag{2}$$

where the maximum is taken over all pairs x, y of unit vectors in $\mathbb{C}^n$. If $M \neq 0$, the maximum is attained if and only if x and y are orthonormal eigenvectors of M corresponding to the eigenvalues λ_1 and λ_n, respectively.

Throughout this paper, unless stated otherwise, all graphs G will be taken undirected and simple with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. Furthermore, for $i = 1, 2, \cdots, n$, the *degree* of a vertex v_i in $V(G)$ will be denoted by d_i.

Let $A(G)$ denote the adjacency matrix of G. Since $A(G)$ is symmetric, the eigenvalues of it can be arranged as $\rho_1(G) \geqslant \rho_2(G) \geqslant \cdots \geqslant \rho_n(G)$. Then, by [13], the *adjacency spread* of the graph G is defined as

$$SA(G) = \rho_1(G) - \rho_n(G) .$$

One can be referred to the papers [10, 13, 19, 30, 31] for some studies over adjacency spread of the graph G. It is known that we also have the Laplacian matrix (cf. [24, 25]) related to the adjacency and diagonal matrices. In fact, for a diagonal matrix $D(G)$ whose (i,i)-entry is d_i, the Laplacian matrix $L(G)$ of G is defined as $L(G) = D(G) - A(G)$. By [20], since $L(G)$ is positive semidefinite, its eigenvalues can be arranged as

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_{n-1}(G) \geqslant \lambda_n(G) = 0 .$$

Thus, by [11], $\lambda_n(G) = 0$ gives the *Laplacian spread* $SL(G)$ of the graph G which is defined as

$$SL(G) = \lambda_1(G) - \lambda_{n-1}(G) .$$

In the literature, there are so many studies about the Laplacian spread (see, for instance, [7, 8, 9, 18, 29]).

Additionally to the Laplacian matrix, we also have the *signless Laplacian matrix* $Q(G) = D(G) + A(G)$ of G (cf. [3, 4, 5, 6, 21, 22]) which is symmetric, non-negative and irreducible (while G is connected). We note that, in some papers ([9, 33]), $Q(G)$ is called the *unoriented Laplacian matrix* of G. For an $n \times m$ vertex-edge incidence matrix M of G, we also write $Q(G) = MM^t$. This implies that $Q(G)$ is positive semidefinite and its eigenvalues can be arranged as

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) \geqslant 0 . \tag{3}$$

In fact, by motivating the definition of adjacency and Laplacian spreads of G, it has been defined the *signless Laplacian spread* $SQ(G)$ of the graph G as

$$SQ(G) = \mu_1(G) - \mu_n(G)$$

(cf. [20]). As an example of signless Laplacian spread, one can consider K_n (the complete graph) and P_n (the path graph). It is known that

$$SQ(K_n) = n \quad \text{and} \quad SQ(P_n) = 2 + \cos\frac{\pi}{n}.$$

Moreover, by [1], for an edge e we have

$$SQ(K_n - e) = \sqrt{n^2 + 4n - 12}.$$

Also in the same reference, for the star graph and complete bipartite graph, it has been obtained that

$$SQ(K_{1,n}) = n \text{ and } SQ(\overline{K_{n,n} + nK_1}) = 3n.$$

In fact, in the literature there is a limited number of papers on this matter (cf. [20]). The present study has been undertaken bearing this fact in mind.

For simplicity, each of the eigenvalues $\mu_i(G)$ of the signless Laplacian matrix $Q(G)$ will only be denoted by μ_i in the rest of this paper.

As a final preliminary material of this section, we note the following two facts which will be needed in the construction of some results of this paper.

- In Eq. (1), if we set x to be a unit vector with equal entries, then we obtain the lower bound $\mu_1 \geqslant \frac{4m}{n}$ such that the equality holds if and only if G is regular.
- We have the following equalities related to the trace of signless Laplacian matrix $Q(G)$:

$$tr(Q(G)) = \sum_i \mu_i = 2m \quad \text{and} \quad tr(Q(G)^2) = \sum_i \mu_i^2 = \sum_i d_i + \sum_i d_i^2. \tag{4}$$

2. An upper bound for $SQ(G)$ in terms of first Zagreb index

Before presenting our main results, let us first state the following proposition over spread of signless Laplacian $SQ(G)$ related to the adjacency spread $SA(G)$ for a k-regular graph G.

We remind that a graph G is k-regular if $d_1 = d_2 = \cdots = d_n = k$. Moreover, if G is k-regular, then it is easy to see that $\mu_1(G) = \rho_1(G) + k$ and $\mu_n(G) = \rho_n(G) + k$. Hence we get $SQ(G) = SA(G)$. In addition, if G is k-regular, then

$$SA(G) = k - \rho_n = \lambda_1(G) \leqslant n$$

with equality holds if and only if the complement $\overline{G}$ of G is disconnected. These facts imply the following proposition.

PROPOSITION 2.1. *For a k-regular graph G, we have*

$$SQ(G) \leqslant n.$$

Moreover $SQ(G) = n$ if and only if the complement $\overline{G}$ of G is disconnected.

As an example of Proposition 2.1, one can consider G as the generalized Petersen or Kneser graphs. In fact, the generalized Petersen graphs $GP(n,k)$ are 3-regular ([12]), and so it is easy to see that

$$SQ(GP(n,k)) = 3 - \cos\left(\frac{2\pi t}{n}\right) - \cos\left(\frac{2\pi tk}{n}\right) + \sqrt{\left(\cos\left(\frac{2\pi t}{n}\right) - \cos\left(\frac{2\pi tk}{n}\right)\right)^2 + 1},$$

where $t = \left[\frac{n}{2k}\right]$. Moreover, by [32], for $k = \binom{n-r}{r}$-regular Kneser graphs $K(n,r)$, there exists

$$SQ(K(n,r)) = \binom{n-r}{r} + \binom{n-r-1}{r-1}.$$

Now let us denote $\sum_i d_i^2$ by $\mathscr{M}_1$. Then we obtain the first main result of this paper.

THEOREM 2.2. *For a graph G with n vertices and m edges, it is always true that*

$$SQ(G) \leqslant \mu_1 + \sqrt{2m + \mathscr{M}_1 - \mu_1^2} \leqslant 2\sqrt{m + \frac{\mathscr{M}_1}{2}}. \tag{5}$$

Moreover the first inequality becomes equality if and only if G is K_2.

Proof. From equations in (4), it is true that $\mu_1^2 + \mu_n^2 \leqslant 2m + \mathscr{M}_1$. So

$$SQ(G) = \mu_1 - \mu_n \leqslant \mu_1 + \mu_n \leqslant \mu_1 + \sqrt{2m + \mathscr{M}_1 - \mu_1^2},$$

as required in the first inequality of (5). Furthermore, by considering μ_1 as a function, we get that the expression $\mu_1 + \sqrt{2m + \mathscr{M}_1 - \mu_1^2}$ is strictly increasing when $\mu_1 \leqslant \sqrt{m + \frac{\mathscr{M}_1}{2}}$, and strictly decreasing when $\mu_1 \geqslant \sqrt{m + \frac{\mathscr{M}_1}{2}}$. These prove the validity of the second inequality in (5).

Suppose that the first equality holds in (5), in other words,

$$SQ(G) = \mu_1 - \mu_n = \mu_1 + \mu_n = \mu_1 + \sqrt{2m + \mathscr{M}_1 - \mu_1^2}.$$

Thus, from the first part of this last equality, we have $\mu_n = 0$, and so G will be a bipartite graph ([5, Proposition 2.1]). Also, from the second part, we obtain $\sum_{i=2}^{n-1} \mu_i^2 = 0$ which implies that G is actually K_2. The sufficient part ($\Leftarrow$) is clear.

Hence the result. □

By setting suitable variables in Proposition 1.1, we also have the following upper bound for $SQ(G)$ which is different than the bound presented in Theorem 2.2.

COROLLARY 2.3. *The signless Laplacian spread has an upper bound*

$$SQ(G) \leqslant 2\sqrt{m + \frac{\mathscr{M}_1}{2} - \frac{2m^2}{n}}. \tag{6}$$

REMARK 2.4. Since $2\sqrt{m + \frac{\mathscr{M}_1}{2} - \frac{2m^2}{n}} \leqslant 2\sqrt{m + \frac{\mathscr{M}_1}{2}}$, it is easy to see that the bound in (6) is better than the second upper bound in (5) .

We recall that the Perron-Frobenius theorem [2, p. 26] implies that $\mu_1(G)$ has a unique positive unit eigenvector if G is connected. Further, by considering $\mu_1(G)$ has the monotonicity property ([2, p. 27]), we then get the following lemma.

LEMMA 2.5. *If H is a subgraph of G, then $\mu_1(H) \leqslant \mu_1(G)$. Nevertheless, if G is connected and H is a proper subgraph of G, then $\mu_1(H) = \mu_1(G)$.*

A subgraph H of G is *induced* if it is obtained from G by deleting a proper subset U of the vertices of G. This is often written as $H = G \backslash U$. Hence, by considering Lemma 2.5 and the definition of induced subgraph, we can state and prove the following theorem.

THEOREM 2.6. *If H is an induced subgraph of G, then $\mu_n(G) \leqslant \mu_n(H)$. Therefore $SQ(H) \leqslant SQ(G)$ with strict inequality if G is connected and H is a proper induced subgraph of G.*

Proof. If U is a proper subset of the vertex set of G, then the signless Laplacian matrix of $H = G \backslash U$ is a principal submatrix of the signless Laplacian matrix of G. Thus, by eigenvalue interlacing [4, p. 19], [14] and [15, p. 189], we get $\mu_n(G) \leqslant \mu_n(G \backslash U)$. Also, by Lemma 2.5, $\mu_1(G) \geqslant \mu_1(G \backslash U)$ with strict inequality if G is connected and U is proper and non-empty. Thus, $SQ(H) \leqslant SQ(G)$ with strict inequality if H is a proper induced subgraph of connected G, as desired. □

3. Bounds for $SQ(G)$ in terms of clique and independence numbers

Let ω and α denote the *clique* and *independence numbers* of G which are defined as the number of vertices of the largest clique and the largest independence sets in G, respectively. Suppose that the graph G has the maximum degree Δ and the minimum degree δ. Also let us consider the eigenvalues in (3) of the signless Laplacian matrix $Q(G)$ with the normalized eigenvectors $u^1 > 0, u^2, \cdots, u^n$, respectively. Finally, let us assume that $\beta \in (-1, +1)$ and e denotes the vector of all ones in $\mathbb{R}^n$. By considering these parameters, in [23, Theorem 3.1 and Theorem 3.2], Maden et al. obtained new lower bounds for ω and α as follows:

$$\omega \geqslant \max_{i=2,3,..,n} \frac{n}{n-[\beta^2\mu_i+(1-\beta^2)\,\mu_1-\Delta]} \tag{7}$$

and

$$\alpha \geqslant \max_{i=2,3,\cdots,n} \frac{U_i^2}{\beta^2\mu_i+(1-\beta^2)\,\mu_1-\delta+U_i^2}, \tag{8}$$

where $s_i = e^T u^i$ and $U_i = \beta s_i + s_1\sqrt{1-\beta^2}$.

It can be easily deduced that, for $i = 2,\cdots,n$, the best bounds of ω and α are the maximum values of the ratios in (7) and (8), respectively. Therefore, by setting $i = n$ in these two inequalities, we obtain the following two results as bounds for $SQ(G)$ in terms of ω and α.

THEOREM 3.1. *Let G be a graph with n vertices and maximum degree Δ. Then*

$$SQ(G) \geqslant \frac{\mu_1+\frac{n}{\omega}-n-\Delta}{\beta^2},$$

where $\beta \in (-1,+1)$.

THEOREM 3.2. *Let G be a graph with n vertices and minimum degree δ. Then*

$$SQ(G) \leqslant \frac{\mu_1+U_i^2\left(1-\frac{1}{\alpha}\right)-\delta}{\beta^2},$$

where, for $1 \leqslant i \leqslant n$, $U_i = \beta s_i + s_1\sqrt{1-\beta^2}$ and $s_i = e^T u^i$ such that u^i is the i-th entry of the normalized eigenvector u.

In the following, we will present a different bound for $SQ(G)$ in terms of only ω. In fact this bound is a generalization of Theorem 2.2.

THEOREM 3.3. *Let the graph G (with m edges) has a clique of order $\omega > 1$. Then*

$$SQ(G) \leqslant \mu_1 + \sqrt{2m+\mathscr{M}_1-\mu_1^2-(n-2)(\omega-2)^2}. \tag{9}$$

Moreover, inequality becomes equality if and only if G is K_2.

Proof. When $\omega = 2$, the result follows from Theorem 2.2. So let us suppose that $\omega \geqslant 3$.

Since K_ω is an induced subgraph of G which has the signless eigenvalue $\omega-2$ with multiplicity $\omega-1$, it follows by interlacing [14] or [15, p. 189] that the total $\omega-1$ smallest eigenvalues of G (including μ_n) must all be less than or equal to $\omega-2$. Thus, at least $n-2$ eigenvalues other than μ_n must be less than or equal to $\omega-2$. By using (4), we then have $\mu_1^2+\mu_n^2+(n-2)(\omega-2)^2 \leqslant 2m+\mathscr{M}_1$ or, equivalently,

$$SQ(G) = \mu_1-\mu_n \leqslant \mu_1+\mu_n \leqslant \mu_1+\sqrt{2m+\mathscr{M}_1-\mu_1^2-(n-2)(\omega-2)^2}.$$

Now assume that equality holds in above. Then, by the fact $\mu_1 - \mu_n = \mu_1 + \mu_n$, we clearly have $\mu_n = 0$. So G is actually bipartite, i.e., $\omega = 2$. Therefore, by a similar idea as in Theorem 2.2, G must be K_2. In fact the reverse part is clear.

Hence the result. □

The upper bound in Theorem 3.3 can also be presented as in the following corollary.

COROLLARY 3.4. *With the same conditions as in Theorem 3.3,*

$$SQ(G) \leqslant 2\sqrt{\frac{2m + \mathscr{M}_1 - \mu_1^2 - (n-2)(\omega-2)^2}{2}}. \tag{10}$$

Proof. By (9), if we consider the function

$$f(x) = x + \sqrt{2m + \mathscr{M}_1 - x^2 - \mu_1^2 - (n-2)(\omega-2)^2},$$

then the maximum value of $f(x)$ occurs actually in

$$x = \sqrt{\frac{2m + \mathscr{M}_1 - \mu_1^2 - (n-2)(\omega-2)^2}{2}}.$$

This completes the proof. □

REMARK 3.5. We strictly note that while the bound in (9) is better than the first bound in (5), the bound in (10) is better than the second bound in (5). To strengthen this theory, let us consider the graph $G = K_4 + e$ as drawn in Figure 1. It is clear that $SQ(G) = 5.7446$. Moreover, while the upper bound in (5) is 9.7478, the bound in (9) is 7.9195.

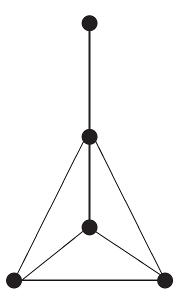

Figure 1: *The graph* $K_4 + e$

4. Bounds for $SQ(G)$ in terms of the tensor product of graphs

This section mainly investigates some new upper bounds for the spread of signless Laplacian over tensor products of simple graphs. We remind that, for any two graphs G and H, the *tensor product* $G \otimes H$ is a graph such that the vertex set of $G \otimes H$ is the Cartesian product $V(G) \times V(H)$; and any two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \otimes H$ if and only if g_1 is adjacent with g_2 and h_1 is adjacent with h_2. It is also equivalent to the *Krönecker product* (which is denoted by the same notation "$\otimes$") of the adjacency matrices of the graphs ([35]). This connection implies the following lemma.

LEMMA 4.1. ([6, 35]) *Let us consider the graphs G and H with their adjacency matrices $A(G)$ and $A(H)$, respectively. Then the adjacency matrix $A(G \otimes H)$ of $G \otimes H$ is equal to the $A(G) \otimes A(H)$.*

As a next step of Lemma 4.1, by considering the above definition of the tensor product, one can easily see that

$$D(G \otimes H) = D(G) \otimes D(H), \tag{11}$$

where $D(.)$ is the diagonal degree matrix.

It is known that, for any two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{kl}]_{p \times q}$, the Krönecker product $A \otimes B$ is defined as the block matrix $[a_{ij}B]_{mp \times nq}$ (see [6, pg. 44]). Hence, for any matrices X, Y, Z and T, a simple calculation gives that

$$(X+Y) \otimes (Z+T) = (X \otimes Z) + (X \otimes T) + (Y \otimes Z) + (Y \otimes T). \tag{12}$$

This above material will be needed for the following lemma which will be used in the proof of the main result of this section.

LEMMA 4.2. *Let G and H be two graphs. Then*

$$Q(G \otimes H) = Q(G) \otimes Q(H) - D(G) \otimes A(H) - A(G) \otimes D(H), \tag{13}$$

where $D(.)$ and $Q(.)$ are the diagonal degree and signless Laplacian matrices, respectively, as previously.

Proof. Since $Q(.) = D(.) + A(.)$, we clearly have

$$\begin{aligned} Q(G \otimes H) &= D(G \otimes H) + A(G \otimes H) \\ &= D(G) \otimes D(H) + A(G) \otimes A(H) \\ &\qquad \text{by Eq. (11) and Lemma 4.1.} \end{aligned}$$

On the other hand, in Eq. (12), if we replace X by $D(G)$, Y by $A(G)$, Z by $D(H)$ and T by $A(H)$, then we obtain

$$\begin{aligned} [D(G) \otimes D(H)] + [A(G) \otimes A(H)] &= [D(G) + A(G)] \otimes [D(H) + A(H)] \\ &\quad - [D(G) \otimes A(H) + A(G) \otimes D(H)]. \end{aligned}$$

Again, by using the definition of $Q(.)$, we finally obtain

$$Q(G\otimes H)=Q(G)\otimes Q(H)-D(G)\otimes A(H)-A(G)\otimes D(H),$$

as required. □

Let $eig(M)$ denotes the set of eigenvalues of an arbitrary matrix M. Then the following proposition is obtained.

PROPOSITION 4.3. ([17]) *Let A_i and B_i be matrices, where $1\leqslant i\leqslant n$. For all $1\leqslant i<j\leqslant n$, if $A_iA_j=A_jA_i$ and $B_iB_j=B_jB_i$ hold, then*

$$eig\left(\sum_{i=1}^{n}A_i\otimes B_i\right)=\sum eig(A_i\otimes B_i).$$

After all these above material, we can present the following main result of this section.

THEOREM 4.4. *Let G and H be two simple graphs with n and m vertices, respectively. Then there exists an upper bound*

$$SQ(G\otimes H)\leqslant SQ(G)\times SQ(H)+\delta(G)[SQ(H)-SA(H)]+\delta(H)[SQ(G)-SA(G)],$$

where $\delta(.)$ and $SA(.)$ denote the minimum degree and the adjacency spread of related graphs, respectively.

Proof. By Proposition 4.3 and taking into account Eq. (13), we have

$$\mu_1(G\otimes H)=\mu_1(G)\mu_1(H)-\Delta(G)\rho_1(H)-\rho_1(G)\Delta(H),$$

(where $\Delta(.)$ denotes the maximum degree of G) and

$$\mu_{nm}(G\otimes H)=\mu_n(G)\mu_m(H)-\delta(G)\rho_m(H)-\rho_n(G)\delta(H).$$

Then we get

$$\begin{aligned}SQ(G\otimes H)&=\mu_1(G\otimes H)-\mu_{nm}(G\otimes H)\\&=[\mu_1(G)-\mu_n(G)][\mu_1(H)-\mu_m(H)]\\&\quad+\mu_n(G)[\mu_1(H)-\mu_m(H)]+\mu_n(H)[\mu_1(G)-\mu_n(G)]\\&\quad-\Delta(G)\rho_1(H)-\rho_1(G)\Delta(H)+\delta(G)\rho_m(H)+\rho_n(G)\delta(H).\end{aligned}\tag{14}$$

By the meaning of δ and Δ (in other words, $\delta\leqslant\Delta$), the above equality can be written as

$$\begin{aligned}SQ(G\otimes H)&\leqslant SQ(G)SQ(H)+\mu_n(G)SQ(H)+\mu_m(H)SQ(G)\\&\quad-\delta(G)\rho_1(H)-\rho_1(G)\delta(H)+\delta(G)\rho_m(H)+\rho_n(G)\delta(H)\\&=SQ(G)SQ(H)+\mu_n(G)SQ(H)+\mu_m(H)SQ(G)\\&\quad-\delta(G)SA(H)-\delta(H)SA(G).\end{aligned}$$

Moreover, by using the fact $\mu_n(G) \leqslant \delta(G)$ and $\mu_m(H) \leqslant \delta(H)$ (cf. [21, Corollary 2.1]), we obtain the required inequality over $SQ(G \otimes H)$.

Hence the result. □

One can give the following consequences of Theorem 4.4 by considering regular and bipartite graphs.

COROLLARY 4.5. *Suppose that G and H are k and t-regular graphs, respectively. Then*

$$SQ(G \otimes H) \leqslant SQ(G)SQ(H).$$

Proof. By Proposition 2.1, the proof is clear. □

COROLLARY 4.6. *Let G be a k-regular and H be a t-regular graphs. Also, let us suppose that G is bipartite. Then*

$$SQ(G \otimes H) = tSQ(G) - kSQ(H).$$

Proof. Suppose that G is a k-regular, bipartite graph with n vertices, and H is a t-regular graph with m vertices. By Eq. (14), since $SQ(G \otimes H) = \mu_1(G \otimes H) - \mu_{nm}(G \otimes H)$, we get

$$\begin{aligned} SQ(G \otimes H) = \; & \mu_1(G)\mu_1(H) - k\lambda_1(H) - t\lambda_1(G) - \mu_n(G)\mu_m(H) \\ & + k\lambda_m(H) + t\lambda_n(G). \end{aligned}$$

Bipartivity assumption on G implies that $\mu_n(G) = 0$ (cf. [5, Proposition 2.1]). Therefore the above equality over $SQ(G \otimes H)$ can be written as

$$SQ(G \otimes H) = \mu_1(G)\mu_1(H) - k(\lambda_1(H) - \lambda_m(H)) - t(\lambda_1(G) - \lambda_n(G)).$$

Furthermore, by Proposition 2.1, we conclude that

$$SQ(G \otimes H) = SQ(G)(\mu_1(H) - t) - kSQ(H) = tSQ(G) - kSQ(H),$$

as required. □

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