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DOMINANCE OF ORDINAL SUMS OF THE ŁUKASIEWICZ AND THE PRODUCT TRIANGULAR NORM

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Abstract. In this paper we provide a simple characterization of the dominance in two classes of continuous triangular norms. In particular, we solve the dominance of (i) ordinal sum t-norms that use the Łukasiewicz t-norm as the only summand operation and (ii) ordinal sum t-norms that use the product t-norm as the only summand operation. In both cases, the dominance relation is characterized by a simple property of the idempotent elements of the dominating t-norm. We also introduce the notion of the axis of a conjunctor and, as a side result, we characterize dominance of continuous conjunctors in terms of their axes.

1. Introduction

Dominance as a binary relation on the set of all binary operations defined on a common poset was originally introduced within the theory of probabilistic metric spaces [19, 21] where it was motivated by the study of Cartesian products of such spaces. Later dominance was studied in connection with the construction of many-valued equivalence relations [3, 4, 22] and many-valued orderings [2]. Recently, the concept of dominance was found to be important for constructions of T-transitive Cartesian products of T-transitive many-valued relations [16]. The mathematical curiosity also motivated the study of dominance as a concept of interest on its own right [14, 15, 20]. Simultaneously with the growing importance of the dominance relation, also the diversity of the considered operations became broader. For example, while in the original framework of probabilistic metric spaces the natural question to deal with is the dominance of triangle functions and of triangular norms, in the later contexts the authors consider dominance of aggregation functions [16] which is a markedly larger class.

DEFINITION 1. Let $A,B\colon [0,1]^2 \to [0,1]$ be two binary operations and let S be a subset of $[0,1]^4$. We say that A *dominates* B *on the set* S, if the inequality

$$A\big(B(x,y),B(u,v)\big) \geqslant B\big(A(x,u),A(y,v)\big) \tag{D}$$

holds for all $x,y,u,v \in [0,1]$ with $(x,u,y,v) \in S$. We say that A *dominates* B, and we write $A \gg B$, if A dominates B on the set $[0,1]^4$.

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By a *conjunctor* we will understand any nondecreasing binary operation on the unit interval with the neutral element 1. A conjunctor which is commutative and associative is called a *triangular norm* or, shortly, a *t-norm* [8, 19]. Within this paper we pay attention mainly to these prototypical t-norms: the minimum $T_{\mathbf{M}}(x,y) = \min\{x,y\}$, the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max\{0, x+y-1\}$ and the product $T_{\mathbf{P}}(x,y) = xy$. Thanks to associativity and commutativity every t-norm dominates itself; hence the dominance of t-norms is a reflexive relation [8]. Moreover, the commutativity together with the fact that all t-norms share the common neutral element implies that dominance is a subrelation of the standard point-wise order of t-norms; alternatively $T_1 \geqslant T_2$ is a necessary condition for $T_1 \gg T_2$. As a consequence the dominance of t-norms is an antisymmetric relation. It was an open problem whether the dominance of t-norms is also transitive [19, Problem 12.11.3] and only recently has been answered negatively [20]. Within the present paper we considerably strengthen the methods and ideas which were used in order to come up with this negative answer.

We will often utilize the concept of an *affine transformation* of a conjunctor which is a special case of the notion of an isomorphic image of an algebra [8]. Given an interval $[a,b]$ and a conjunctor C we define

$$\langle a,b,C\rangle\colon\ [a,b]^2 \to [a,b] : (x,y) \mapsto a + (b-a)C\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}\right)$$

and we refer to the new operation as an affine transform of C. Notice that the algebraic structures $([0,1],C)$ and $([a,b],\langle a,b,C\rangle)$ are isomorphic via the unique affine order isomorphism from the unit interval to $[a,b]$.

Let $(C_i)_{i\in I}$ be a family of conjunctors indexed by an at most countable index set I and let $([a_i,b_i])_{i\in I}$ be a system of subintervals of $[0,1]$ with pairwise disjoint interiors. The *ordinal sum* given by these two ingredients is the binary operation

$$C\colon\ [0,1]^2 \to [0,1]\colon (x,y) \mapsto \begin{cases} \langle a_i,b_i,C_i\rangle(x,y) & \text{if } x,y \in\,]a_i,b_i[, \\ T_{\mathbf{M}}(x,y) & \text{otherwise.} \end{cases}$$

By the abuse of notation we identify the ordinal sum with the sequence $(\langle a_i,b_i,C_i\rangle)_{i\in I}$ of affine-transformed operations. We refer to the intervals $[a_i,b_i]$ as *summand carriers* and to the individual conjunctors C_i as *summand operations*, or simply *summands*. Clearly, the ordinal sum of conjunctors is a conjunctor again. Moreover, if all the summands are (continuous) t-norms then also the ordinal sum is a (continuous) t-norm. If we have two t-norms $T_1 \geqslant T_2$ where $T_2 = (\langle a_i,b_i,T_{2,i}\rangle)_{i\in I}$, then there exists a family of t-norms $(T_{1,i})_{i\in I}$ such that $T_1 = (\langle a_i,b_i,T_{1,i}\rangle)_{i\in I}$. In other words, T_1 can be expressed as an ordinal sum with the same structure of summand carriers as T_2 but with possibly different summands. Since the dominance of t-norms implies their comparability, this applies also to the situation when $T_1 \gg T_2$. By the next important theorem, the question of dominance of two ordinal sum t-norms boils down to the question of dominance of their respective summands [13].

THEOREM 1. *Let T_1 and T_2 be ordinal sum t-norms*

$$T_1 = (\langle a_i,b_i,T_{1,i}\rangle)_{i\in I}, \qquad T_2 = (\langle a_i,b_i,T_{2,i}\rangle)_{i\in I}$$

with a common system of summand carriers but with possibly different systems of summands, $(T_{1,i})_{i\in I}$ *and* $(T_{2,i})_{i\in I}$ *respectively. Then* $T_1 \gg T_2$ *if and only if for each* $i \in I$ *it holds that* $T_{1,i} \gg T_{2,i}$.

For a t-norm T_s we will use the symbol OS_{T_s} to denote the class of all ordinal sums of type $(\langle a_i, b_i, T_s\rangle)_{i\in I}$; in other words, it is the class of all ordinal sums that are constructable using exclusively T_s as the summand operation. By Theorem 1, in order to describe the structure of dominance on OS_{T_s} it is sufficient to characterize all T in OS_{T_s} which dominate T_s.

Recall that x from the unit interval is said to be an *idempotent element* of a t-norm T if $T(x,x) = x$. By the symbol $\mathfrak{I}_T$ we denote the set of all idempotent elements of T. For every t-norm T it holds that $\{0,1\} \subseteq \mathfrak{I}_T$. That is why we refer to 0 and 1 as *trivial idempotent elements*. A pair of t-norms T_1, T_2 with $T_1 \geqslant T_2$ satisfies $\mathfrak{I}_{T_1} \supseteq \mathfrak{I}_{T_2}$. The same inclusion holds also when $T_1 \gg T_2$ as the dominance of t-norms implies their comparability, but in this case even more can be said [13].

THEOREM 2. *If a t-norm* T_1 *dominates a t-norm* T_2 *then* $\mathfrak{I}_{T_1}$ *is closed with respect to* T_2.

The core result of this paper is a strengthening of Theorem 2 in the case of t-norms from either the class $\mathrm{OS}_{T_\mathbf{L}}$ or $\mathrm{OS}_{T_\mathbf{P}}$. Namely, we prove the following characterization (in fact two of them: one for $T_s = T_\mathbf{L}$ and the other one for $T_s = T_\mathbf{P}$).

THEOREM 3. *Let* T_s *be either* $T_\mathbf{L}$ *or* $T_\mathbf{P}$. *A t-norm* $T \in \mathrm{OS}_{T_s}$ *dominates* T_s *if and only if* $\mathfrak{I}_T$ *is closed with respect to* T_s.

A *continuous Archimedean t-norm* is a continuous t-norm T which has no nontrivial idempotent elements. The class of continuous Archimedean t-norms splits naturally into the class of *strict* and the class of *continuous nilpotent* t-norms. A t-norm T is strict if it is *order-isomorphic* to $T_\mathbf{P}$, i.e., the monoids $([0,1],T)$ and $([0,1],T_\mathbf{P})$ are isomorphic via an order isomorphism $\varphi\colon [0,1] \to [0,1]$. Similarly, a t-norm is continuous nilpotent if it is order-isomorphic to $T_\mathbf{L}$. In this sense $T_\mathbf{P}$ and $T_\mathbf{L}$ are two prototypical continuous Archimedean t-norms [1, 8, 19].

The dominance relation is invariant under order isomorphism: given an order isomorphism $\varphi\colon [0,1] \to [0,1]$ and four t-norms T_1, T_1', T_2, T_2' where T_i is order-isomorphic to T_i' via φ ($i = 1,2$), the condition $T_1 \gg T_2$ is equivalent with $T_1' \gg T_2'$. Therefore, in order to solve the dominance of all continuous Archimedean t-norms it is enough to characterize the continuous Archimedean t-norms that dominate $T_\mathbf{P}$ or $T_\mathbf{L}$. Moreover, since every continuous t-norm is an ordinal sum of continuous Archimedean t-norms [1, 8, 19], by Theorem 1 it follows that the solution of dominance of all continuous t-norms is equivalent to the characterization of continuous t-norms that dominate $T_\mathbf{P}$ or $T_\mathbf{L}$. In view of the latter fact the results of the present paper are just first steps towards a characterization of dominance of continuous t-norms.

Given two conjunctors $A \geqslant B$ it is generally not an easy task to decide whether A dominates B. Any geometric insight into the structure of dominance is more than welcome. For the special case when the dominated operation is either $T_\mathbf{L}$ or $T_\mathbf{P}$ such a geometric intuition has been provided by De Baets, Saminger and the present author [18, Section 3.2]. It is this geometric insight that motivated the main results of the present paper.

The paper is organized as follows. First, in Section 2, we prove two auxiliary results on dominance of continuous conjunctors. Then, with the help of these results we prove the main results in Section 3. Later in Section 4 we provide some counterexamples to transitivity of dominance of t-norms and we show how to use these results in order to solve dominance in some more complicated situations. The paper is concluded in Section 5.

2. Dominance of continuous conjunctors

By the *axis* of a conjunctor C we mean the set $\mathfrak{A}_C$ defined via

$$\mathfrak{A}_C = \{(x,y) \in]0,1]^2 \mid x = y \text{ or } C(x,y) \neq T_{\mathbf{M}}(x,y)\}. \tag{1}$$

Clearly, every continuous conjunctor C behaves on the border of its axis as the minimum t-norm, i.e., $C \upharpoonright_{\partial\mathfrak{A}_C} = T_{\mathbf{M}} \upharpoonright_{\partial\mathfrak{A}_C}$. Notice that all points with zero coordinates are excluded from the axes by definition. As we will see later, this small technicality simplifies some considerations.

LEMMA 1. *Let C be a continuous conjunctor and let $(x,y) \in]0,1]^2$. Then there is a point $(x_*,y_*) \in \mathrm{cl}(\mathfrak{A}_C)$ such that $x_* \leqslant x$, $y_* \leqslant y$, and $C(x_*,y_*) = C(x,y)$.*

Proof. Pick an arbitrary pair $(x,y) \in]0,1]^2$. We will distinguish two mutually exclusive cases: (x,y) either is or is not a member of $\mathrm{cl}(\mathfrak{A}_C)$.

In the first case, trivially, $(x_*,y_*) = (x,y)$. If, on the other hand, (x,y) is not an element of $\mathrm{cl}(\mathfrak{A}_C)$ then $x \neq y$. We will assume $y < x$; the other case $x < y$ would be treated analogously. Since $(x,y) \notin \mathfrak{A}_C$ we can conclude $C(x,y) = T_{\mathbf{M}}(x,y) = y$. Put

$$\begin{aligned} y_* &= y, \\ x_* &= \sup\{\alpha \in [y_*,x] \mid \alpha = y_* \text{ or } C(\alpha,y_*) \neq T_{\mathbf{M}}(\alpha,y_*)\}. \end{aligned} \tag{2}$$

The set under supremum in (2) is nonempty, whence x_* exists. Clearly $(x_*,y_*) \in \partial\mathfrak{A}_C$ which entails the equality $C(x_*,y_*) = T_{\mathbf{M}}(x_*,y_*)$ and settles the claim $(x_*,y_*) \in \mathrm{cl}(\mathfrak{A}_C)$. The relations $x_* \leqslant x$ and $y_* \leqslant y$ are established trivially and, by the definition of x_*, it also holds that $y_* \leqslant x_*$. Therefore $C(y_*,x_*) = T_{\mathbf{M}}(y_*,x_*) = y_* = y = C(x,y)$. □

In what follows we show that the points in the axis of a conjunctor or, eventually, in its topological closure are, in a sense, the only important ones in order to establish or disprove the dominance over another conjunctor.

THEOREM 4. *Let A be a continuous conjunctor and let B be a conjunctor. Then $A \gg B$ if and only if A dominates B on the set $\mathrm{cl}(\mathfrak{A}_A) \times \mathrm{cl}(\mathfrak{A}_A)$.*

Proof. Dominance trivially implies dominance on any subset of $[0,1]^4$. In the other way round, observe that (D) is satisfied whenever any of the variables attains the value 0; this follows from the fact that 0 is the absorbing element of every conjunctor. Therefore let x,y,u,v be arbitrary elements of $]0,1]$. Let (x_*,u_*), (y_*,v_*) be two points in $\mathrm{cl}\,(\mathfrak{A}_A)$ with

$$\begin{aligned} x_* \leqslant x, \quad u_* \leqslant u, \quad A(x_*,u_*) = A(x,u),\\ y_* \leqslant y, \quad v_* \leqslant v, \quad A(y_*,v_*) = A(y,v). \end{aligned}$$

Existence of such points is guaranteed by Lemma 1. We have the following chain of inequalities

$$\begin{aligned} A\big(B(x,y),B(u,v)\big) \geqslant A\big(B(x_*,y_*),B(u_*,v_*)\big) \geqslant \\ \geqslant B\big(A(x_*,u_*),A(y_*,v_*)\big) = B\big(A(x,u),A(y,v)\big). \end{aligned}$$

The first comparison follows from nondecreasingness of A and B, the second one by the assumption of dominance on the set $\mathrm{cl}\,(\mathfrak{A}_A) \times \mathrm{cl}\,(\mathfrak{A}_A)$. Finally, the last equality follows directly from the defining properties of the starred variables, which are listed above. □

COROLLARY 1. *Let A and B be continuous conjunctors. Then $A \gg B$ if and only if A dominates B on the set $\mathfrak{A}_A \times \mathfrak{A}_A$.*

Proof. Dominance trivially implies dominance on any subset of $[0,1]^4$. In the other way round, suppose that A dominates B on the set $\mathfrak{A}_A \times \mathfrak{A}_A$. Since A and B are continuous, so are the functions on both sides of the inequality (D). Therefore A dominates B also on the set $\mathrm{cl}\,(\mathfrak{A}_A) \times \mathrm{cl}\,(\mathfrak{A}_A)$ and the proof is concluded invoking Theorem 4. □

3. Proofs of the main results

3.1. Dominance on the class $\mathrm{OS}_{T_{\mathbf{L}}}$

For a binary operation $O\colon [0,1]^2 \to [0,1]$ we define its *diagonal* as the mapping

$$\delta_O\colon [0,1] \to [0,1]\colon x \mapsto O(x,x).$$

Triangular norms from $\mathrm{OS}_{T_{\mathbf{L}}}$ are known to be expressible in terms of their diagonals in a simple form [1, p. 154].

THEOREM 5. *Every t-norm $T \in \mathrm{OS}_{T_{\mathbf{L}}}$ satisfies*

$$T(x,y) = \min\left\{x, y, \delta_T\left(\frac{x+y}{2}\right)\right\}$$

for all $x,y \in [0,1]$.

Recall that a real function f is said to be *subadditive* if the domain of f is closed with respect to the standard addition and

$$f(x)+f(y) \geqslant f(x+y) \tag{3}$$

holds for all $x,y \in \mathrm{Dom}(f)$ (a nice introduction to subadditive functions was given by Hille and Phillips [7]). In this paper we are interested mainly in functions defined on the unit interval. Since the unit interval is not closed with respect to the standard addition, the notion of subadditivity has to be modified slightly; a function $f\colon [0,1] \to [0,1]$ is said to be subadditive if it satisfies (3) for all $x,y \in [0,1]$ with $x+y \leqslant 1$. Analogously, we say that the operation $O\colon [0,1]^2 \to [0,1]$ is subadditive if

$$O(x,y)+O(u,v) \geqslant O(x+u,y+v)$$

holds for all $x,y,u,v \in [0,1]$ with $x+u \leqslant 1$ and $y+v \leqslant 1$.

Given a t-norm T we define its *dual t-conorm* (or simply *dual*) as the binary operation

$$T_d\colon [0,1]^2 \to [0,1]\colon (x,y) \mapsto 1-T(1-x,1-y).$$

The t-conorm dual to $T_{\mathbf{L}}$ is denoted $S_{\mathbf{L}}$ and satisfies $S_{\mathbf{L}}(x,y) = \max\{x+y,1\}$. One important tool utilized in our proofs is the following result characterizing all t-norms dominating $T_{\mathbf{L}}$ by means of subadditivity and duality [12, 16].

THEOREM 6. *A t-norm T dominates $T_{\mathbf{L}}$ if and only if its dual t-conorm T_d is subadditive.*

Simple manipulations of the defining expressions reveal that the relationship between the diagonal of a t-norm T and the diagonal of its dual t-conorm T_d is given by the equality $\delta_{T_d}(x) = 1-\delta_T(1-x)$. Moreover, in the case $T \in \mathrm{OS}_{T_{\mathbf{L}}}$ we can invoke Theorem 5 and derive

$$T_d(x,y) = \max\left\{x,y,\delta_{T_d}\left(\frac{x+y}{2}\right)\right\} \tag{4}$$

which is a characterization dual to that of Theorem 5. By the following lemma, subadditivity of binary operations of this type is related to the subadditivity of their diagonals.

LEMMA 2. *If $\delta\colon [0,1] \to [0,1]$ is a subadditive function then so is the binary operation*

$$F\colon [0,1]^2 \to [0,1]\colon (x,y) \mapsto \max\left\{x,y,\delta\left(\frac{x+y}{2}\right)\right\}.$$

Proof. Let δ be subadditive; we have to prove $F(x,y)+F(u,v) \geqslant F(x+u,y+v)$ for all $x,y,u,v \in [0,1]$ with $x+u \leqslant 1$ and $y+v \leqslant 1$. For the sake of notational simplicity we write $c=(x+y)/2$ and $d=(u+v)/2$. Let us start from the left-hand side of the subadditivity inequality:

$$\begin{aligned} F(x,y)+F(u,v) &= \max\{x,y,\delta(c)\}+\max\{u,v,\delta(d)\} \\ &= \max\left\{\begin{array}{l} x+u,\, x+v,\, x+\delta(d), \\ y+u,\, y+v,\, y+\delta(d), \\ \delta(c)+u,\, \delta(c)+v,\, \delta(c)+\delta(d) \end{array}\right\} \\ &\geqslant \max\{x+u, y+v, \delta(c)+\delta(d)\} \\ &\geqslant \max\{x+u, y+v, \delta(c+d)\} \\ &= F(x+u, y+v). \end{aligned}$$

Note that the second inequality follows from the subadditivity of δ. □

Taking into account Theorem 5, Theorem 6, and Lemma 2, for every t-norm T in $\mathrm{OS}_{T_\mathbf{L}}$ the subadditivity of δ_{T_d} implies $T \gg T_\mathbf{L}$. In order to examine this subadditivity neatly we will utilize another tool which allows to express δ_{T_d} in an exceptionally handy way.

LEMMA 3. *If* $T \in \mathrm{OS}_{T_\mathbf{L}}$ *then the equations*

$$\delta_T(x) = x - \inf_{t\in\mathfrak{I}_T} |x-t| \qquad \textit{and} \qquad \delta_{T_d}(x) = x + \inf_{t\in\mathfrak{I}_{T_d}} |x-t|$$

hold for every $x \in [0,1]$.

Proof. We only show the first equality. The second one follows for free by duality.

Let $T = (\langle a_i, b_i, T_\mathbf{L}\rangle)_{i\in I}$. For $i \in I$ denote $c_i = \frac{a_i+b_i}{2}$. Closer inspection of the defining expression for $T_\mathbf{L}$ reveals that the diagonal of T is given by

$$\delta_T(x) = \begin{cases} a_i & \text{if } x \in\,]a_i, c_i[, \\ 2x-b_i & \text{if } x \in [c_i, b_i[, \\ x & \text{otherwise.} \end{cases}$$

Alternatively, for the function $\theta_T = \mathrm{id}_{[0,1]} - \delta_T$ we have

$$\theta_T(x) = \begin{cases} x-a_i & \text{if } x \in\,]a_i, c_i[, \\ b_i - x & \text{if } x \in [c_i, b_i[, \\ 0 & \text{otherwise.} \end{cases}$$

The proof will be carried out by showing that $\theta(x)$ is the distance of x from the set $\mathfrak{I}_T$, that is $\theta_T(x) = \inf_{t\in\mathfrak{I}_T} |x-t|$. By the definition we have $x \in \mathfrak{I}_T$ if and only if $\delta_T(x) = x$ which is further equivalent to $\theta_T(x) = 0$. Now let $x \notin \mathfrak{I}_T$. There exists $i \in I$ such that $x \in\,]a_i, b_i[$. If $x \in\,]a_i, c_i[$ then the idempotent element closest to x is a_i and their distance is $x - a_i$. Analogously, if $x \in\,]c_i, b_i[$, the closest idempotent element is b_i and the distance is $b_i - x$. Finally, if $x = c_i$ there exist two different idempotent elements that have the minimal distance to x. In order to compute this distance one can use for example the distance from b_i. □

LEMMA 4. *Let M be a set where $\{0,1\} \subseteq M \subseteq [0,1]$. If M is closed with respect to $S_{\mathbf{L}}$ then*

$$f_M \colon [0,1] \to \mathbf{R} \colon x \mapsto \inf_{t \in M} |x - t|$$

is a subadditive function.

Proof. For arbitrary $x, y \in [0,1]$ with $x + y \leqslant 1$ we have

$$\begin{aligned} f_M(x) + f_M(y) &= \inf_{r \in M} |x - r| + \inf_{s \in M} |y - s| \\ &= \inf_{r,s \in M} \big(|x - r| + |y - s|\big) \\ &\geqslant \inf_{r,s \in M} |x + y - (r + s)|. \end{aligned}$$

The value 1 is attained by the term $r + s$ for example in the form $0 + 1$. Since $x + y \leqslant 1$ cutting the value $r + s$ off by 1 does not affect the value of the latter infimum:

$$\inf_{r,s \in M} |x + y - (r + s)| = \inf_{r,s \in M} |x + y - \min\{r + s, 1\}|$$

Notice that $\min\{r + s, 1\} = S_{\mathbf{L}}(r,s)$. Moreover, $S_{\mathbf{L}}(M,M) = M$; indeed, one inclusion follows from the assumption that M is closed with respect to $S_{\mathbf{L}}$, the other follows from the fact that 0 is a neutral element of $S_{\mathbf{L}}$ and $0 \in M$. Whence

$$\inf_{r,s \in M} |x + y - \min\{r + s, 1\}| = \inf_{t \in M} |x + y - t| = f_M(x + y)$$

prooving that f_M is subadditive. □

Now we prove the two main results of this section. First we formulate a particular instance of Theorem 3. Next, as a corollary, we characterize dominance within the whole class $\mathrm{OS}_{T_{\mathbf{L}}}$.

THEOREM 7. *A t-norm $T \in \mathrm{OS}_{T_{\mathbf{L}}}$ dominates $T_{\mathbf{L}}$ if and only if the set $\mathfrak{I}_T$ is closed with respect to $T_{\mathbf{L}}$.*

Proof. (*i*) If $T \gg T_{\mathbf{L}}$, then $\mathfrak{I}_T$ is closed with respect to $T_{\mathbf{L}}$ by Theorem 2.

(*ii*) Assume that $\mathfrak{I}_T$ is closed with respect to $T_{\mathbf{L}}$. By means of duality, $\mathfrak{I}_{T_d}$ is closed with respect to $S_{\mathbf{L}}$. By Lemma 3 and Lemma 4, the diagonal δ_{T_d} is a sum of $\mathrm{id}_{[0,1]}$ and a subadditive function. Therefore δ_{T_d} is subadditive. Next, by equation (4) and Lemma 2 also T_d is subadditive. Invoking Theorem 6 the t-norm T dominates $T_{\mathbf{L}}$. □

COROLLARY 2. *For t-norms $T_1, T_2 \in \mathrm{OS}_{T_{\mathbf{L}}}$ we have $T_1 \gg T_2$ if and only if $\mathfrak{I}_{T_1}$ is closed with respect to T_2.*

Proof. Straightforward application of Theorem 7 and Theorem 1. □

3.2. Dominance on the class $\mathrm{OS}_{T_{\mathbf{P}}}$

Before we formulate the first auxiliary result of this section, we would like to recall that a *sector function* (or a *sector* for brevity) of a binary operation $O\colon [0,1]^2 \to [0,1]$ is any mapping of type $t \mapsto O(t,y)$ or $t \mapsto O(x,t)$ with $x,y \in [0,1]$. In the first case we sometimes speak about a *horizontal* and in the second case about a *vertical sector function*.

LEMMA 5. *Let f be a binary operation on the unit interval and let all sectors of f be nondecreasing and concave. If a subinterval $[a,b]$ of the unit interval satisfies $f(a,a) \geqslant a$ and $f(b,b) \geqslant b$ then $f\restriction_{[a,b]^2} \geqslant \langle a,b,T_{\mathbf{P}}\rangle$.*

Proof. For the sake of legibility, we will write g instead of $f\restriction_{[a,b]^2}$ and h in place of $\langle a,b,T_{\mathbf{P}}\rangle$. Now our claim reads as $g \geqslant h$. The proof will be carried out in two steps. First we show that the inequality is satisfied at the corners of $[a,b]^2$. Then, utilizing the concavity assumption, we establish the inequality on the entire $[a,b]^2$.

Direct evaluation of h at the corners reveals $h(x,y) = \min\{x,y\}$ for all possible choices of x,y in $\{a,b\}$. At the corners (a,a) and (b,b) is our claim satisfied by the assumption. Moreover, since the inequality is satisfied at (a,a) and f is assumed to have nondecreasing sector functions, our claim holds also at the remaining corners (a,b) and (b,a).

Observe that while $x \mapsto g(x,a)$ and $x \mapsto g(x,b)$ are concave by assumption, the functions $x \mapsto h(x,a)$ and $x \mapsto h(x,b)$ are affine. The latter fact is easy to see as h is a composition of the bilinear function $T_{\mathbf{P}}$ with two affine maps. Now, taking into account that $g \geqslant h$ holds in the corners of R and invoking the standard definition of a concave function, we conclude that the inequality is satisfied also at the points (x,a) and (x,b) for every x in $[a,b]$. Finally, considering the vertical sector function $y \mapsto f(x,y)$ and iterating an analogous argument once again, we prove that the inequality is actually satisfied for every $(x,y) \in [a,b]^2$. $\square$

Axes of t-norms in $\mathrm{OS}_{T_{\mathbf{P}}}$ (and generally, of ordinal sums with arbitrary strict summands) are of special structure. We express such axes as unions of two sets. Given a t-norm $T = (\langle a_i,b_i,T_{\mathbf{P}}\rangle)_{i\in I}$ we define the *slim part* and the *thick part* of $\mathfrak{A}_T$ via

$$\mathfrak{A}_T^{\mathrm{slim}} = \{(x,x) \,|\, x \in \mathfrak{I}_T \setminus \{0\}\} \qquad \text{and} \qquad \mathfrak{A}_T^{\mathrm{thick}} = \bigcup_{i\in I}]a_i,b_i[^2$$

respectively. Since $T_{\mathbf{P}}$ is strictly weaker than $T_{\mathbf{M}}$ on the open unit square, every operation $\langle a_i,b_i,T_{\mathbf{P}}\rangle$ is so on the open square $]a_i,b_i[^2$. Taking into account the definition of ordinal sum t-norms, T is strictly weaker than $T_{\mathbf{M}}$ exactly on the set $\mathfrak{A}_T^{\mathrm{thick}}$. Comparing this fact with (1) yields

$$\mathfrak{A}_T = \mathfrak{A}_T^{\mathrm{slim}} \cup \mathfrak{A}_T^{\mathrm{thick}}. \tag{5}$$

Since summand carriers are intervals with mutually disjoint interiors, the union in the definition of $\mathfrak{A}_T^{\mathrm{thick}}$ is disjoint. As a consequence, for every pair $(x,y) \in \mathfrak{A}_T^{\mathrm{thick}}$ there exists a unique index i in I such that $(x,y) \in]a_i,b_i[^2$; the values a_i and b_i given this way will be denoted $a_{(x,y)}$ and $b_{(x,y)}$ respectively. By the definition of ordinal sum t-norms, the identity

$$T(x,y) = \langle a_{(x,y)}, b_{(x,y)}, T_{\mathbf{P}} \rangle (x,y) \tag{6}$$

holds for every $T \in \mathrm{OS}_{T_{\mathbf{P}}}$ and every pair $(x,y) \in \mathfrak{A}_T^{\text{thick}}$.

In the proof of the next lemma we will invoke a quite obvious claim that all horizontal and vertical sector functions of t-norms from $\mathrm{OS}_{T_{\mathbf{P}}}$ are concave. One possibility how to establish this claim is to use a recent result by Durante and Papini according to which an ordinal sum of copulas with concave sector functions is again a copula with concave sectors [6, Proposition 2.5]. Interested reader can find the definition of copula functions for example in the monograph by Schweizer and Sklar [19, Chapter 6]. The important fact at this point is that $T_{\mathbf{P}}$ is indeed a copula [8, Example 9.5.i] and has concave (actually even linear) sectors.

LEMMA 6. *Let T be a member of $\mathrm{OS}_{T_{\mathbf{P}}}$ where $\mathfrak{I}_T$ has an accumulation point in 0 and let $S \subseteq \mathfrak{A}_T$. If T dominates $T_{\mathbf{P}}$ on the set $S \times \mathfrak{A}_T^{\text{slim}}$ then so it does also on the set $S \times \mathfrak{A}_T$.*

Proof. Let (x,u) be an arbitrary but fixed pair in S. Being a member of the class $\mathrm{OS}_{T_{\mathbf{P}}}$, the t-norm T assumes positive values for pairs of positive arguments. Therefore $T(x,u)$ is a positive number and we can define the function

$$f\colon [0,1]^2 \to \mathbf{R}\colon (y,v) \mapsto \frac{T(xy,uv)}{T(x,u)}.$$

The satisfaction of (D) is equivalent to $f(y,v) \geqslant T(y,v)$; all cases of dominance within the proof will be considered in this form. Now it is enough to establish this inequality for every $(y,v) \in \mathfrak{A}_T^{\text{thick}}$. Indeed, assuming that the inequality holds whenever $(y,v) \in \mathfrak{A}_T^{\text{slim}}$, by (5) it then holds also for $(y,v) \in \mathfrak{A}_T$ and, as the choice of $(x,u) \in S$ is arbitrary, we prove our claim.

Let (y,v) be a pair in $\mathfrak{A}_T^{\text{thick}}$. The interval $\left[a_{(y,v)}, b_{(y,v)}\right]$ is a summand carrier and $a_{(y,v)}, b_{(y,v)}$ are idempotent elements of T. Since $\mathfrak{I}_T$ is accumulated in 0, both $a_{(y,v)}$ and $b_{(y,v)}$ are positive. As a consequence, the points $(a_{(y,v)}, a_{(y,v)})$ and $(b_{(y,v)}, b_{(y,v)})$ are elements of $\mathfrak{A}_T^{\text{slim}}$ and, by the assumption, satisfy the dominance inequality in the form

$$\begin{aligned} f(a_{(y,v)}, a_{(y,v)}) &\geqslant T(a_{(y,v)}, a_{(y,v)}) = a_{(y,v)}\,, \\ f(b_{(y,v)}, b_{(y,v)}) &\geqslant T(b_{(y,v)}, b_{(y,v)}) = b_{(y,v)}\,. \end{aligned} \tag{7}$$

Being a member of $\mathrm{OS}_{T_{\mathbf{P}}}$, our t-norm T has nondecreasing and concave sector functions. Clearly, f inherits this property which, taking (7) into account, allows to invoke Lemma 5 and conclude $f(y,v) \geqslant \langle a_{(y,v)}, b_{(y,v)}, T_{\mathbf{P}} \rangle (y,v)$. Finally, combining the latter inequality with (6) establishes the dominance inequality for the pair (y,v) and concludes the proof. □

Now we formulate and prove the two main results of this section. First we formulate a particular instance of Theorem 3. Next, as a corollary, we characterize dominance within the whole class $\mathrm{OS}_{T_{\mathbf{P}}}$.

THEOREM 8. *A t-norm T from the class $\mathrm{OS}_{T_\mathbf{P}}$ dominates $T_\mathbf{P}$ if and only if $\mathfrak{I}_T$ is closed with respect to $T_\mathbf{P}$.*

Proof. (*i*) If $T \gg T_\mathbf{P}$, then $\mathfrak{I}_T$ is closed with respect to $T_\mathbf{P}$ by Theorem 2.

(*ii*) Let us take arbitrary $(x,u),(y,v) \in \mathfrak{A}_T^{\mathrm{slim}}$. By the definition of the slim part of the axis we have $x = u$, $y = v$, and $x,y \in \mathfrak{I}_T$. By the assumption $xy \in \mathfrak{I}_T$. Therefore

$$T(xy,uv) = T(xy,xy) = xy = T(x,x)T(y,y) = T(x,u)T(y,v).$$

meaning that T dominates $T_\mathbf{P}$ on the set $\mathfrak{A}_T^{\mathrm{slim}} \times \mathfrak{A}_T^{\mathrm{slim}}$.

Now we distinguish two mutually exclusive cases: the intersection $\mathfrak{I}_T \cap]0,1[$ is either empty or nonempty. In the first case $\mathfrak{I}_T$ is simply $\{0,1\}$, the t-norm T is actually $T_\mathbf{P}$ and the dominance is satisfied trivially. In the other case the closeness with respect to $T_\mathbf{P}$ implies that 0 is an accumulation point of $\mathfrak{I}_T$. Since $\mathfrak{A}_T^{\mathrm{slim}} \subseteq \mathfrak{A}_T$ we can use Lemma 6 and conclude that T dominates $T_\mathbf{P}$ also on the set $\mathfrak{A}_T^{\mathrm{slim}} \times \mathfrak{A}_T$. Thanks to the commutativity of t-norms, both sides of (D) are invariant under the permutation $(x,u,y,v) \mapsto (y,v,x,u)$. Whence T dominates $T_\mathbf{P}$ also on the set $\mathfrak{A}_T \times \mathfrak{A}_T^{\mathrm{slim}}$. Invoking Lemma 6 once again, the dominance is extended to the set $\mathfrak{A}_T \times \mathfrak{A}_T$ and, as T and $T_\mathbf{P}$ are both continuous conjunctors, the latter implies $T \gg T_\mathbf{P}$ via Corollary 1. □

COROLLARY 3. *Two t-norms T_1 and T_2 from the class $\mathrm{OS}_{T_\mathbf{P}}$ satisfy $T_1 \gg T_2$ if and only if $\mathfrak{I}_{T_1}$ is closed with respect to T_2.*

Proof. Straightforward application of Theorem 8 and Theorem 1. □

4. The non-transitivity of dominance

Several consequences of the here presented results have already been promoted informally. For example the family of Mayor-Torrens [9] and Dubois-Prade [5] t-norms were shown to admit only a sparse relation of dominance [18, 17]. It was also shown that certain slightly modified versions of these families are linearly ordered by dominance [18, 17].

As we have already mentioned in the introduction, the question whether dominance of triangular norms is a transitive relation remained open for almost three decades. In view of the present results it is very easy to construct three continuous t-norms T_1, T_2, and T_3 such that $T_1 \gg T_2$, $T_2 \gg T_3$ and $T_1 \not\gg T_3$. For example, the counterexample [20] was based on the choice

$$T_1 = (\langle 0, \tfrac{1}{2}, T_\mathbf{L} \rangle), \quad T_2 = (\langle 0, \tfrac{1}{2}, T_\mathbf{L} \rangle, \langle \tfrac{1}{2}, 1, T_\mathbf{L} \rangle), \quad T_3 = T_\mathbf{L}.$$

The tools used to establish this counterexample were much less general than those of the present paper. In view of our results all these three relations follow from Corollary 2.

Although we already know that dominance of continuous t-norms is not transitive, it remains an open question whether it is true also in the case of strict t-norms. With the help of Corollary 3 we can design a counterexample to transitivity which involves two strict t-norms. Before we do so, let us recall that the Aczél-Alsina t-norm [8] with parameter $\lambda \in]0,\infty[$ is a strict t-norm defined via

$$T_\lambda^{\mathbf{AA}}\colon (x,y) \mapsto \exp\left(-\left(|\log(x)|^\lambda + |\log(y)|^\lambda\right)^{1/\lambda}\right).$$

Now, in order to establish the desired counterexample, it is enough to set

$$T_1 = (\langle \tfrac{1}{8}, \tfrac{1}{4}, T_{\mathbf{P}}\rangle, \langle \tfrac{1}{4}, \tfrac{1}{2}, T_{\mathbf{P}}\rangle, \langle \tfrac{1}{2}, 1, T_{\mathbf{P}}\rangle), \quad T_2 = T_{\mathbf{P}}, \quad T_3 = T_{0.9}^{\mathbf{AA}}.$$

The relation $T_2 \gg T_3$ is known to hold thanks to a peculiar algebraic relationship of the Aczél-Alsina t-norms to the product t-norm [8]. In order to disprove $T_1 \gg T_3$ it is enough to show that $\mathfrak{I}_{T_1}$ is not closed with respect T_1 and invoke Theorem 2. This is an easy task as $\frac{1}{2} \in \mathfrak{I}_{T_1}$ satisfies $T_3(\frac{1}{2}, \frac{1}{2}) \sim 0.2237 \notin \mathfrak{I}_{T_1}$. Finally, the relation $T_1 \gg T_2$ is established by means of Corollary 3 since the set $\mathfrak{I}_{T_1}$ is clearly closed with respect to the standard multiplication.

5. Two concluding remarks

The tool developed in Section 2 (or better its generalization for n-ary conjunctors on the dominating side) could find applications in design of T-transitivity preserving conjunctive aggregation procedures for sets of many-valued relations; every such procedure is based on a conjunctor that dominates a given t-norm T [16, Theorem 3.1].

Next, it is interesting that both main results of the present paper (Theorem 7 and Theorem 8) are of the same structure. Actually Theorem 3 formulates both of them at the same place. Therefore it is natural to ask which t-norms T_s other than $T_{\mathbf{L}}$ and $T_{\mathbf{P}}$ satisfy formally the same theorem. First brief investigations in this direction revealed that there exist continuous Archimedean t-norms that violate the condition in question [17].

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REFERENCES

[1] C. ALSINA, M. J. FRANK, AND B. SCHWEIZER, *Associative Functions: Triangular Norms and Copulas*, World Scientific Publishing Company, Singapore, (2006).

[2] U. BODENHOFER, *A Similarity-based Generalization of Fuzzy Orderings*, Universitätsverlag Rudolf Trauner, Schriftereihe der Johannes-Kepler-Universität Linz, Volume **C 26**, Linz, (1999).

[3] B. DE BAETS AND R. MESIAR, *Pseudo-metrics and T-equivalences*, J. Fuzzy Math., Vol. 5, **2** (1997), p. 471–481.

[4] B. DE BAETS AND R. MESIAR, *T-partitions*, Fuzzy Sets and Systems, Vol. 97, **2** (1998), p. 211–223.

[5] D. DUBOIS AND H. PRADE, *New Results about Properties and Semantics of Fuzzy-Set Theoretic Operators*, p. 59–75, in Fuzzy sets: Theory and Applications to Policy Analysis and Information Systems (P. P. Wang and S. K. Chang Eds.) Plenum Press, 1980, New York.

[6] F. DURANTE AND P. L. PAPINI, *Componentwise Concave Copulas and their Asymmetry Kybernetika*, Vol. 45, **6** (2009), p. 1003–1011.
[7] E. HILLE AND R. S. PHILLIPS, *Functional Analysis and Semi-groups*, American Mathematical Society, Providence, Rhode Island, (1957).
[8] E. P. KLEMENT, R. MESIAR, AND E. PAP, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, (2000).
[9] G. MAYOR AND J. TORRENS, *On a Family of t-Norms*, Fuzzy Sets and Systems, Vol. 41, **2** (1991), p. 161–166.
[10] R. MESIAR AND S. SAMINGER, *Domination of Ordered Weighted Averaging Operators over t-norms*, Soft Computing, Vol. 8, **8** (2004), p. 562–570.
[11] H. P. MULHOLLAND, *On Generalizations of Minkovski's Inequality in the Form of a Triangle Inequality*, Proc. London Math. Soc., Vol. 51, **1** (1950), p. 294–307.
[12] A. PRADERA AND E. TRILLAS, *A note on pseudometrics aggregation*, Int. J. General Systems, Vol. 31, **1** (2002), p. 41–51.
[13] S. SAMINGER, B. DE BAETS AND H. DE MEYER, *On the Dominance Relation Between Ordinal Sums of Conjunctors*, Kybernetika, Vol. 42, **3** (2006), p. 337–350.
[14] S. SAMINGER, B. DE BAETS AND H. DE MEYER, *A Generalization of the Mulholland Inequality for Continuous Archimedean t-norms*, J. Math. Anal. Appl., Vol. 345, **2** (2008), p. 607–614.
[15] S. SAMINGER, B. DE BAETS AND H. DE MEYER, *Differential Inequality Conditions for Dominance Between Continuous Archimedean t-norms*, Math. Inequal. Appl., Vol. 12, **1** (2009), p. 191–208.
[16] S. SAMINGER, R. MESIAR AND U. BODENHOFER, *Domination of Aggregation Operators and Preservation of Transitivity*, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems, Vol. 10, **1** (2002), p. 11–35.
[17] S. SAMINGER AND P. SARKOCI, *Dominance of Ordinal Sums of $T_{\mathbf{L}}$ and $T_{\mathbf{P}}$*, Proceedings of the 5$^{\text{th}}$ EUSFLAT Conference, Vol. 1, 2007, Ostrava. p. 35–39.
[18] S. SAMINGER, P. SARKOCI AND B. DE BAETS, *The Dominance Relation on the Class of Continuous T-Norms from an Ordinal Sum Point of View*, Lecture Notes in Artificial Intelligence, Vol. 4342, **1** (2006), p. 334–354.
[19] B. SCHWEIZER AND A. SKLAR, *Probabilistic Metric Spaces*, North-Holland, New York, (1983).
[20] P. SARKOCI, *Dominance is not Transitive on Continuous Triangular Norms*, Aequationes Mathematicae, **3**, Vol. 75, (2008), p. 201–207.
[21] R. M. TARDIFF, *Topologies for Probabilistic Metric Spaces*, Pacific J. Math., **1**, Vol. 65, (1976), p. 233–251.
[22] L. VALVERDE, *On the Structure of F-indistinguishability Operators*, Fuzzy Sets and Systems, **3**, Vol. 17, (1985), p. 313–328.

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