Mathematical Inequalities & Applications
Volume 17, Number 1 (2014), 367–388
doi:10.7153/mia-17-28

# A MORE ACCURATE HALF–DISCRETE HILBERT–TYPE INEQUALITY WITH A GENERAL NON–HOMOGENEOUS KERNEL AND OPERATOR EXPRESSIONS

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(*Communicated by Josip Pečarić*)

*Abstract.* In this paper, by the use of the methods of weight functions and technique of real analysis, a more accurate half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best possible constant factor is given. The equivalent forms and some reverses are obtained. We also consider the operator expressions with the norm and some particular examples.

## 1. Introduction

Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)$, $g(y)\geqslant 0$, $f\in L^p(\mathbf{R}_+)$, $g\in L^q(\mathbf{R}_+)$,

$$||f||_p=\{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}}>0,$$

$||g||_q>0$. We have the following Hardy-Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}||f||_p||g||_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m,b_n\geqslant 0$, $a=\{a_m\}_{m=1}^\infty\in l^p$, $b=\{b_n\}_{n=1}^\infty\in l^q$, $||a||_p=\{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}}>0$, $||b||_q>0$, then we still have the following discrete variant of the above inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}||a||_p||b||_q, \tag{2}$$

which is named Hardy-Hilbert's inequality (cf. [1]). Inequalities (1) and (2) are important in Analysis and its applications (cf. [1], [2], [3], [4], [5], [6]).

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*Mathematics subject classification* (2010): 26D15, 47A07.
*Keywords and phrases*: Half-discrete Hilbert-type inequality, weight function, equivalent form, reverse, Hilbert-type operator.
This work is supported by 2012 Knowledge Construction Special Foundation Item of Guangdong Institution of Higher Learning College and University (No. 2012KJCX0079).

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [7] gave an extension of (1) at $p = q = 2$. In 2009 and 2011, Yang [3], [4] gave some extensions of (1) and (2) as follows: If $\lambda_1, \lambda_2, \lambda \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(y) = y^{q(1-\lambda_2)-1}$, $f(x)$, $g(y) \geqslant 0$,

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}$, $||g||_{q,\psi} > 0$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) f(x) g(y) dx dy < k(\lambda_1) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_m, b_n \geqslant 0$,

$$a \in l_{p,\phi} = \left\{ a; ||a||_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $||a||_{p,\phi}$, $||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n) a_m b_n < k(\lambda_1) ||a||_{p,\phi} ||b||_{q,\psi}, \tag{4}$$

where, the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (3) reduces to (1), while (4) reduces to (2). Some other results including the multidimensional Hilbert-type integral inequalities are provided by [8]–[19].

About the topic of half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the the constant factors are the best possible. However, Yang [20] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ by introducing a variable and proved that the constant factor is the best possible. In 2011, Yang [21] gave the following half-discrete Hardy-Hilbert's inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx < B(\lambda_1, \lambda_2) ||f||_{p,\phi} ||a||_{q,\psi}, \tag{5}$$

where, $\lambda_1 \lambda_2 > 0$, $0 \leqslant \lambda_2 \leqslant 1$, $\lambda_1 + \lambda_2 = \lambda$,

$$B(u,v) = \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1} dt \quad (u,v > 0)$$

is the beta function. Zhong et al. ([22]–[17]) investigated several half-discrete Hilbert-type inequalities with particular kernels.

Using the way of weight functions and the techniques of discrete and integral Hilbert-type inequalities with some additional conditions on the kernel, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ is obtained as follows:

$$\int_0^\infty f(x)\sum_{n=1}^\infty k_\lambda(x,n)a_n dx < k(\lambda_1)||f||_{p,\phi}||a||_{q,\psi}, \tag{6}$$

which is an extension of (5) (see Yang and Chen [29]). At the same time, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [30].

REMARK 1. (1) Many different kinds of Hilbert-type discrete, half-discrete and integral inequalities with applications are presented in recent twenty years. Special attention is given to new results proved during 2009–2012. Included are many generalizations, extensions and refinements of Hilbert-type discrete, half-discrete and integral inequalities involving many special functions such as beta, gamma, hypergeometric, trigonometric, hyperbolic, zeta, Bernoulli functions, Bernoulli numbers and Euler constant et al.

(2) In his five books, Yang ([3], [5], [4], [6], [31]) presented many new results on Hilbert-type operators with general homogeneous kernels of degree of real numbers and two pairs of conjugate exponents as well as the related inequalities. These research monographs contained recent developments of discrete, half-discrete and integral types of operators and inequalities with proofs, examples and applications.

In this paper, by the use of the methods of weight functions and technique of real analysis, a more accurate half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best possible constant factor is given. The equivalent forms and some reverses are obtained. We also consider the operator expressions with the norm and some particular examples.

## 2. Some lemmas

LEMMA 1. *Suppose that* $(-1)^i h^{(i)}(t) > 0$ $(t > 0;\ i = 0,1,2)$. *Then for* $\int_{\frac{1}{2}}^\infty h(t)dt < \infty$, *we have*

$$\int_1^\infty h(t)dt < \sum_{n=1}^\infty h(n) < \int_{\frac{1}{2}}^\infty h(t)dt. \tag{7}$$

*Proof.* Since $h(t)$ is a decreasing convex function, by the decreasing property and Hermite-Hadamard's inequality (cf. [32]), we have

$$\int_n^{n+1} h(t)dt < h(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(t)dt \quad (n \in \mathbf{N}),$$

and then

$$\int_1^{\infty} h(t)dt = \sum_{n=1}^{\infty} \int_n^{n+1} h(t)dt < \sum_{n=1}^{\infty} h(n)$$
$$< \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(t)dt = \int_{\frac{1}{2}}^{\infty} h(t)dt.$$

Hence (7) follows. The lemma is proved. □

NOTE. If $h(t) = g_1(t)g_2(t)$ and $(-1)^i g_j^{(i)}(t) > 0$ $(t > 0;\ i = 0,1,2,\ j = 1,2)$, then it is evident that $(-1)^i h^{(i)}(t) > 0$ $(t > 0;\ i = 0,1,2)$.

LEMMA 2. *Suppose that $h(t)$ is a non-negative measurable function in $\mathbf{R}_+$, $a \in \mathbf{R}$, and there exists a constant $\delta_0 > 0$, such that for any $\delta \in [0,\delta_0)$,*

$$k(a \pm \delta) := \int_0^{\infty} h(t)t^{(a\pm\delta)-1}dt \in \mathbf{R}.$$

*Then we have*

$$k(a \pm \delta) = k(a) + o(1) \ \ (\delta \to 0^+). \tag{8}$$

*Proof.* For any $\delta \in [0, \frac{\delta_0}{2})$, it follows

$$h(t)t^{(a\pm\delta)-1} \leqslant g(t) := \begin{cases} h(t)t^{(a-\frac{\delta_0}{2})-1}, & t \in (0,1], \\ h(t)t^{(a+\frac{\delta_0}{2})-1}, & t \in (1,\infty). \end{cases}$$

Since we find

$$\begin{aligned} 0 &\leqslant \int_0^{\infty} g(t)dt \\ &= \int_0^1 h(t)t^{(a-\frac{\delta_0}{2})-1}dt + \int_1^{\infty} h(t)t^{(a+\frac{\delta_0}{2})-1}dt \\ &\leqslant \int_0^{\infty} h(t)t^{(a-\frac{\delta_0}{2})-1}dt + \int_0^{\infty} h(t)t^{(a+\frac{\delta_0}{2})-1}dt \\ &= k\left(a - \frac{\delta_0}{2}\right) + k\left(\alpha + \frac{\delta_0}{2}\right) \in \mathbf{R}, \end{aligned}$$

then by Lebesgue control convergence theorem (cf. [33]), it follows

$$\begin{aligned} k(a\pm\delta) &= \int_0^{\infty} h(t)t^{(a\pm\delta)-1}dt \\ &= \int_0^{\infty} h(t)t^{a-1}dt + o(1) \ \ (\delta \to 0^+), \end{aligned}$$

namely, (8) follows. The lemma is proved. □

DEFINITION 1. For $x \in \mathbf{R}_+$, $n \in \mathbf{N}$, $\tau \in [0, \frac{1}{2}]$, $\sigma \in \mathbf{R}$, $h(t)$ is a non-negative measurable function in $\mathbf{R}_+$, define two weight coefficients $w(\sigma,n)$ and $W(\sigma,x)$ as follows:

$$w(\sigma,n) := (n-\tau)^{\sigma} \int_{\mathbf{R}_+} h(x(n-\tau)) \frac{dx}{x^{1-\sigma}}, \tag{9}$$

$$W(\sigma,x) := x^{\sigma} \sum_{n=1}^{\infty} h(x(n-\tau)) \frac{1}{(n-\tau)^{1-\sigma}}. \tag{10}$$

LEMMA 3. *As the assumptions of Definition 1, if there exists a constant* $K > 0$, *such that* $0 < w(\sigma,n)$, $W(\sigma,x) < K$, $f(x) \geqslant 0$, $a_n \geqslant 0$, *then*

*(i) for* $p > 1$, *we have the following inequality:*

$$\widehat{J}_1 := \left\{ \sum_{n=1}^{\infty} \frac{(n-\tau)^{p\sigma-1}}{[w(\sigma,n)]^{p-1}} \left( \int_{\mathbf{R}_+} h(x(n-\tau)) f(x) dx \right)^p \right\}^{\frac{1}{p}}$$
$$\leqslant \left\{ \int_{\mathbf{R}_+} W(\sigma,x) x^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{11}$$

$$\widehat{J}_2 := \left\{ \int_{\mathbf{R}_+} \frac{x^{q\sigma-1}}{[W(\sigma,x)]^{q-1}} \left( \sum_{n=1}^{\infty} h(x(n-\tau)) a_n \right)^q dx \right\}^{\frac{1}{q}}$$
$$\leqslant \left\{ \sum_{n=1}^{\infty} w(\sigma,n) (n-\tau)^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}}; \tag{12}$$

*(ii) for* $p < 0$, *or* $0 < p < 1$, *we have the reverses of (11) and (12).*

*Proof.* (i) For $p > 1$, by Hölder's inequality with weight (cf. [32]), it follows

$$\int_{\mathbf{R}_+} h(x(n-\tau)) f(x) dx$$
$$= \int_{\mathbf{R}_+} h(x(n-\tau) \left[ \frac{x^{(1-\sigma)/q} f(x)}{(n-\tau)^{(1-\sigma)/p}} \right] \left[ \frac{(n-\tau)^{(1-\sigma)/p}}{x^{(1-\sigma)/q}} \right] dx$$
$$\leqslant \left\{ \int_{\mathbf{R}_+} h(x(n-\tau)) \frac{x^{(1-\sigma)(p-1)}}{(n-\tau)^{1-\sigma}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_{\mathbf{R}_+} h(x(n-\tau)) \frac{(n-\tau)^{(1-\sigma)(q-1)}}{||x||_{\alpha}^{i_0-\sigma}} \right\}^{\frac{1}{q}}$$
$$= [w(\sigma,n)]^{\frac{1}{q}} (n-\tau)^{\frac{1}{p}-\sigma} \left\{ \int_{\mathbf{R}_+} h(x(n-\tau)) \frac{x^{(1-\sigma)(p-1)}}{(n-\tau)^{1-\sigma}} f^p(x) dx \right\}^{\frac{1}{p}}. \tag{13}$$

Then by Lebesgue term by term integration theorem (cf. [33]), we have

$$\begin{aligned}\widehat{J}_1 &\leqslant \left\{\sum_{n=1}^{\infty}\int_{\mathbf{R}_+} h(x(n-\tau))\frac{x^{(1-\sigma)(p-1)}}{(n-\tau)^{1-\sigma}}f^p(x)dx\right\}^{\frac{1}{p}} \\ &= \left\{\int_{\mathbf{R}_+}\sum_{n=1}^{\infty} h(x(n-\tau))\frac{x^{(1-\sigma)(p-1)}}{(n-\tau)^{1-\sigma}}f^p(x)dx\right\}^{\frac{1}{p}} \\ &= \left\{\int_{\mathbf{R}_+} W(\sigma,x)x^{p(1-\sigma)-1}f^p(x)dx\right\}^{\frac{1}{p}}. \end{aligned} \tag{14}$$

Hence, (11) follows.

By the same way, we have

$$\sum_{n=1}^{\infty} h(x(n-\tau))a_n \leqslant [W(\sigma,x)]^{\frac{1}{p}} x^{\frac{1}{q}-\sigma}\left\{\sum_{n=1}^{\infty} h(x(n-\tau))\frac{(n-\tau)^{(1-\sigma)(q-1)}}{x^{1-\sigma}}a_n^q\right\}^{\frac{1}{q}}, \tag{15}$$

then by Lebesgue term by term integration theorem and the same way as in obtaining (14), we have (12).

(ii) For $p<0$, or $0<p<1$, by the reverse Hölder's inequality with weight (cf. [32]), we obtain the reverses of (13) and (15). Then by Lebesgue term by term integration theorem, we still can obtain the reverses of (11) and (12). The lemma is proved. □

LEMMA 4. *As the assumptions of Lemma 3, then*
*(i) for $p>1$, we have the following inequality equivalent to (11) and (12):*

$$\begin{aligned}\widehat{I} &:= \sum_{n=1}^{\infty}\int_{\mathbf{R}_+} h(x(n-\tau))a_n f(x)dx \\ &\leqslant \left\{\int_{\mathbf{R}_+} W(\sigma,x)x^{p(1-\sigma)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} w(\sigma,n)(n-\tau)^{q(1-\sigma)-1}a_n^q\right\}^{\frac{1}{q}}; \end{aligned} \tag{16}$$

*(ii) for $p<0$, or $0<p<1$, we have the reverse of (16) equivalent to the reverses of (11) and (12).*

*Proof.* (i) For $p>1$, by Hölder's inequality (cf. [32]), it follows

$$\begin{aligned}\widehat{I} &= \sum_{n=1}^{\infty}\frac{(n-\tau)^{\frac{1}{q}-(1-\sigma)}}{[w(\sigma,n)]^{\frac{1}{q}}}\left[\int_{\mathbf{R}_+} h(x(n-\tau))f(x)dx\right]\left[[w(\sigma,n)]^{\frac{1}{q}}(n-\tau)^{(1-\sigma)-\frac{1}{q}}a_n\right] \\ &\leqslant \widehat{J}_1\left\{\sum_{n=1}^{\infty} w(\sigma,n)(n-\tau)^{q(1-\sigma)-1}a_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{17}$$

Then by (11), we have (16).

On the other hand, assuming that (16) is valid, we set

$$b_n := \frac{(n-\tau)^{p\sigma-1}}{[w(\sigma,n)]^{p-1}} \left( \int_{\mathbf{R}_+} h(x(n-\tau)) f(x) dx \right)^{p-1}, \quad n \in \mathbf{N}.$$

Then it follows

$$\widehat{J}_1^p = \sum_{n=1}^{\infty} w(\sigma,n)(n-\tau)^{q(1-\sigma)-1} a_n^q.$$

If $\widehat{J}_1 = 0,$ then (11) is trivially valid; if $\widehat{J}_1 = \infty,$ then by (14), (11) keeps the form of equality $(=\infty)$. Suppose that $0 < \widehat{J}_1 < \infty$. By (16), we have

$$\begin{aligned} 0 &< \sum_{n=1}^{\infty} w(\sigma,n)(n-\tau)^{q(1-\sigma)-j_0} a_n^q = \widehat{J}_1^p = \widehat{I} \\ &\leqslant \left\{ \int_{\mathbf{R}_+} W(\sigma,x) ||x||_{\alpha}^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} w(\sigma,n)(n-\tau)^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}} < \infty. \end{aligned}$$

It follows

$$\begin{aligned} \widehat{J}_1 &= \left\{ \sum_{n=1}^{\infty} w(\sigma,n)(n-\tau)^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{p}} \\ &\leqslant \left\{ \int_{\mathbf{R}_+} W(\sigma,x) x^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned}$$

and then (11) follows. Hence, (16) and (11) are equivalent.

By Hölder's inequality and the same way, we can obtain

$$\widehat{I} \leqslant \left\{ \int_{\mathbf{R}_+} W(\sigma,x) x^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \widehat{J}_2.$$

Then by (12), we have (16). On the other hand, assuming that (16) is valid, we set

$$f(x) = \frac{x^{q\sigma-1}}{[W(\sigma,x)]^{q-1}} \left( \sum_{n=1}^{\infty} h(x(n-\tau)) a_n \right)^{q-1} \quad (x \in \mathbf{R}_+).$$

Then it follows

$$\widehat{J}_2^q = \int_{\mathbf{R}_+} W(\sigma,x) x^{p(1-\sigma)-1} f^p(x) dx.$$

By (16) and the same way, we can obtain

$$\begin{aligned} \widehat{J}_2 &= \left\{ \int_{\mathbf{R}_+} W(\sigma,x) ||x||_{\alpha}^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{q}} \\ &\leqslant \left\{ \sum_{n=1}^{\infty} w(\sigma,n) ||n-\tau||_{\beta}^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}}, \end{aligned}$$

and then (16) and (12) are equivalent.

Hence, (11), (12) and (16) are equivalent.

(ii) For $p<0$, or $0<p<1$, by the same way, we have the reverse of (16) equivalent to the reverses of (11) and (12). The lemma is proved. □

LEMMA 5. *If* $\widetilde{\sigma} \in \mathbf{R}$,

$$\widehat{k}(\widetilde{\sigma}) := \int_0^\infty h(u)u^{\widetilde{\sigma}-1}du \in \mathbf{R}_+,$$

*then we have*

$$w(\widetilde{\sigma},n) = \widehat{k}\ (\widetilde{\sigma}) \in \mathbf{R}_+ (n \in \mathbf{N}). \tag{18}$$

*Moreover, if*

$$\frac{d}{dt}(h(t)t^{\widetilde{\sigma}-1}) < 0, \quad \frac{d^2}{dt^2}(h(t)t^{\widetilde{\sigma}-1}) > 0,$$

*there exists constants* $L>0$, *and* $\eta_0 < \widetilde{\sigma}$, *satisfying*

$$h(t) \leqslant \frac{L}{t^{\eta_0}} \ \ (t \in (0,\infty)), \tag{19}$$

*then we have*

$$\widehat{k}(\widetilde{\sigma})(1-\vartheta_{\widetilde{\sigma}}(x)) < W(\widetilde{\sigma},x) < \widehat{k}(\widetilde{\sigma}) \ \ (x \in \mathbf{R}_+). \tag{20}$$

*where,*

$$\begin{aligned} \vartheta_{\widetilde{\sigma}}(x) &:= \frac{1}{\widehat{k}(\widetilde{\sigma})}\int_0^x h(t)t^{\widetilde{\sigma}-1}dt \\ &= O(x^{\rho(\widetilde{\sigma})}) \in (0,1) \ \ (\rho(\widetilde{\sigma}) = \widetilde{\sigma}-\eta_0 > 0). \end{aligned} \tag{21}$$

*Proof.* By (9), setting $t = x(n-\tau)$, we find

$$w(\widetilde{\sigma},n) = \int_0^\infty h(t)t^{\widetilde{\sigma}-1}dt.$$

Hence, we have (18).

Moreover, by (7), setting $t = x(y-\tau)$, we obtain

$$\begin{aligned} W(\widetilde{\sigma},x) &< x^{\widetilde{\sigma}}\int_{\frac{1}{2}}^\infty h(x(y-\tau))\frac{dy}{(y-\tau)^{1-\widetilde{\sigma}}} \\ &= \int_{x(\frac{1}{2}-\tau)}^\infty h(t)t^{\widetilde{\sigma}-1}dt \leqslant \int_0^\infty h(t)t^{\widetilde{\sigma}-1}dt = \widehat{k}(\widetilde{\sigma}), \\ W(\widetilde{\sigma},x) &> x^{\widetilde{\sigma}}\int_{1+\tau}^\infty h(x(y-\tau))\frac{dy}{(y-\tau)^{1-\widetilde{\sigma}}} \\ &= \int_x^\infty h(t)t^{\widetilde{\sigma}-1}dt = \int_0^\infty h(t)t^{\widetilde{\sigma}-1}dt - \int_0^x h(t)t^{\widetilde{\sigma}-1}dt \\ &= \widehat{k}(\widetilde{\sigma})(1-\vartheta_{\widetilde{\sigma}}(x)) > 0, \end{aligned}$$

$$0 < \vartheta_{\widetilde{\sigma}}(x) = \frac{1}{\widehat{k}(\widetilde{\sigma})}\int_0^x h(t)t^{\widetilde{\sigma}-1}dt$$
$$\leqslant \frac{L}{\widehat{k}(\widetilde{\sigma})}\int_0^x t^{\widetilde{\sigma}-\eta_0-1}dt = \frac{L}{\widehat{k}(\widetilde{\sigma})(\widetilde{\sigma}-\eta_0)}x^{\widetilde{\sigma}-\eta_0}.$$

Hence, we have (20) and (21). The lemma is proved. □

## 3. Main results and some reverses

Setting

$$\Phi_\sigma(x) := x^{p(1-\sigma)-1}(x \in \mathbf{R}_+), \Psi_\sigma(n) := (n-\tau)^{q(1-\sigma)-1} \quad (n \in \mathbf{N}),$$
$$\widetilde{\Phi}_\sigma(x) := (1-\vartheta_\sigma(x))x^{p(1-\sigma)-1} \quad (\vartheta_\sigma(x) \in (0,1);\ x \in \mathbf{R}_+),$$

we have

THEOREM 1. *Suppose that* $p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $\tau \in [0,\frac{1}{2}]$, $\sigma \in \mathbf{R}$, $h(t)$ *is a non-negative finite measurable function in* $\mathbf{R}_+$, *there exists a constant* $\delta_0 > 0$, *such that for any* $\widetilde{\sigma} \in (\sigma-\delta_0, \sigma+\delta_0)$,

$$\widehat{k}(\widetilde{\sigma}) = \int_0^\infty h(t)t^{\widetilde{\sigma}-1}dt \in \mathbf{R}_+,$$
$$\frac{d}{dt}(h(t)t^{\widetilde{\sigma}-1}) < 0, \quad \frac{d^2}{dt^2}(h(t)t^{\widetilde{\sigma}-1}) > 0,$$

*and there exists constants* $L > 0$ *and* $\eta_0 < \widetilde{\sigma}$, *satisfying*

$$h(t) \leqslant \frac{L}{t^{\eta_0}} \quad (t \in (0,\infty)). \tag{22}$$

*If* $p > 1$, $f(x) \geqslant 0$, $a_n \geqslant 0$,

$$0 < ||f||_{p,\Phi_\sigma} = \{\int_{\mathbf{R}_+}\Phi_\sigma(x)f^p(x)dx\}^{\frac{1}{p}} < \infty,$$
$$0 < ||a||_{q,\Psi_\sigma} = \{\sum_{n=1}^\infty \Psi_\sigma(n)a_n^q\}^{\frac{1}{q}} < \infty,$$

*then we have the following equivalent inequalities with the best possible constant factor* $\widehat{k}(\sigma)$*:*

$$\widehat{I} = \sum_{n=1}^\infty \int_{\mathbf{R}_+} h(x(n-\tau))a_n f(x)dx < \widehat{k}(\sigma)||f||_{p,\Phi_\sigma}||a||_{q,\Psi_\sigma}, \tag{23}$$

$$\widehat{J} := \left\{\sum_{n=1}^\infty (n-\tau)^{p\sigma-1}\left(\int_{\mathbf{R}_+} h(x(n-\tau))f(x)dx\right)^p\right\}^{\frac{1}{p}} < \widehat{k}(\sigma)||f||_{p,\Phi_\sigma}, \tag{24}$$

$$\widehat{H} := \left\{ \int_{\mathbf{R}_+} x^{q\sigma-1} \left( \sum_{n=1}^{\infty} h(x(n-\tau)) a_n \right)^q dx \right\}^{\frac{1}{q}} < \widehat{k}(\sigma) ||a||_{q,\Psi_\sigma}, \tag{25}$$

*where,* $\widehat{k}(\sigma) = \int_0^\infty h(t) t^{\sigma-1} dt$.

*In particular, for* $\tau = 0$, *setting* $\psi_\sigma(n) := n^{q(1-\sigma)-1}$, *we have the following equivalent inequalities with the best possible constant factor* $\widehat{k}(\sigma)$*:*

$$\sum_{n=1}^{\infty} \int_{\mathbf{R}_+} h(xn) a_n f(x) dx < \widehat{k}(\sigma) ||f||_{p,\Phi_\sigma} ||a||_{q,\psi_\sigma}, \tag{26}$$

$$\left\{ \sum_{n=1}^{\infty} n^{p\sigma-1} \left( \int_{\mathbf{R}_+} h(xn) f(x) dx \right)^p \right\}^{\frac{1}{p}} < \widehat{k}(\sigma) ||f||_{p,\Phi_\sigma}, \tag{27}$$

$$\left\{ \int_{\mathbf{R}_+} x^{q\sigma-1} \left( \sum_{n=1}^{\infty} h(xn) a_n \right)^q dx \right\}^{\frac{1}{q}} < \widehat{k}(\sigma) ||a||_{q,\psi_\sigma}. \tag{28}$$

*Proof.* By Lemma 3, Lemma 4 and Lemma 5, we have equivalent inequalities (23), (24) and (25). By Hölder's inequality, we still have

$$\widehat{I} \leqslant \widehat{J} \left\{ \sum_{n=1}^{\infty} (n-\tau)^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}}, \tag{29}$$

$$\widehat{I} \leqslant \left\{ \int_{\mathbf{R}_+} x^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \widehat{H}. \tag{30}$$

For $0 < \varepsilon < q\delta_0$, we set $\widetilde{f}(x)$, $\widetilde{a}_n$ as follows:

$$\widetilde{f}(x) := \begin{cases} x^{\sigma+\frac{\varepsilon}{p}-1}, & 0 < x \leqslant 1, \\ 0, & x > 1, \end{cases}$$

$$\widetilde{a}_n := (n-\tau)^{\sigma-\frac{\varepsilon}{q}-1}, \ n \in \mathbf{N}.$$

Then for $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{q}$, in view of (20), we find

$$\begin{aligned} ||\widetilde{f}||_{p,\Phi_\sigma} ||\widetilde{a}||_{q,\Psi_\sigma} &= \left\{ \int_0^\infty x^{-1+\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ (1-\tau)^{-1-\varepsilon} + \sum_{n=2}^{\infty} (n-\tau)^{-1-\varepsilon} \right\}^{\frac{1}{q}} \\ &< \left\{ \frac{1}{\varepsilon} \right\}^{\frac{1}{p}} \left\{ (1-\tau)^{-1-\varepsilon} + \int_1^\infty (y-\tau)^{-1-\varepsilon} dy \right\}^{\frac{1}{q}} \\ &= \left\{ \frac{1}{\varepsilon} \right\}^{\frac{1}{p}} \left\{ (1-\tau)^{-1-\varepsilon} + \frac{1}{\varepsilon} (1-\tau)^{-\varepsilon} \right\}^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} [\varepsilon (1-\tau)^{-1-\varepsilon} + (1-\tau)^{-\varepsilon}]^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned}\widetilde{I} &:= \int_{\mathbf{R}_+} \sum_{n=1}^{\infty} h(x(n-\tau))\widetilde{a}_n \widetilde{f}(x)dx = \int_0^1 x^{-1+\varepsilon} W(\widetilde{\sigma},x)dx \\ &\geqslant \widehat{k}(\widetilde{\sigma}) \int_0^1 x^{-1+\varepsilon}(1-O(x^{\rho(\widetilde{\sigma})}))dx = \frac{1}{\varepsilon}\widehat{k}(\widetilde{\sigma})\left[1-\varepsilon O_{\widetilde{\sigma}}(1)\right].\end{aligned}$$

If there exists a constant $K \leqslant \widehat{k}(\sigma)$, such that (23) is valid when replacing $\widehat{k}(\sigma)$ by $K$, then in particular, we have

$$\begin{aligned}\widehat{k}(\widetilde{\sigma})\left[1-\varepsilon O_{\widetilde{\sigma}}(1)\right] &\leqslant \varepsilon\widetilde{I} = \varepsilon K||\widetilde{f}||_{p,\Phi}||\widetilde{a}||_{q,\Psi} \\ &< K[\varepsilon(1-\tau)^{-1-\varepsilon}+(1-\tau)^{-\varepsilon}]^{\frac{1}{q}},\end{aligned}$$

and then by (8), $\widehat{k}(\sigma) \leqslant K(\varepsilon \to 0^+)$. Hence $K = \widehat{k}(\sigma)$ is the best possible constant factor of (26).

By the equivalency, the constant factor $\widehat{k}(\sigma)$ in (24) ((25)) is the best possible. Otherwise, we would reach a contradiction by (29) ((30)) that the constant factor $\widehat{K}(\sigma)$ in (23) is not the best possible. The theorem is proved. □

THEOREM 2. *As the assumptions of Theorem 1, if* $p<0$ $(0<q<1)$, $f(x) \geqslant 0$, $a_n \geqslant 0$, $0<||f||_{p,\Phi_\sigma}<\infty$, $0<||a||_{q,\Psi_\sigma}<\infty$, *then we have the following equivalent inequalities with the best possible constant factor* $\widehat{k}(\sigma)$*:*

$$\sum_{n=1}^{\infty} \int_{\mathbf{R}_+} h(x(n-\tau))a_n f(x)dx > \widehat{k}(\sigma)||f||_{p,\Phi_\sigma}||a||_{q,\Psi_\sigma}, \tag{31}$$

$$\left\{\sum_{n=1}^{\infty}(n-\tau)^{p\sigma-1}\left(\int_{\mathbf{R}_+} h(x(n-\tau))f(x)dx\right)^p\right\}^{\frac{1}{p}} > \widehat{k}(\sigma)||f||_{p,\Phi_\sigma}, \tag{32}$$

$$\left\{\int_{\mathbf{R}_+} x^{q\sigma-1}\left(\sum_{n=1}^{\infty} h(x(n-\tau))a_n\right)^q dx\right\}^{\frac{1}{q}} > \widehat{k}(\sigma)||a||_{q,\Psi_\sigma}. \tag{33}$$

*In particular, for* $\tau=0$, *setting* $\psi_\sigma(n)$ *as Theorem 1, we have the following equivalent inequalities with the best possible constant factor* $\widehat{k}(\sigma)$*:*

$$\sum_{n=1}^{\infty} \int_{\mathbf{R}_+} h(xn)a_n f(x)dx > \widehat{k}(\sigma)||f||_{p,\Phi_\sigma}||a||_{q,\psi_\sigma}, \tag{34}$$

$$\left\{\sum_{n=1}^{\infty} n^{p\sigma-1}\left(\int_{\mathbf{R}_+} h(xn)f(x)dx\right)^p\right\}^{\frac{1}{p}} > \widehat{k}(\sigma)||f||_{p,\Phi_\sigma}, \tag{35}$$

$$\left\{\int_{\mathbf{R}_+} x^{q\sigma-1}\left(\sum_{n=1}^{\infty} h(xn)a_n\right)^q dx\right\}^{\frac{1}{q}} > \widehat{k}(\sigma)||a||_{q,\psi_\sigma}. \tag{36}$$

*Proof.* We only prove that the constant factor $\widehat{K}(\sigma)$ in (31) is the best possible. The rests are omitted. For $0<\varepsilon<q\delta_0$, $\widetilde{\sigma}=\sigma-\frac{\varepsilon}{q}$, we set $\widetilde{f}(x)$, $\widetilde{a}_n$ as follows Theorem 1. If there exists a constant $K\geqslant \widehat{k}(\sigma)$, such that (31) is valid when replacing $\widehat{k}(\sigma)$ by $K$, then in particular, by (20), we have

$$\begin{aligned}
K\{(1-\tau)^{-\varepsilon}\}^{\frac{1}{q}} &= \varepsilon K\left\{\int_0^1 x^{-1+\varepsilon}dx\right\}^{\frac{1}{p}}\left\{\int_1^\infty (y-\tau)^{-1-\varepsilon}\right\}^{\frac{1}{q}}\\
&< \varepsilon K\left\{\int_0^1 x^{-1+\varepsilon}dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty (n-\tau)^{-1-\varepsilon}\right\}^{\frac{1}{q}}\\
&= \varepsilon K||\widetilde{f}||_{p,\Phi_\sigma}||\widetilde{a}||_{q,\Psi_\sigma} < \varepsilon\widetilde{I} = \varepsilon\int_0^1 x^{-1+\varepsilon}W(\widetilde{\sigma},x)dx\\
&< \varepsilon\widehat{k}(\widetilde{\sigma})\int_0^1 x^{-1+\varepsilon}dx = \widehat{k}(\widetilde{\sigma}),
\end{aligned}$$

and then by (8), $K\leqslant \widehat{k}(\sigma)$ $(\varepsilon\to 0^+)$. Hence $K=\widehat{k}(\sigma)$ is the best possible constant factor of (31). The theorem is proved. □

THEOREM 3. *As the assumptions of Theorem 1, if* $0<p<1$ $(q<0)$, $f(x)\geqslant 0$, $a_n\geqslant 0$, $0<||f||_{p,\widetilde{\Phi}_\sigma}<\infty$, $0<||a||_{q,\Psi_\sigma}<\infty$, *then we have the following equivalent inequalities with the best possible constant factor* $\widehat{k}(\sigma)$*:*

$$\sum_{n=1}^\infty \int_{\mathbf{R}_+} h(x(n-\tau))a_n f(x)dx > \widehat{k}(\sigma)||f||_{p,\widetilde{\Phi}_\sigma}||a||_{q,\Psi_\sigma}, \tag{37}$$

$$\left\{\sum_{n=1}^\infty (n-\tau)^{p\sigma-1}\left(\int_{\mathbf{R}_+} h(x(n-\tau))f(x)dx\right)^p\right\}^{\frac{1}{p}} > \widehat{k}(\sigma)||f||_{p,\widetilde{\Phi}_\sigma}, \tag{38}$$

$$\left\{\int_{\mathbf{R}_+} \frac{x^{q\sigma-1}}{(1-\vartheta_\sigma(x))^{q-1}}\left(\sum_{n=1}^\infty h(x(n-\tau))a_n\right)^q dx\right\}^{\frac{1}{q}} > \widehat{k}(\sigma)||a||_{q,\Psi_\sigma}. \tag{39}$$

*In particular, for* $\tau=0$, *setting* $\psi_\sigma(n)$ *as Theorem 1, we have the following equivalent inequalities with the best possible constant factor* $\widehat{k}(\sigma)$*:*

$$\sum_{n=1}^\infty \int_{\mathbf{R}_+} h(xn)a_n f(x)dx > \widehat{k}(\sigma)||f||_{p,\widetilde{\Phi}_\sigma}||a||_{q,\psi_\sigma}, \tag{40}$$

$$\left\{\sum_{n=1}^\infty n^{p\sigma-1}\left(\int_{\mathbf{R}_+} h(xn)f(x)dx\right)^p\right\}^{\frac{1}{p}} > \widehat{k}(\sigma)||f||_{p,\widetilde{\Phi}_\sigma}, \tag{41}$$

$$\left\{\int_{\mathbf{R}_+} \frac{x^{q\sigma-1}}{(1-\vartheta_\sigma(x))^{q-1}}\left(\sum_{n=1}^\infty h(xn)a_n\right)^q dx\right\}^{\frac{1}{q}} > \widehat{k}(\sigma)||a||_{q,\psi_\sigma}. \tag{42}$$

*Proof.* We only prove that the constant factor $\widehat{k}(\sigma)$ in (37) is the best possible. The rests are omitted. For $0<\varepsilon<|q|\delta_0$, $\widetilde{\sigma}=\sigma-\frac{\varepsilon}{q}$, we set $\widetilde{f}(x)$, $\widetilde{a}_n$ as Theorem 1. Then we obtain

$$
\begin{aligned}
||\widetilde{f}||_{p,\widetilde{\Phi}_\sigma}||\widetilde{a}||_{q,\Psi_\sigma} &= \left\{\int_0^1 x^{-1+\varepsilon}(1-O(x^{\rho(\sigma)}))dx\right\}^{\frac{1}{p}}\left\{(1-\tau)^{-1-\varepsilon}+\sum_{n=2}^{\infty}(n-\tau)^{-1-\varepsilon}\right\}^{\frac{1}{q}} \\
&> \left\{\frac{1}{\varepsilon}-\widetilde{O}(1)\right\}^{\frac{1}{p}}\left\{(1-\tau)^{-1-\varepsilon}+\int_1^{\infty}(y-\tau)^{-1-\varepsilon}dy\right\}^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left\{1-\varepsilon\widetilde{O}(1)\right\}^{\frac{1}{p}}[\varepsilon(1-\tau)^{-1-\varepsilon}+(1-\tau)^{-\varepsilon}]^{\frac{1}{q}}.
\end{aligned}
$$

If there exists a constant $K\geqslant\widehat{k}(\sigma)$, such that (37) is valid when replacing $\widehat{k}(\sigma)$ by $K$, then in particular, by (20), we have

$$
\begin{aligned}
&K\left\{1-\varepsilon\widetilde{O}(1)\right\}^{\frac{1}{p}}[\varepsilon(1-\tau)^{-1-\varepsilon}+(1-\tau)^{-\varepsilon}]^{\frac{1}{q}} \\
&<\varepsilon K||\widetilde{f}||_{p,\widetilde{\Phi}_\sigma}||\widetilde{a}||_{q,\Psi_\sigma}<\varepsilon\widetilde{I}=\varepsilon\int_0^1 x^{-1+\varepsilon}W(\widetilde{\sigma},x)dx \\
&<\varepsilon\widehat{k}(\widetilde{\sigma})\int_0^1 x^{-1+\varepsilon}dx=\widehat{k}(\widetilde{\sigma}),
\end{aligned}
$$

and then by (8), $K\leqslant\widehat{k}(\sigma)$ $(\varepsilon\to 0^+)$. Hence $K=\widehat{k}(\sigma)$ is the best possible constant factor of (37). The theorem is proved. □

NOTE. For $\tau=0$ in Theorem 1-3, we don't need the assumption that $\frac{d^2}{dt^2}(h(t)t^{\widetilde{\sigma}-1})$ $>0$.

## 4. Operator expressions and some examples

For $p>1$, we still set

$$\Phi_\sigma(x)=x^{p(1-\sigma)-1}\ (x\in\mathbf{R}_+),\quad \Psi(n)=(n-\tau)^{q(1-\sigma)-1}\ (n\in\mathbf{N}),$$

wherefrom

$$[\Psi_\sigma(n)]^{1-p}=(n-\tau)^{p\sigma-1},\qquad [\Phi_\sigma(x)]^{1-q}=x^{q\sigma-1}.$$

We define two real weight normal spaces $\mathbf{L}_{p,\Phi_\sigma}(\mathbf{R}_+)$ and $\mathbf{l}_{q,\Psi_\sigma}$ as follows:

$$
\begin{aligned}
\mathbf{L}_{p,\Phi_\sigma}(\mathbf{R}_+) &:= \left\{f;||f||_{p,\Phi_\sigma}=\{\int_{\mathbf{R}_+}\Phi_\sigma(x)|f(x)|^p dx\}^{\frac{1}{p}}<\infty\right\}, \\
\mathbf{l}_{q,\Psi_\sigma} &:= \left\{a=\{a_n\};||a||_{q,\Psi_\sigma}=\{\sum_{n=1}^{\infty}\Psi_\sigma(n)|a_n|^q\}^{\frac{1}{q}}<\infty\right\}.
\end{aligned}
$$

As the assumptions of Theorem 1, in view of

$$\widehat{J}<\widehat{k}(\sigma)||f||_{p,\Phi_\sigma},\quad \widehat{H}<\widehat{k}(\sigma)||a||_{q,\Psi_\sigma},$$

we give the following definition:

DEFINITION 2. Define a first kind of half-discrete Hilbert-type operator

$$\widehat{T}_1 : \mathbf{L}_{p,\Phi_\sigma}(\mathbf{R}_+) \to \mathbf{l}_{p,\Psi_\sigma^{1-p}} \tag{43}$$

as follows: For $f \in \mathbf{L}_{p,\Phi_\sigma}(\mathbf{R}_+)$, there exists an unique representation $\widehat{T}_1 f \in \mathbf{l}_{p,\Psi_\sigma^{1-p}}$, satisfying

$$(\widehat{T}_1 f)(n) := \int_{\mathbf{R}_+} h(x(n-\tau))f(x)dx (n \in \mathbf{N}). \tag{44}$$

For $a \in \mathbf{l}_{q,\Psi_\sigma}$, we define the following formal inner product of $\widehat{T}_1 f$ and $a$ as follows:

$$(\widehat{T}_1 f, a) := \sum_{n=1}^{\infty} a_n \int_{\mathbf{R}_+} h(x(n-\tau))f(x)dx. \tag{45}$$

Also we define a second kind of half-discrete Hilbert-type operator

$$\widehat{T}_2 : \mathbf{l}_{q,\Psi_\sigma} \to \mathbf{L}_{q,\Phi_\sigma^{1-q}}(\mathbf{R}_+)$$

as follows: For $a \in \mathbf{l}_{q,\Psi_\sigma}$, there exists an unique representation $\widehat{T}_2 a \in \mathbf{L}_{q,\Phi_\sigma^{1-q}}(\mathbf{R}_+)$, satisfying

$$(\widehat{T}_2 a)(x) := \sum_{n=1}^{\infty} h(x(n-\tau))a_n (x \in \mathbf{R}_+). \tag{46}$$

For $f \in \mathbf{L}_{p,\Phi_\sigma}(\mathbf{R}_+)$, we define the following formal inner product of $f$ and $\widehat{T}_2 a$ as follows:

$$(f, \widehat{T}_2 a) := \int_{\mathbf{R}_+} \sum_{n=1}^{\infty} h(x(n-\tau))a_n f(x)dx. \tag{47}$$

Then by Theorem 1, for $0 < ||f||_{p,\Phi_\sigma}$, $||a||_{q,\Psi_\sigma} < \infty$, we have the following equivalent inequalities:

$$(\widehat{T}_1 f, a) = (f, \widehat{T}_2 a) < \widehat{k}(\sigma)||f||_{p,\Phi_\sigma}||a||_{q,\Psi_\sigma}, \tag{48}$$

$$||\widehat{T}_1 f||_{p,\Psi_\sigma^{1-p}} < \widehat{k}(\sigma)||f||_{p,\Phi_\sigma}, \tag{49}$$

$$||\widehat{T}_2 a||_{q,\Phi_\sigma^{1-q}} < \widehat{k}(\sigma)||a||_{q,\Psi_\sigma}. \tag{50}$$

It follows that $\widehat{T}_1$ and $\widehat{T}_2$ are bounded with

$$||\widehat{T}_1|| := \sup_{f(\neq\theta)\in\mathbf{L}_{p,\Phi_\sigma}(\mathbf{R}_+)} \frac{||\widehat{T}_1 f||_{p,\Psi_\sigma^{1-p}}}{||f||_{p,\Phi_\sigma}} \leqslant \widehat{k}(\sigma),$$

$$||\widehat{T}_2|| := \sup_{a(\neq\theta)\in\mathbf{l}_{q,\Psi_\sigma}} \frac{||\widehat{T}_2 a||_{q,\Phi_\sigma^{1-q}}}{||a||_{q,\Psi_\sigma}} \leqslant \widehat{k}(\sigma).$$

Since the constant factor $\widehat{K}(\sigma)$ in (49) and (50) is the best possible, we have

$$||\widehat{T}_1|| = ||\widehat{T}_2|| = \widehat{k}(\sigma) = \int_0^{\infty} h(t)t^{\sigma-1}dt \in \mathbf{R}_+. \tag{51}$$

EXAMPLE 1. (i) We set $h(t)=\frac{1}{t^{\lambda}+1}$ $(0<\sigma<\lambda\leqslant 1)$. For $\delta_0=\frac{1}{2}\min\{\sigma,\lambda-\sigma\}>0$, and $\widetilde{\sigma}\in(\sigma-\delta_0,\sigma+\delta_0)$, it follows

$$\begin{aligned}\widehat{k}(\widetilde{\sigma}) &= \int_0^\infty \frac{1}{t^\lambda+1}t^{\widetilde{\sigma}-1}dt\\ &\overset{v=t^\lambda}{=} \frac{1}{\lambda}\int_0^\infty \frac{1}{v+1}v^{\frac{\widetilde{\sigma}}{\lambda}-1}dv\\ &= \frac{\pi}{\lambda\sin\pi(\widetilde{\sigma}/\lambda)}\in\mathbf{R}_+,\end{aligned}$$

and by the Note of Lemma 1, for $0<\widetilde{\sigma}<\sigma+\delta_0<\lambda\leqslant 1$,

$$\frac{d}{dt}\Big(\frac{1}{t^\lambda+1}t^{\widetilde{\sigma}-1}\Big)<0,\qquad \frac{d^2}{dt^2}\Big(\frac{1}{t^\lambda+1}t^{\widetilde{\sigma}-1}\Big)>0.$$

Setting $\eta_0\in(0,\sigma-\delta_0)$, then we find $\eta_0<\widetilde{\sigma}$ $(<\lambda)$. Since

$$\frac{t^{\eta_0}}{t^\lambda+1}\to 0\ \ (t\to 0^+),\qquad \frac{t^{\eta_0}}{t^\lambda+1}\to 0\ \ (t\to\infty),$$

there exists a constant $L>0$, such that $h(t)=\frac{1}{t^\lambda+1}\leqslant\frac{L}{t^{\eta_0}}$ $(t\in(0,\infty))$.

Then by Theorem 1 and (51), we have

$$||\widehat{T}_1||=||\widehat{T}_2||=\frac{\pi}{\lambda\sin(\frac{\pi\sigma}{\lambda})}. \tag{52}$$

(ii) We set $h(t)=\frac{1}{(t+1)^\lambda}$ $(0<\sigma<\min\{\lambda,1\})$. For $\delta_0=\frac{1}{2}\min\{\sigma,\lambda-\sigma,1-\sigma\}>0$, and $\widetilde{\sigma}\in(\sigma-\delta_0,\sigma+\delta_0)$, it follows

$$\widehat{k}(\widetilde{\sigma})=\int_0^\infty\frac{1}{(t+1)^\lambda}t^{\widetilde{\sigma}-1}dt=B(\widetilde{\sigma},\lambda-\widetilde{\sigma})\in\mathbf{R}_+,$$

and by the Note of Lemma 1, for $0<\widetilde{\sigma}<\sigma+\delta_0<1$,

$$\frac{d}{dt}\Big(\frac{1}{(t+1)^\lambda}t^{\widetilde{\sigma}-1}\Big)<0,\qquad \frac{d^2}{dt^2}\Big(\frac{1}{(t+1)^\lambda}t^{\widetilde{\sigma}-1}\Big)>0.$$

Setting $\eta_0\in(0,\sigma-\delta_0)$, then we find $\eta_0<\widetilde{\sigma}$ $(<\lambda)$. Since

$$\frac{t^{\eta_0}}{(t+1)^\lambda}\to 0\ \ (t\to 0^+),\qquad \frac{t^{\eta_0}}{(t+1)^\lambda}\to 0\ \ (t\to\infty),$$

there exists a constant $L>0$, such that

$$k_\lambda(t,1)=\frac{1}{(t+1)^\lambda}\leqslant\frac{L}{t^{\eta_0}}\ \ (t\in(0,\infty)).$$

Then by Theorem 1 and (51), we have

$$||\widehat{T}_1||=||\widehat{T}_2||=B(\sigma,\lambda-\sigma). \tag{53}$$

(iii) We set $h(t)=\frac{\ln t}{t^{\lambda}-1}$ $(0<\sigma<\lambda\leqslant 1)$. For $\delta_0=\frac{1}{2}\min\{\sigma,\lambda-\sigma\}$, and $\widetilde{\sigma}\in(\sigma-\delta_0,\sigma+\delta_0)$, it follows

$$\begin{aligned}\widehat{k}(\widetilde{\sigma}) &= \int_0^{\infty}\frac{\ln t}{t^{\lambda}-1}t^{\widetilde{\sigma}-1}dt\\ &\overset{v=t^{\lambda}}{=} \frac{1}{\lambda^2}\int_0^{\infty}\frac{\ln v}{v-1}v^{\frac{\widetilde{\sigma}}{\lambda}-1}dv\\ &= \left[\frac{\pi}{\lambda\sin\pi(\widetilde{\sigma}/\lambda)}\right]^2\in\mathbf{R}_+,\end{aligned}$$

and by the Note of Lemma 1, for $0<\widetilde{\sigma}<\sigma+\delta_0<\lambda\leqslant 1$,

$$\frac{d}{dt}\left(\frac{\ln t}{t^{\lambda}-1}t^{\widetilde{\sigma}-1}\right)<0,\qquad \frac{d^2}{dt^2}\left(\frac{\ln t}{t^{\lambda}-1}t^{\widetilde{\sigma}-1}\right)>0.$$

Setting $\eta_0\in(0,\sigma-\delta_0)$, then we find $\eta_0<\widetilde{\sigma}(<\lambda)$. Since

$$\frac{(\ln t)t^{\eta_0}}{t^{\lambda}-1}\to 0\ \ (t\to 0^+),\qquad \frac{(\ln t)t^{\eta_0}}{t^{\lambda}-1}\to 0\ \ (t\to\infty),$$

there exists a constant $L>0$, such that $h(t)=\frac{\ln t}{t^{\lambda}-1}\leqslant\frac{L}{t^{\eta_0}}$ $(t\in(0,\infty))$.

Then by Theorem 1 and (51), we have

$$||\widehat{T}_1||=||\widehat{T}_2||=\left[\frac{\pi}{\lambda\sin\pi(\frac{\sigma}{\lambda})}\right]^2. \tag{54}$$

LEMMA 6. *If* $\mathbf{C}$ *is the set of complex numbers and* $\mathbf{C}_\infty=\mathbf{C}\cup\{\infty\}$, $z_k\in\mathbf{C}\backslash\{z|\mathrm{Re}z\geqslant 0,\ \mathrm{Im}z=0\}$ $(k=1,2,\cdots,n)$ *are different points, the function* $f(z)$ *is analytic in* $\mathbf{C}_\infty$ *except for* $z_i$ $(i=1,2,\cdots,n)$, *and* $z=\infty$ *is a zero point of* $f(z)$ *whose order is not less than 1, then for* $\alpha\in\mathbf{R}$, *we have*

$$\int_0^{\infty}f(x)x^{\alpha-1}dx=\frac{2\pi i}{1-e^{2\pi\alpha i}}\sum_{k=1}^{n}\mathrm{Res}[f(z)z^{\alpha-1},z_k], \tag{55}$$

*where,* $0<\mathrm{Im}\ln z=\arg z<2\pi$. *In particular, if* $z_k$ $(k=1,\cdots,n)$ *are all poles of order 1, setting* $\varphi_k(z)=(z-z_k)f(z)$ $(\varphi_k(z_k)\neq 0)$, *then*

$$\int_0^{\infty}f(x)x^{\alpha-1}dx=\frac{\pi}{\sin\pi\alpha}\sum_{k=1}^{n}(-z_k)^{\alpha-1}\varphi_k(z_k). \tag{56}$$

*Proof.* By [34] (p. 118), we have (55). We find

$$\begin{aligned}1-e^{2\pi\alpha i}&=1-\cos 2\pi\alpha-i\sin 2\pi\alpha\\ &=-2i\sin\pi\alpha(\cos\pi\alpha+i\sin\pi\alpha)=-2ie^{i\pi\alpha}\sin\pi\alpha.\end{aligned}$$

In particular, since $f(z)z^{\alpha-1}=\frac{1}{z-z_k}(\varphi_k(z)z^{\alpha-1})$, it is obvious that

$$\mathrm{Res}[f(z)z^{\alpha-1},-a_k]={z_k}^{\alpha-1}\varphi_k(z_k)=-e^{i\pi\alpha}(-z_k)^{\alpha-1}\varphi_k(z_k).$$

Then by (55), we obtain (56). The lemma is proved. □

EXAMPLE 2. (i) For $s \in \mathbf{N}$, we set

$$h(t) = \frac{1}{\prod_{k=1}^{s}(t^{\lambda/s}+a_k)} \quad (0<a_1<\cdots<a_s,\ 0<\lambda \leqslant s,\ 0<\sigma<\min\{\lambda,1\}).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \lambda-\sigma, 1-\sigma\} > 0$, and $\widetilde{\sigma} \in (\sigma-\delta_0, \sigma+\delta_0)$, by (56), it follows

$$\begin{aligned}\widehat{k}(\widetilde{\sigma}) &= \int_0^\infty \frac{1}{\prod_{k=1}^{s}(t^{\lambda/s}+a_k)} t^{\widetilde{\sigma}-1}dt \\ &= \frac{s}{\lambda}\int_0^\infty \frac{1}{\prod_{k=1}^{s}(u+a_k)} u^{\frac{s\widetilde{\sigma}}{\lambda}-1}du \\ &= \frac{\pi s}{\lambda \sin(\frac{\pi s \widetilde{\sigma}}{\lambda})}\sum_{k=1}^{s} a_k^{\frac{s\widetilde{\sigma}}{\lambda}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{a_j-a_k} \in \mathbf{R}_+,\end{aligned}$$

and by the Note of Lemma 1, for $0<\lambda \leqslant s$, $0<\widetilde{\sigma}<\sigma+\delta_0<1$,

$$\frac{d}{dt}\left(\frac{t^{\widetilde{\sigma}-1}}{\prod_{k=1}^{s}(t^{\lambda/s}+a_k)}\right)<0, \qquad \frac{d^2}{dt^2}\left(\frac{t^{\widetilde{\sigma}-1}}{\prod_{k=1}^{s}(t^{\lambda/s}+a_k)}\right)>0.$$

Setting $\eta_0 \in (0, \sigma-\delta_0)$, then we find $\eta_0 < \widetilde{\sigma}(<\lambda)$. Since

$$\frac{t^{\eta_0}}{\prod_{k=1}^{s}(t^{\lambda/s}+a_k)} \to 0 \ \ (t \to 0^+),$$
$$\frac{t^{\eta_0}}{\prod_{k=1}^{s}(t^{\lambda/s}+a_k)} \to 0 \ \ (t \to \infty),$$

there exists a constant $L>0$, such that

$$h(t) = \frac{1}{\prod_{k=1}^{s}(t^{\lambda/s}+a_k)} \leqslant \frac{L}{t^{\eta_0}} \quad (t \in (0,\infty)).$$

Then by Theorem 1 and (51), we have

$$||\widehat{T}_1|| = ||\widehat{T}_2|| = \frac{\pi s}{\lambda \sin(\frac{\pi s \sigma}{\lambda})}\sum_{k=1}^{s} a_k^{\frac{s\sigma}{\lambda}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{a_j-a_k}. \tag{57}$$

(ii) We set

$$h(t) = \frac{1}{t^\lambda + \sqrt{c}t^{\lambda/2}\cos\gamma + \frac{c}{4}} \quad (c>0,\ 0<\gamma<\frac{\pi}{2},\ 0<\sigma<\lambda \leqslant 1).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \lambda-\sigma\} > 0$, and $\widetilde{\sigma} \in (\sigma-\delta_0, \sigma+\delta_0)$, setting $z_1 = -\frac{\sqrt{c}}{2}e^{i\gamma}$, $z_2 = -\frac{\sqrt{c}}{2}e^{-i\gamma}$, by (56), it follows

$$\begin{aligned}k(\widetilde{\sigma}) &= \int_0^\infty \frac{t^{\widetilde{\sigma}-1}}{t^\lambda + \sqrt{c}t^{\lambda/2}\cos\gamma + \frac{c}{4}}dt = \frac{2}{\lambda}\int_0^\infty \frac{u^{\frac{2\widetilde{\sigma}}{\lambda}-1}}{u^2+\sqrt{c}u\cos\gamma+\frac{c}{4}}du \\ &= \frac{2}{\lambda}\int_0^\infty \frac{u^{\frac{2\widetilde{\sigma}}{\lambda}-1}}{(u-z_1)(u-z_2)}du\end{aligned}$$

$$= \frac{2\pi}{\lambda\sin(\frac{2\pi\widetilde{\sigma}}{\lambda})}\left[\left(\frac{\sqrt{c}}{2}e^{i\gamma}\right)^{\frac{2\widetilde{\sigma}}{\lambda}-1}\frac{\sqrt{c}}{2(e^{-i\gamma}-e^{i\gamma})}+\left(\frac{\sqrt{c}}{2}e^{-i\gamma}\right)^{\frac{2\widetilde{\sigma}}{\lambda}-1}\frac{\sqrt{c}}{2(e^{i\gamma}-e^{-i\gamma})}\right]$$
$$= \left(\frac{\sqrt{c}}{2}\right)^{\frac{2\widetilde{\sigma}}{\lambda}}\frac{2\pi\sin\gamma(1-\frac{2\widetilde{\sigma}}{\lambda})}{\lambda\sin\gamma\sin(\frac{2\pi\widetilde{\sigma}}{\lambda})}\in\mathbf{R}_+,$$

and by the Note of Lemma 1, for $0<\widetilde{\sigma}<\sigma+\delta_0<\lambda\leqslant 1$,

$$\frac{d}{dt}\left(\frac{t^{\widetilde{\sigma}-1}}{t^{\lambda}+\sqrt{c}t^{\lambda/2}\cos\gamma+\frac{c}{4}}\right)<0,$$
$$\frac{d^2}{dt^2}\left(\frac{t^{\widetilde{\sigma}-1}}{t^{\lambda}+\sqrt{c}t^{\lambda/2}\cos\gamma+\frac{c}{4}}\right)>0.$$

Setting $\eta_0\in(0,\sigma-\delta_0)$, then we find $\eta_0<\widetilde{\sigma}\ (<\lambda)$. Since

$$\frac{t^{\eta_0}}{t^{\lambda}+\sqrt{c}t^{\lambda/2}\cos\gamma+\frac{c}{4}}\to 0\ \ (t\to 0^+),$$
$$\frac{t^{\eta_0}}{t^{\lambda}+\sqrt{c}t^{\lambda/2}\cos\gamma+\frac{c}{4}}\to 0\ \ (t\to\infty),$$

there exists a constant $L>0$, such that

$$h(t)=\frac{1}{t^{\lambda}+\sqrt{c}t^{\lambda/2}\cos\gamma+\frac{c}{4}}\leqslant\frac{L}{t^{\eta_0}}\ \ (t\in(0,\infty)).$$

Then by Theorem 1 and (51), we have

$$||\widehat{T}_1||=||\widehat{T}_2||=\left(\frac{\sqrt{c}}{2}\right)^{\frac{2\sigma}{\lambda}}\frac{2\pi\sin\gamma(1-\frac{2\sigma}{\lambda})}{\lambda\sin\gamma\sin(\frac{2\pi\sigma}{\lambda})}. \tag{58}$$

EXAMPLE 3. (i) We set

$$h(t)=\ln\left(\frac{b+t^{\gamma}}{a+t^{\gamma}}\right)\quad(0\leqslant a<b,\ 0<\sigma<\gamma\leqslant 1).$$

For $\delta_0=\frac{1}{2}\min\{\sigma,\gamma-\sigma\}>0$, and $\widetilde{\sigma}\in(\sigma-\delta_0,\sigma+\delta_0)$, it follows

$$\widehat{k}(\widetilde{\sigma})=\int_0^{\infty}\ln\left(\frac{b+t^{\gamma}}{a+t^{\gamma}}\right)t^{\widetilde{\sigma}-1}dt$$
$$=\int_0^{\infty}\ln\left(\frac{by^{\gamma}+1}{ay^{\gamma}+1}\right)y^{-\widetilde{\sigma}-1}dy$$
$$=\frac{1}{\widetilde{\sigma}}\left(b^{\frac{\widetilde{\sigma}}{\gamma}}-a^{\frac{\widetilde{\sigma}}{\gamma}}\right)\frac{\pi}{\sin\pi(\frac{\widetilde{\sigma}}{\gamma})}\in\mathbf{R}_+,$$

and by the Note of Lemma 1, for $0<\widetilde{\sigma}<\sigma+\delta_0<\gamma\leqslant 1$,

$$\frac{d}{dt}\left(\ln\left(\frac{b+t^{\gamma}}{a+t^{\gamma}}\right)t^{\widetilde{\sigma}-1}\right)<0,\qquad\frac{d^2}{dt^2}\left(\ln\left(\frac{b+t^{\gamma}}{a+t^{\gamma}}\right)t^{\widetilde{\sigma}-1}\right)>0.$$

Setting $\eta_0 \in (0, \sigma - \delta_0)$, then we find $\eta_0 < \widetilde{\sigma} < \gamma$. Since

$$t^{\eta_0} \ln\left(\frac{b+t^\gamma}{a+t^\gamma}\right) \to 0 \ \ (t \to 0^+), \qquad t^{\eta_0} \ln\left(\frac{b+t^\gamma}{a+t^\gamma}\right) \to 0 \ \ (t \to \infty),$$

there exists a constant $L > 0$, such that

$$h(t) = \ln\left(\frac{b+t^\gamma}{a+t^\gamma}\right) \leqslant \frac{L}{t^{\eta_0}} \ \ (t \in (0,\infty)).$$

Then by Theorem 1 and (51), we have

$$||\widehat{T}_1|| = ||\widehat{T}_2|| = \frac{\left(b^{\frac{\sigma}{\gamma}} - a^{\frac{\sigma}{\gamma}}\right)\pi}{\sigma \sin \pi(\frac{\sigma}{\gamma})}. \tag{59}$$

(ii) We set $h(t) = e^{-\rho t^\gamma}$ $(\rho > 0,\ 0 < \sigma < \gamma \leqslant 1)$. For $\delta_0 = \frac{1}{2}\min\{\sigma, \gamma - \sigma\} > 0$, and $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, it follows

$$\begin{aligned}\widehat{k}(\widetilde{\sigma}) &= \int_0^\infty e^{-\rho t^\gamma} t^{\widetilde{\sigma}-1} dt \overset{u=\rho t^\gamma}{=} \frac{1}{\gamma}\rho^{-\frac{\widetilde{\sigma}}{\gamma}} \int_0^\infty e^{-u} u^{\frac{\widetilde{\sigma}}{\gamma}-1} du \\ &= \frac{1}{\gamma \rho^{\frac{\widetilde{\sigma}}{\gamma}}} \Gamma\left(\frac{\widetilde{\sigma}}{\gamma}\right) \in \mathbf{R}_+,\end{aligned}$$

and by the Note of Lemma 1, for $\rho > 0,\ 0 < \widetilde{\sigma} < \sigma + \delta_0 < \gamma \leqslant 1$,

$$\frac{d}{dt}(e^{-\rho t^\gamma} t^{\widetilde{\sigma}-1}) < 0, \qquad \frac{d^2}{dt^2}(e^{-\rho t^\gamma} t^{\widetilde{\sigma}-1}) > 0.$$

Setting $\eta_0 \in (0, \sigma - \delta_0)$, then we find $\eta_0 < \widetilde{\sigma}$. Since

$$t^{\eta_0} e^{-\rho t^\gamma} \to 0 \ \ (t \to 0^+), \qquad t^{\eta_0} e^{-\rho t^\gamma} \to 0 \ \ (t \to \infty),$$

there exists a constant $L > 0$, such that $h(t) = e^{-\rho t^\gamma} \leqslant \frac{L}{t^{\eta_0}}$ $(t \in (0,\infty))$.

Then by Theorem 1 and (51), we have

$$||\widehat{T}_1|| = ||\widehat{T}_2|| = \frac{1}{\gamma \rho^{\frac{\sigma}{\gamma}}} \Gamma\left(\frac{\sigma}{\gamma}\right). \tag{60}$$

(iiiv) We set $h(t) = \arctan \rho t^{-\gamma}$ $(\rho > 0,\ 0 < \sigma < \min\{1, \gamma\})$. For $\delta_0 = \frac{1}{2}\min\{\sigma, 1 - \gamma, 1 - \sigma\} > 0$, and $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, it follows

$$\begin{aligned}\widehat{k}(\widetilde{\sigma}) &= \int_0^\infty t^{\widetilde{\sigma}-1} (\arctan \rho t^{-\gamma}) dt = \frac{1}{\widetilde{\sigma}} \int_0^\infty (\arctan \rho t^{-\gamma}) dt^{\widetilde{\sigma}} \\ &= \frac{1}{\widetilde{\sigma}} \left[ (\arctan \rho t^{-\gamma}) t^{\widetilde{\sigma}} |_0^\infty + \int_0^\infty \frac{\gamma \rho t^{\widetilde{\sigma}-\gamma-1}}{1 + (\rho t^{-\gamma})^2} dt \right]\end{aligned}$$

$$= \frac{\rho^{\frac{\widetilde{\sigma}}{\gamma}}}{2\widetilde{\sigma}} \int_0^\infty \frac{1}{1+u} u^{(-\frac{\widetilde{\sigma}}{2\gamma}+\frac{1}{2})-1} du$$
$$= \frac{\rho^{\frac{\widetilde{\sigma}}{\gamma}}\pi}{2\widetilde{\sigma}\sin\pi(-\frac{\widetilde{\sigma}}{2\gamma}+\frac{1}{2})} = \frac{\rho^{\frac{\widetilde{\sigma}}{\gamma}}\pi}{2\widetilde{\sigma}\cos\pi(\frac{\widetilde{\sigma}}{2\gamma})} \in \mathbf{R}_+,$$

and by the Note of Lemma 1, for $0 < \widetilde{\sigma} < \sigma + \delta_0 < 1$,

$$\frac{d}{dt}(t^{\widetilde{\sigma}-1}\arctan \rho t^{-\gamma}) < 0, \qquad \frac{d^2}{dt^2}(t^{\widetilde{\sigma}-1}\arctan \rho t^{-\gamma}) > 0.$$

Setting $\eta_0 \in (0, \sigma - \delta_0)$, then we find $\eta_0 < \widetilde{\sigma}$. Since

$$t^{\eta_0}\arctan \rho t^{-\gamma} \to 0 \ \ (t \to 0^+), \qquad t^{\eta_0}\arctan \rho t^{-\gamma} \to 0 \ \ (t \to \infty),$$

there exists a constant $L > 0$, such that

$$h(t) = \arctan \rho t^{-\gamma} \leqslant \frac{L}{t^{\eta_0}} \ \ (t \in (0,\infty)).$$

Then by Theorem 1 and (51), we have

$$||\widehat{T}_1|| = ||\widehat{T}_2|| = \frac{\rho^{\frac{\sigma}{\gamma}}\pi}{2\sigma\cos\pi(\frac{\sigma}{2\gamma})}. \tag{61}$$

EXAMPLE 4. We set

$$h(t) = \frac{(\min\{t,1\})^\gamma}{(\max\{t,1\})^{\lambda+\gamma}} \ \ (-\gamma < \sigma < \lambda + \gamma < 1 - \gamma).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma+\gamma, \lambda+\gamma-\sigma, 1-\sigma-\gamma\} > 0$, and $\widetilde{\sigma} \in (\sigma-\delta_0, \sigma+\delta_0)$, it follows

$$\widehat{k}(\widetilde{\sigma}) = \int_0^\infty \frac{(\min\{t,1\})^\gamma}{(\max\{t,1\})^{\lambda+\gamma}} t^{\widetilde{\sigma}-1} dt$$
$$= \frac{\lambda+2\gamma}{(\widetilde{\sigma}+\gamma)(\lambda-\widetilde{\sigma}+\gamma)} \in \mathbf{R}_+,$$

and

$$h(t)t^{\widetilde{\sigma}-1} = \frac{(\min\{t,1\})^\gamma t^{\widetilde{\sigma}-1}}{(\max\{t,1\})^{\lambda+\gamma}} = \begin{cases} t^{\gamma+\widetilde{\sigma}-1}, & 0 < t < 1, \\ \frac{1}{t^{\lambda+\gamma-\widetilde{\sigma}+1}}, & t \geqslant 1, \end{cases}$$

is strict decreasing with respect to $t \in \mathbf{R}_+$.

There exists a constant $\eta_0 \in (-\gamma, \min\{\sigma-\delta_0, \lambda+\gamma\})$, such that $\eta_0 < \widetilde{\sigma}$, $\gamma+\eta_0 > 0$ and $\lambda+\gamma-\eta_0 > 0$. In view of

$$t^{\eta_0}h(t) = \frac{t^{\eta_0}(\min\{t,1\})^\gamma}{(\max\{t,1\})^{\lambda+\gamma}} = \begin{cases} t^{\gamma+\eta_0}, & 0 < t < 1, \\ \frac{1}{t^{\lambda+\gamma-\eta_0}}, & t \geqslant 1, \end{cases}$$

we have $t^{\eta_0}h(t) \to 0 \ (t \to 0^+)$, and $t^{\eta_0}h(t) \to 0 \ (t \to \infty)$. Hence, there exists a constant $L > 0$, satisfying $h(t) \leqslant \frac{L}{t^{\eta_0}} \ (t \in (0,\infty))$.

Therefore, by the Note of Theorem 3, for $\tau = 0$ in Theorem 1 and (51), we have

$$||\widehat{T}_1|| = ||\widehat{T}_2|| = \frac{\lambda + 2\gamma}{(\sigma + \gamma)(\lambda - \sigma + \gamma)}. \tag{62}$$

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(Received November 29, 2012)

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