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ON A CLASS OF PUNCTUAL CONVEX FUNCTIONS

AURELIA FLOREA AND EUGEN PĂLTĂNEA

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Abstract. The aim of this paper is to show that the inequality of Jensen for real functions holds under a weaker condition than the usual convexity on an interval. Thus, we introduce the concept of *convexity at a point*. We present and discuss the basic properties of the class of functions satisfying the punctual convexity. This concept is further extended to the *lateral convexity at a point*. The interest in these notions is the extensions of some inequalities, as illustrated in this paper. It should be noted that the usual convexity on intervals does not provide a direct answer for these problems.

1. Introduction

Jensen's inequality is usually stated in the context of convex functions but we can find easily examples where this inequality still works for some nonconvex functions and some convex combinations of points "well placed" in the interval of definition. Indeed, let us consider the inequality

$$(\lambda_1 x_1 + \cdots + \lambda_n x_n) b^{\lambda_1 x_1 + \cdots + \lambda_n x_n} \leqslant \lambda_1 x_1 b^{x_1} + \cdots + \lambda_n x_n b^{x_n}, \tag{1}$$

where $b > 1$ and $\lambda_1, \cdots \lambda_n \in [0,1]$, such that $\sum_{i=1}^{n} \lambda_i = 1$.

Since the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = xb^x$, is convex only on the interval $[-2\log^{-1} b, \infty)$, the inequality (1) holds for $x_1, \cdots, x_n \geqslant -2\log^{-1} b$. However, we will show that this inequality can be extended to all real numbers x_i, such that their convex combination $\sum_{i=1}^{n} \lambda_i x_i$ is at least equal to $-\log^{-1} b$ (see Example 3, Section 4).

A second example illustrating this phenomenon is as follows. For $n+1$ positive numbers $a_1, \ldots, a_n$ and p, let us consider the inequality

$$\sqrt{\frac{a_1}{a_2 + pa_1}} + \sqrt{\frac{a_2}{a_3 + pa_2}} + \cdots + \sqrt{\frac{a_{n-1}}{a_n + pa_{n-1}}} + \sqrt{\frac{a_n}{a_1 + pa_n}} \leqslant \frac{n}{\sqrt{1+p}}. \tag{2}$$

Letting $x_i = \log \frac{a_{i+1}}{a_i}$, $i = 1, \ldots, n$, where $a_{n+1} = a_1$, we are led to consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = -\frac{1}{\sqrt{e^x + p}},$$

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where $p>0$ is a fixed parameter. Then the inequality (2) is equivalent with

$$f(0)=f\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)\leqslant\frac{\sum_{i=1}^{n}f(x_i)}{n}, \tag{3}$$

for real numbers $x_1,\cdots,x_n$, with $\sum_{i=1}^{n}x_i=0$. The function f is convex only on the interval $(-\infty,\log(2p)]$. Then (3) results from the usual inequality of Jensen under the assumptions $x_i\leqslant\log(2p)$, for $i=1,\cdots,n$, such that $\sum_{i=1}^{n}x_i=0$, where $p\geqslant 1/2$. Hence (2) holds for $p\geqslant 1/2$ and $0<a_{i+1}/a_i\leqslant 2p$, $i=1,\cdots,n$ (with $a_{n+1}=a_1$). But we will show that the inequality (2) is also valid for $p\geqslant 1/2$ and $0<a_1\leqslant a_2\leqslant\cdots\leqslant a_n$ (see Example 5, Section 4).

The two examples above illustrate a new concept of weak convexity (called by us *convexity at a point*), which we will introduce and describe in this paper.

DEFINITION 1. Let I be an open interval and $a\in I$. We say that a function $f:I\to\mathbb{R}$ is convex at the point a if, for all $x,y\in I$, such that $x\leqslant a\leqslant y$, we have

$$f(a)+f(x+y-a)\leqslant f(x)+f(y). \tag{4}$$

A function f is called concave at the point a provided that $-f$ is convex at that point.

Denote by $\mathrm{Conv}_a(I)$ the set of all convex functions at a, defined on an open interval I, such that $a\in I$. Then every $f\in\mathrm{Conv}_a(I)$ verifies the inequality

$$2f(a)\leqslant f(a+t)+f(a-t),\ \text{for all } t \text{ such that } a-t,a+t\in I. \tag{5}$$

The class $\mathrm{Conv}_a(I)$ may contain non-continuous (hence nonconvex) functions. For example, the function $f:\mathbb{R}\to\mathbb{R}$, defined by:

$$f(x)=\begin{cases} p, & \text{if } x<a\\ q, & \text{if } x=a\\ r, & \text{if } x>a\end{cases},$$

is convex at the point a (i.e. $f\in\mathrm{Conv}_a(\mathbb{R})$) if and only if $q\leqslant\min\{p,r\}$. Remark that, if $q<\min\{p,r\}$, then f is convex only at the point a. Indeed, for $b<a$, we have $2f(b)=2p>p+q=f(2b-a)+f(a)$, so $f\notin\mathrm{Conv}_b(\mathbb{R})$, and, for $b>a$, $2f(b)=2r>r+q=f(2b-a)+f(a)$, so $f\notin\mathrm{Conv}_b(\mathbb{R})$.

Clearly, a convex function f defined on an open real interval I is convex at each $a\in I$. One can show easily that a continuous function $f:I\to\mathbb{R}$ which is convex at all the points of a dense subset of I is convex on the whole interval I.

In the next section we will show that for continuous functions in the class $\mathrm{Conv}_a(I)$, Jensen's inequality holds (at the point a). A characterization of convexity at a point in the presence of differentiability is provided by Theorem 2 below.

Section 3 is devoted to an extended concept, called *the lateral convexity at a point*. In Section 4, we will illustrate by examples the usefulness of these notions and results. Also, a possible extension of the punctual convexity in real Banach spaces is commented.

It is worth noticing that there are many other works treating the existence of Jensen's type inequalities for nonconvex functions (but in a different manner). So is the case of *convex-concave symmetric functions* studied for example by Czinder and Páles [1] and Florea and Niculescu [2], and the case of left almost convex functions, first considered by Niculescu and Spiridon in [5]. Also, Minuţă [3] studies an alternative concept of punctual convexity, called x*-convexity*.

2. The class of punctual convex functions

We start by noticing that the set of convex functions at a point is closed under addition and under multiplication by positive reals. Also, this class is closed under translations. These closure properties and other characterizations are stated in the next lemma.

LEMMA 1. *Let I be an open interval and $a \in I$. The following statements are true:*

1. *the class* $\mathrm{Conv}_a(I)$ *is a convex pointed cone;*
2. $f \in \mathrm{Conv}_0(I)$ *if and only if* $f_a \in \mathrm{Conv}_a(I+a)$, *where* $0 \in I$, $I+a = \{x+a | x \in I\}$ *and* $f_a(x) = f(x-a)$, $x \in I+a$;
3. *if* $f \in \mathrm{Conv}_a(I)$ *and* $g(x) = f(x)+bx+c$, $x \in I$, *where b and c are real constants, then* $g \in \mathrm{Conv}_a(I)$;
4. *for* $f : \mathbb{R} \to \mathbb{R}$, *if* $f(x+a)-f(a)$ *is subadditive, then* $f \in \mathrm{Conv}_a(\mathbb{R})$;
5. *an even function* $f \in \mathrm{Conv}_0(\mathbb{R})$ *has the origin as a global minimum point.*

The following lemma states a special case of Jensen's inequality under the presence of punctual convexity.

LEMMA 2. *Assume* $f \in \mathrm{Conv}_a(I)$, *where I is an open interval, with $a \in I$. Then, for all* $x_1, x_2, \cdots, x_n \in I$, *such that* $\sum_{i=1}^{n} x_i = na$, *we have*

$$nf(a) \leqslant f(x_1) + f(x_2) + \cdots + f(x_n). \tag{6}$$

Proof. We will prove the lemma by induction. For $n = 2$, the inequality (6) becomes directly from the relation (5). Suppose that the inequality (6) holds for $n \geqslant 2$ numbers. Let us consider $x_1, x_2, \ldots, x_{n+1} \in I$, with $\sum_{i=1}^{n+1} x_i = (n+1)a$. Without loss of generality, we can assume $x_i \leqslant x_{i+1}$, $i = 1, 2, \ldots, n$. Then, clearly, $x_1 \leqslant a \leqslant x_{n+1}$. Therefore,

$$f(a) + f(x_1 + x_{n+1} - a) \leqslant f(x_1) + f(x_{n+1}).$$

But, from the induction assumption, we have

$$nf(a) \leqslant f(x_1 + x_{n+1} - a) + f(x_2) + \cdots + f(x_n).$$

Summing these inequalities, we obtain $(n+1)f(a) \leqslant \sum_{k=1}^{n+1} f(x_k)$. So, we get the conclusion. □

A useful application is indicated by the following corollary.

COROLLARY 1. *If* $f \in \mathrm{Conv}_0(\mathbb{R})$*, then* $g = f \circ \log$ *satisfies the inequality:*

$$ng(1) \leqslant g(u_1) + g(u_2) + \cdots + g(u_n),\ \forall\, u_1, u_2, \cdots, u_n > 0,\ \prod_{i=1}^{n} u_i = 1. \tag{7}$$

The discrete case of Jensen's inequality holds in full generality for continuous functions belonging to the class $\mathrm{Conv}_a(I)$.

LEMMA 3. *Let* $f \in \mathrm{Conv}_a(I)$ *be a continuous function. For all positive real numbers* $\lambda_1, \cdots, \lambda_n$*, with* $\sum\limits_{i=1}^{n} \lambda_i = 1$*, and for all* $x_1, \cdots, x_n \in I$*, such that* $\sum\limits_{i=1}^{n} \lambda_i x_i = a$*, we have*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leqslant \sum_{i=1}^{n} \lambda_i f(x_i). \tag{8}$$

Proof. Firstly, let us prove (8) for positive rational numbers λ_i.

Assume $\lambda_1, \cdots, \lambda_n \in \mathbb{Q}_+$ and $x_1, \ldots, x_n \in I$, such that $\sum\limits_{i=1}^{n} \lambda_i = 1$ and $\sum_{i=1}^{n} \lambda_i x_i = a$. There are $n+1$ positive integers $p_1, \cdots, p_n$ and q, such that $\lambda_i = \frac{p_i}{q}$, $i = 1, \ldots, n$. Then $\sum\limits_{i=1}^{n} p_i x_i = aq = a \sum\limits_{i=1}^{n} p_i$. Hence, Lemma 2 ensures

$$qf(a) = \left(\sum_{i=1}^{n} p_i\right) f(a) \leqslant \sum_{i=1}^{n} p_i f(x_i).$$

Therefore, the inequality (8) holds.

Now, let us treat the general case. Assume n positive numbers $\lambda_1, \ldots, \lambda_n$, with $\sum\limits_{i=1}^{n} \lambda_i = 1$, and consider $x_1, \ldots, x_n \in I$, such that $\sum_{i=1}^{n} \lambda_i x_i = a$. For $1 \leqslant i \leqslant n-1$, we choose a sequence of positive rational numbers $(\lambda_k^{(i)})_{k \geqslant 1}$, such that $\lambda_k^{(i)} \leqslant \lambda_i$ and $\lim\limits_{k\to\infty} \lambda_k^{(i)} = \lambda_i$. Now, for each positive integer k, let us denote $\lambda_k^{(n)} = 1 - \sum\limits_{i=1}^{n-1} \lambda_k^{(i)}$ and $z_k = \frac{a - \sum_{i=1}^{n-1} \lambda_k^{(i)} x_i}{\lambda_k^{(n)}}$. Then, for all positive integers k, we have $\sum\limits_{i=1}^{n} \lambda_k^{(i)} = 1$, with $\lambda_k^{(n)} \geqslant 1 - \sum\limits_{i=1}^{n-1} \lambda_i = \lambda_n > 0$, and $\lambda_k^{(n)} z_k + \sum\limits_{i=1}^{n-1} \lambda_k^{(i)} x_i = a$. In addition, $\lim\limits_{k\to\infty} \lambda_k^{(n)} = \lambda_n$ and $\lim\limits_{k\to\infty} z_k = x_n$. In this case, there is a positive integer k_0 such that $z_k \in I$, $\forall\, k \geqslant k_0$. Therefore, the firstly treated case ensures

$$f(a) \leqslant \lambda_k^{(n)} f(z_k) + \sum_{i=1}^{n-1} \lambda_k^{(i)} f(x_i),\ \forall\, k \geqslant k_0.$$

As follows, by using the continuity of f at x_n, we find

$$f(a) \leqslant \lim_{k\to\infty}\left(\lambda_k^{(n)} f(z_k) + \sum_{i=1}^{n-1} \lambda_k^{(i)} f(x_i)\right) = \sum_{i=1}^{n} \lambda_i f(x_i).$$

Thus, the lemma is proved. □

In particular, for $n = 2$ and $a = 0$, we obtain a relevant consequence of above lemma.

COROLLARY 2. *For a continuous function* $f \in \mathrm{Conv}_0(I)$*, we have*

$$(y-x)f(0) \leqslant yf(x) - xf(y), \text{ for all } x,y \in I, \text{ such that } x \leqslant 0 \leqslant y. \tag{9}$$

Remark that the reciprocal statement is not true (see Example 1, Section 4).

A continuous function on an open interval which is convex at a given point admits a support line at that point. More details on this notion can be found, for example, in the monograph of Niculescu and Persson [4].

LEMMA 4. *If* $f \in Conv_a(I)$ *is a continouous function, then there exists* $\lambda \in \mathbb{R}$ *such that*

$$f(x) \geqslant f(a) + \lambda(x-a), \text{ for every } x \in I.$$

Proof. Let us consider $x, y \in I$ such that $x < a < y$. From Lemma 3 we obtain

$$f(a) \leqslant \frac{y-a}{y-x} f(x) + \frac{a-x}{y-x} f(y).$$

As follows,

$$\frac{f(a)-f(x)}{a-x} \leqslant \frac{f(y)-f(a)}{y-a}, \text{ for all } x,y \in I, \text{such that } x < a < y. \tag{10}$$

Denote $\alpha = \sup\limits_{x\in I;\ x<a} \frac{f(a)-f(x)}{a-x}$ and $\beta = \inf\limits_{y\in I;\ y>a} \frac{f(y)-f(a)}{y-a}$, The relation (10) provides $\alpha, \beta \in \mathbb{R}$ and $\alpha \leqslant \beta$. More that, we have

$$\frac{f(a)-f(x)}{a-x} \leqslant \alpha \leqslant \beta \leqslant \frac{f(y)-f(a)}{y-a}, \text{ for all } x,y \in I, \text{such that } x < a < y.$$

Then, for $\lambda \in [\alpha, \beta]$, the following inequality holds

$$f(x) \geqslant f(a) + \lambda(x-a), \text{ for all } x \in I,$$

i.e. f admits a support line at the point a. □

Now, let us derive the general form of Jensen's inequality.

THEOREM 1. *Let $f \in Conv_a(I)$ be a continuous function on the open interval I. Then, for every interval $[\alpha,\beta] \subset I$ including the point a, and for every Borel probability measure μ on $[\alpha,\beta]$ whose barycenter is a, we have*

$$f(a) \leqslant \int_{\alpha}^{\beta} f(x)d\mu(x).$$

Proof. We apply Lemma 3 and the approximation result on Borel probability measures established by Lemma 4.1.10 in Niculescu and Persson [4]. □

The functions $f \in \mathrm{Conv}_a(I)$ with lateral derivatives on I enjoy a specific property.

LEMMA 5. *Assume a function $f : I \to \mathbb{R}$ having lateral derivatives on the open interval I. Suppose $a \in I$. If $f \in \mathrm{Conv}_a(I)$, then $f'(x+0) \leqslant f'(a+0)$, for all $x \in I$, such that $x < a$, and $f'(a-0) \leqslant f'(x-0)$, for all $x \in I$, such that $x > a$.*

Proof. Let $f \in \mathrm{Conv}_0(I)$ be a function with lateral derivatives on I. Let us consider $x \in I \cap (-\infty, a)$. From (4), we have:

$$\frac{f(x+y-a)-f(x)}{y-a} \leqslant \frac{f(y)-f(a)}{y-a}, \ \forall\, y \in I \cap (a,\infty).$$

Therefore,

$$f'(x+0) = \lim_{y\downarrow a} \frac{f(x+y-a)-f(x)}{y-a} \leqslant \lim_{y\downarrow a} \frac{f(y)-f(a)}{y-a} = f'(a+0).$$

Similarly, for a fixed $y \in I \cap (a,\infty)$, we have

$$f'(a-0) = \lim_{x\uparrow a} \frac{f(a)-f(x)}{a-x} \leqslant \lim_{x\uparrow a} \frac{f(y)-f(x+y-a)}{a-x} = f'(y-0).$$

So the lemma is proved. □

Remark that the converse implication is not true (see Example 1, Section 4). Now, let us characterize the differentiable functions of the class $\mathrm{Conv}_a(I)$.

THEOREM 2. *Let $f : I \to \mathbb{R}$ be a differentiable function, where I is an open interval, with $a \in I$. The following statements are equivalent:*

1. $f \in \mathrm{Conv}_a(I)$;
2. $f'(x) \leqslant f'(a) \leqslant f'(y),\ \forall\, x,y \in I,\ x \leqslant a \leqslant y.$

Proof. Suppose $f \in \mathrm{Conv}_a(I)$. From Lemma 5 we find $f'(x) \leqslant f'(a) \leqslant f'(y)$, for all $x,y \in I$, such that $x \leqslant a \leqslant y$. Conversely, assume that the statement *2)* is true. Let us consider $x,y \in I$, such that $x < a < y$. Suppose that $x+y \leqslant 2a$. From the Mean Value Theorem, there are $b,c \in I$, $x < b < x+y-a \leqslant a < c < y$, such that $f(x+y-a)-f(x) = f'(b)(y-a)$ and $f(y)-f(a) = f'(c)(y-a)$. Hence, from the hypothesis, we obtain $f(x+y-a)-f(x) \leqslant f(y)-f(a)$. Similarly, if $x+y > 2a$, then we find $f(a)-f(x) \leqslant f(y)-f(x+y-a)$. As follows, $f \in \mathrm{Conv}_a(I)$. □

The above theorem can be the source of many interesting examples of nonconvex functions which are still convex at a point.

3. Lateral convexity at a point

We introduce now the concept of *lateral convexity at a point*. This notion is less restrictive than the convexity at a point.

DEFINITION 2. Let I be an open interval and $a \in I$.

We say that a function $f : I \to \mathbb{R}$ is left-convex at the point a if, for all $x, y \in I$, such that $x \leqslant a \leqslant y$ and $x + y \leqslant 2a$, we have $f(a) + f(x + y - a) \leqslant f(x) + f(y)$.

We say that a function $f : I \to \mathbb{R}$ is right-convex at the point a if, for all $x, y \in I$, such that $x \leqslant a \leqslant y$ and $x + y \geqslant 2a$, we have $f(a) + f(x + y - a) \leqslant f(x) + f(y)$.

Clearly, f is convex at a point a if and only if it is left-convex and right-convex at the point a. Since the properties of the right-convexity at points can easily be obtained from that one of the left-convexity at points, we discuss here only this last one concept. Also, note that the definition and the properties of the lateral concavity can be directly obtained from that one of lateral convexity. The class of functions satisfying a lateral convexity (or concavity) condition at a point enjoy a series of similar properties to those presented in Section 2.

Let us denote by $\mathrm{Conv}_a^-(I)$ the class of left-convex functions at the point a. Obviously, $\mathrm{Conv}_a(I) \subset \mathrm{Conv}_a^-(I)$. Now, let us show that Lemma 2 can be reformulated for the class $\mathrm{Conv}_a^-(I)$.

LEMMA 6. *Assume $f \in \mathrm{Conv}_a^-(I)$, where I is an open interval, with $a \in I$. Then, for all $x_1, x_2, \cdots, x_n \in I$, such that $x_1 \leqslant a$, $x_2 \geqslant a$, $x_3 \geqslant a, \cdots$, $x_n \geqslant a$ and $\sum_{i=1}^n x_i = na$, we have*

$$nf(a) \leqslant f(x_1) + f(x_2) + \cdots + f(x_n). \tag{11}$$

Proof. The theorem will be proved by induction. For $n = 2$, the inequality (11) becomes directly from the Definition 2. Suppose that the inequality (11) holds for a positive integer $n \geqslant 2$. Let us consider $x_1, x_2, \cdots, x_{n+1} \in I$, such that $x_1 \leqslant a$ and $x_2 \geqslant a$, $x_3 \geqslant a, \cdots$, $x_{n+1} \geqslant a$. Assume $\sum_{i=1}^{n+1} x_i = (n+1)a$. Then we have $x_1 + x_{n+1} = (n+1)a - \sum_{i=2}^n x_i \leqslant (n+1)a - (n-1)a = 2a$. Therefore,

$$f(a) + f(x_1 + x_{n+1} - a) \leqslant f(x_1) + f(x_{n+1}).$$

But from our assumption of induction,

$$nf(a) \leqslant f(x_1 + x_{n+1} - a) + f(x_2) + \cdots + f(x_n).$$

By summing these inequalities, we obtain

$$(n+1)f(a) \leqslant f(x_1) + f(x_2) + \cdots + f(x_n) + f(x_{n+1}).$$

So, we get the conclusion. $\square$

Now, let us formulate necessary and sufficient conditions, respectively, for the left-convexity at points.

THEOREM 3. *Assume $f : I \to \mathbb{R}$, where I is an open interval, with $a \in I$.*

1. *If f has right-derivatives on $I \cap (-\infty, a]$ and $f \in \mathrm{Conv}_a^-(I)$, then $f'(x+0) \leqslant f'(a+0)$, for all $x \in I \cap (-\infty, a)$.*

2. *If I is symmetrical with respect to the point a, f is convex on the interval $I \cap (-\infty, a]$ and*
$$2f(a) \leqslant f(y) + f(2a-y),\ \forall\, y \in I, \tag{12}$$
then $f \in \mathrm{Conv}_a^-(I)$.

Proof. For 1), we can use the same arguments as in the proof of Lemma 5. Now, let us prove 2). Consider $x, y \in I$ such that $x \leqslant a \leqslant y$ and $x+y \leqslant 2a$. Since $x+y-a, 2a-y \in [x,a] \subset I \cap (-\infty, a]$ and f is convex on $I \cap (-\infty, a]$, we have
$$f(x+y-a) + f(2a-y) \leqslant f(x) + f(a). \tag{13}$$
By summing the inequalities (12) and (13), we find $f(a) + f(x+y-a) \leqslant f(x) + f(y)$. Hence, $f \in \mathrm{Conv}_a^-(I)$. □

Mention that Example 5 in Section 4 (also discussed in the introduction) is based on above results.

4. Examples and comments

The first example show that Corollary 2 and Lemma 5 have not reciprocal statements.

EXAMPLE 1. The nonnegative continuous function $f : \mathbb{R} \to \mathbb{R}$,
$$f(x) = \begin{cases} x+2, & x \in [-2,-1); \\ |x|, & x \in [-1,1); \\ 2-x, & x \in [1,2]; \\ 0, & x \in \mathbb{R} \setminus [-2,2], \end{cases}$$
satisfies (9), since $(y-x)f(0) = 0 \leqslant yf(x) - xf(y)$, for all $x, y \in \mathbb{R}$, such that $x \leqslant 0 \leqslant y$. Also, clearly, $f'(x+0) \leqslant 1 = f'(0+0)$, $\forall\, x \leqslant 0$, and $f'(0-0) = -1 \leqslant f'(x-0)$, $\forall\, x \geqslant 0$. But $f(0) + f(1) = 1 > 0 = f(-2) + f(3)$, hence $f \notin \mathrm{Conv}_0(\mathbb{R})$.

The second example illustrates Lemma 3 and Lemma 5.

EXAMPLE 2. Let us consider the continuous periodic function $f : \mathbb{R} \to \mathbb{R}$, with the period $P = 2$, defined as $f(x) = |x|$, for $x \in [-1,1]$. Let $x \leqslant 0$ and $y \geqslant 0$ be two real numbers. Denote $s = \left[\frac{x+1}{2}\right]$ and $t = \left[\frac{y+1}{2}\right]$, the integer parts of $\frac{x+1}{2}$ and $\frac{y+1}{2}$, respectively. We have $x \in [2s-1, 2s+1)$ and $y \in [2t-1, 2t+1)$. Therefore $u := x - 2s \in [-1,1)$ and $v := y - 2t \in [-1,1)$. From the periodicity of f, we obtain
$$f(x) + f(y) = f(u) + f(v) = |u| + |v|,$$

and

$$f(0)+f(x+y)=f(u+v)=\begin{cases}|u+v|, & u+v\in[-1,1]\\ 2-(u+v), & u,v\in(0,1),\ u+v\in(1,2)\\ u+v+2, & u,v<0,\ u+v\in[-2,-1).\end{cases}$$

Clearly, $f(u+v)\leqslant |u|+|v|=f(u)+f(v)$. Then $f\in \mathrm{Conv}_0(I)$. Since $f(t)\geqslant 0$, $\forall t\in \mathbb{R}$, we have

$$f(0)=0\leqslant \sum_{i=1}^{n}\lambda_i f(x_i),$$

for $\lambda_i>0$, $x_i\in\mathbb{R}$, such that $\sum\limits_{i=1}^{n}\lambda_i=1$ and $\sum_{i=1}^{n}\lambda_i x_i=0$. On the other hand, f has lateral derivatives on $\mathbb{R}$, such that $f'(x+0)\leqslant 1=f'(0+0)$, $\forall\, x\leqslant 0$, and $f'(0-0)=-1\leqslant f'(x-0)$, $\forall\, x\geqslant 0$.

EXAMPLE 3. *Assume* $b>1$ *and* $x_1,\cdots,x_n\in\mathbb{R}$, *such that* $\sum\limits_{i=1}^{n}\lambda_i x_i\geqslant -1/\log b$, *Then*

$$(\lambda_1x_1+\cdots+\lambda_nx_n)b^{\lambda_1x_1+\cdots+\lambda_nx_n}\leqslant \lambda_1x_1b^{x_1}+\cdots+\lambda_nx_nb^{x_n},$$

where $\lambda_1,\cdots\lambda_n\in[0,1]$, *such that* $\sum\limits_{i=1}^{n}\lambda_i=1$.

Let us consider the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=xb^x$. We have $f'(x)=b^x(1+x\log b)$ and $f''(x)=b^x\log b(2+x\log b)$. Since $\lim_{x\to-\infty}f'(x)=0$, $f'(-1/\log b)=0$ and f' is increasing on the interval $[-2/\log b,\infty)$ (i.e. f is convex on $[-2/\log b,\infty)$), we obtain

$$f'(x)\leqslant f'(a)\leqslant f'(y),$$

for all $a\geqslant -1/\log b$ and $x\leqslant a\leqslant y$. Then, from Theorem 2, $f\in\mathrm{Conv}_a(I)$, for all a in the interval $[-1/\log b,\infty)$. The conclusion follows from Lemma 3.

EXAMPLE 4. *The following inequality holds:*

$$e^{-x^2}+e^{-y^2}\leqslant 1+e^{-(x+y)^2},\ \forall\, x\leqslant 0\leqslant y.$$

Consider the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=-e^{-x^2}$, $x\in\mathbb{R}$. Since $f'(x)=2xe^{-x^2}$, we have $f'(x)\leqslant f'(0)\leqslant f'(y)$, for $x\leqslant 0\leqslant y$. Then $f\in\mathrm{Conv}_0(\mathbb{R})$ and the inequality is proved. Note that f is convex on $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$ and concave on the intervals $\left(-\infty,-\frac{1}{\sqrt{2}}\right]$ and $\left[\frac{1}{\sqrt{2}},\infty\right)$.

The last example prove the utility of the concept *left-convexity at the origin*.

EXAMPLE 5. *Let* $n\geqslant 2$ *be an integer number, and let* $a_1\leqslant a_2\leqslant\cdots\leqslant a_n$ *be* n *positive numbers. Then, for* $p\geqslant\frac{1}{2}$, *the following inequality holds*

$$\sqrt{\frac{a_1}{a_2+pa_1}}+\sqrt{\frac{a_2}{a_3+pa_2}}+\cdots+\sqrt{\frac{a_{n-1}}{a_n+pa_{n-1}}}+\sqrt{\frac{a_n}{a_1+pa_n}}\leqslant\frac{n}{\sqrt{1+p}}.$$

Let us denote $x_i = \log \frac{a_{i+1}}{a_i}$, $i = 1, \cdots, n$, where $a_{n+1} = a_1$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = -\frac{1}{\sqrt{e^x + p}}$, where $p \geqslant 1/2$ is a fixed parameter. Then we have to show

$$\sum_{i=1}^{n} f(x_i) \geqslant nf(0),$$

under the conditions $x_i \geqslant 0$, for $i = 1, \cdots, n-1$, $x_n \leqslant 0$ and $\sum_{i=1}^{n} x_i = 0$. From Lemma 6, it is sufficient to verify that f is left-convex at the origin. Firstly, we show that $2f(0) \leqslant f(x) + f(-x)$, $\forall\, x \in \mathbb{R}$, i.e.

$$g(t) := \frac{1}{\sqrt{t+p}} + \frac{1}{\sqrt{\frac{1}{t}+p}} \leqslant \frac{2}{\sqrt{1+p}}, \; \forall\, t > 0. \tag{14}$$

We easily state that $\operatorname{sgn}\{g'(t)\} = \operatorname{sgn}\left\{(1-t^2)[p^3(t^2+1) + t(3p^2-1)]\right\}$, for $t > 0$. But we have

$$p^3(t^2+1) + t(3p^2-1) \geqslant t(2p^3 + 3p^2 - 1) = t(2p-1)(p+1)^2 \geqslant 0, \; \forall\, t > 0.$$

Therefore, $g'(t) > 0$, for $t \in (0,1)$, and $g'(t) < 0$, for $t \in (1, \infty)$. As follows, we have $g(t) \leqslant g(1)$, $\forall\, t > 0$, i.e. (14) holds. Secondly, we verify the convexity of f on $(-\infty, 0]$. We have

$$f''(x) = 4^{-1} e^x (e^x + p)^{-5/2} (2p - e^x), \; x \in \mathbb{R}.$$

Since $2p \geqslant 1$, we find $f''(x) > 0$, $\forall\, x < 0$. Therefore, f is convex on the interval $(-\infty, 0]$. By applying Theorem 3 (for $a = 0$), we obtain $f \in \operatorname{Conv}_0^-(I)$. Thus, the given inequality is proved.

We think that the concept of punctual convexity can be of interest in more general frames. Thus, let E be a real Banach space and let $C \subset E$ be an open convex set. A function $f : C \to \mathbb{R}$ will be called convex at the point $a \in C$ if, for every vector $v \neq 0$ of E,

$$f(a) + f(a + (x+y)v) \leqslant f(a + xv) + f(a + yv),$$

for all $x, y \in \mathbb{R}$, such that $xy < 0$ and $a + xv, a + yv \in C$. Clearly, this definition extends the concept developed in this paper for the real case. The study of the punctual convexity can lead to interesting applications in Banach spaces.

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Aurelia Florea
University of Craiova
Romania
e-mail: aurelia_florea@yahoo.com

Eugen Păltănea
Transilvania University of Braşov
Romania
e-mail: epaltanea@unitbv.ro