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NECESSARY AND SUFFICIENT CONDITIONS FOR THE BOUNDEDNESS OF THE MAXIMAL OPERATOR FROM LEBESGUE SPACES TO MORREY–TYPE SPACES

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Abstract. It is proved that the boundedness of the maximal operator M from a Lebesgue space $L_{p_1}(\mathbb{R}^n)$ to a general local Morrey-type space $LM_{p_2\theta,w}(\mathbb{R}^n)$ is equivalent to the boundedness of the embedding operator from $L_{p_1}(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$ and in its turn to the boundedness of the Hardy operator from $L_{\frac{p_1}{p_2}}(0,\infty)$ to the weighted Lebesgue space $L_{\frac{\theta}{p_2},v}(0,\infty)$ for a certain weight function v determined by the functional parameter w. This allows obtaining necessary and sufficient conditions on the function w ensuring the boundedness of M from $L_{p_1}(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$ for any $0<\theta\leqslant\infty$, $0<p_2\leqslant p_1\leqslant\infty$, $p_1>1$. These conditions with $p_1=p_2=1$ are necessary and sufficient for the boundedness of M from $L_1(\mathbb{R}^n)$ to the weak local Morrey-type space $WLM_{1\theta,w}(\mathbb{R}^n)$.

1. Introduction

For $x\in\mathbb{R}^n$ and $r>0$, let $B(x,r)$ denote the open ball centered at x of radius r. Moreover, let $B_r\equiv B(0,r)$.

Let $f\in L_1^{loc}(\mathbb{R}^n)$. The maximal operator M is defined for all $x\in\mathbb{R}^n$ by

$$Mf(x)=\sup_{t>0}\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f(y)|dy,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

DEFINITION 1. Let $0<p,\ \theta\leqslant\infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w}(\mathbb{R}^n)$ the local Morrey-type space, the space of all functions $f\in L_p^{loc}(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{LM_{p\theta,w}(\mathbb{R}^n)}=\Big\|w(r)\|f\|_{L_p(B_r)}\Big\|_{L_\theta(0,\infty)},$$

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and by $WLM_{p\theta,w}(\mathbb{R}^n)$ the weak local Morrey-type space, the space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{WLM_{p\theta,w}(\mathbb{R}^n)} = \Big\| w(r)\|f\|_{WL_p(B_r)} \Big\|_{L_\theta(0,\infty)},$$

where $\|f\|_{WL_p(B_r)}$ denotes the weak L_p-quasinorm of f on B_r:

$$\|f\|_{WL_p(B_r)} = \|f\chi_{B_r}\|_{WL_p(\mathbb{R}^n)} = \sup_{t>0} t^{\frac{1}{p}} (f\chi_{B_r})^*(t) = \sup_{0<t\leqslant|B_r|} t^{\frac{1}{p}} (f\chi_{B_r})^*(t). \qquad (1)$$

Here χ_Ω is the characteristic function of the set $\Omega \subset \mathbb{R}^n$ and f^* denotes the non-increasing rearrangement of f:

$$f^*(t) = \inf\{\tau : \lambda_f(\tau) \leqslant t\}, \quad t > 0,$$

where $\lambda_f(\tau) = |\{x \in \mathbb{R}^n : |f(x)| > \tau\}|$, $\tau > 0$ is the distribution function of the function f.

Note that

$$\|f\|_{LM_{p\infty,1}(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)}, \quad \|f\|_{WLM_{p\infty,1}(\mathbb{R}^n)} = \|f\|_{WL_p(\mathbb{R}^n)}.$$

DEFINITION 2. Let $0 < \theta \leqslant \infty$. We denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w\|_{L_\theta(t,\infty)} < \infty. \qquad (2)$$

REMARK 1. In [6] it was proved that if w is a non-negative measurable function on $(0,\infty)$ which is not equivalent to 0, then the space $LM_{p\theta,w}(\mathbb{R}^n)$ is non-trivial, i. e. consists not only of functions equivalent to 0 on $\mathbb{R}^n$, if and only if $w \in \Omega_\theta$. For this reason it will always be assumed that $w \in \Omega_\theta$.

Let A, B be some sets and φ, ψ be non-negative functions defined on $A \times B$. (It may happen that $\varphi(\alpha,\beta) = +\infty$ or $\psi(\alpha,\beta) = +\infty$ for some $\alpha \in A, \beta \in B$.) We say that φ *is dominated* by ψ (or ψ *dominates* φ) on $A \times B$ *uniformly in* $\alpha \in A$ and write

$$\varphi(\alpha,\beta) \lesssim \psi(\alpha,\beta) \quad \text{uniformly in} \quad \alpha \in A$$

or

$$\psi(\alpha,\beta) \gtrsim \varphi(\alpha,\beta) \quad \text{uniformly in} \quad \alpha \in A,$$

if for each $\beta \in B$ there exists $c(\beta) > 0$ such that

$$\varphi(\alpha,\beta) \leqslant c(\beta)\,\psi(\alpha,\beta)$$

for all $\alpha \in A$. We also say that φ is *equivalent* to ψ on $A \times B$ *uniformly in* $\alpha \in A$ and write

$$\varphi(\alpha,\beta) \approx \psi(\alpha,\beta) \quad \text{uniformly in} \quad \alpha \in A,$$

if φ and ψ dominate each other on $A \times B$ uniformly in $\alpha \in A$.

LEMMA 1. [7] *Let* $0 < \theta \leqslant \infty$ *and* $w_1,\ w_2 \in \Omega_\theta$. *Then for each* $0 < p \leqslant \infty$ *the equality*

$$LM_{p\theta,w_1}(\mathbb{R}^n) = LM_{p\theta,w_2}(\mathbb{R}^n)$$

holds if and only if [1]

$$\|w_1\|_{L_\theta(t,\infty)} \approx \|w_2\|_{L_\theta(t,\infty)} \quad \text{uniformly in} \quad t \in (0,\infty).$$

The boundedness of the maximal operator from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ for general w_1 and w_2 was studied in [9], [10], [5], [6], and [4]. See detailed exposition of this and related problems in recent survey papers [2], [3]. In [5], [6], and [4], for a certain range of the parameters p, θ_1 and θ_2, necessary and sufficient conditions on w_1 and w_2 were obtained ensuring the boundedness of M from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$, namely the following statement was proved.

THEOREM 1. *If* $n \in \mathbb{N},\ 1 < p < \infty,\ 0 < \theta_1 \leqslant \theta_2 \leqslant \infty,\ w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$, *then the condition*

$$\left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,\infty)} \lesssim \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{3}$$

uniformly in $t \in (0,\infty)$ *is necessary and sufficient for the boundedness of* M *from* $LM_{p\theta_1,w_1}(\mathbb{R}^n)$ *to* $LM_{p\theta_2,w_2}(\mathbb{R}^n)$. *Moreover,*

$$\|M\|_{LM_{p\theta_1,w_1}(\mathbb{R}^n) \to LM_{p\theta_2,w_2}(\mathbb{R}^n)} \approx \sup_{0<t<\infty} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,\infty)} \tag{4}$$

uniformly in $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

If $p = 1$, *then condition* (3) *is necessary and sufficient for the boundedness of* M *from* $LM_{1\theta_1,w_1}(\mathbb{R}^n)$ *to* $WLM_{1\theta_2,w_2}(\mathbb{R}^n)$ *and*

$$\|M\|_{LM_{1\theta_1,w_1}(\mathbb{R}^n) \to WLM_{1\theta_2,w_2}(\mathbb{R}^n)} \approx \sup_{0<t<\infty} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \left\| w_2(r) \left(\frac{r}{t+r} \right)^{n} \right\|_{L_{\theta_2}(0,\infty)} \tag{5}$$

uniformly in $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

In (4) and (5) we assume that $(+\infty)^{-1} = 0$ and $0 \cdot (+\infty) = 0$.

In [5], [6] this was proved under the additional assumption $\theta_1 \leqslant p$. The general case was considered in [4].

If $\theta_2 < \theta_1$, then sufficient conditions on w_1 and w_2 for the boundedness of M from $LM_{p\theta_1,w_1}(\mathbb{R}^n)$ to $LM_{p\theta_2,w_2}(\mathbb{R}^n)$ are given in [4]. However, the challenging problem of finding necessary and sufficient conditions on w_1 and w_2 ensuring the boundedness of M from $LM_{p\theta_1,w_1}(\mathbb{R}^n)$ to $LM_{p\theta_2,w_2}(\mathbb{R}^n)$ for the case $\theta_2 < \theta_1$ is still open. In

[1] By the above convention this means that, given $0 < p \leqslant \infty$, for each $0 < \theta \leqslant \infty$, $w_1,\ w_2 \in \Omega_\theta$ there exist $c_1,\ c_2 > 0$ such that

$$c_1 \|w_1\|_{L_\theta(t,\infty)} \leqslant \|w_2\|_{L_\theta(t,\infty)} \leqslant c_2 \|w_1\|_{L_\theta(t,\infty)}$$

for all $t \in (0,\infty)$. So, for a fixed $0 < p \leqslant \infty$, c_1 and c_2 may depend on $0 < \theta \leqslant \infty$, $w_1,\ w_2 \in \Omega_\theta$, but are independent of $t \in (0,\infty)$. However, for the whole range of the parameter p, c_1 and c_2 may also depend on p.

this paper we give its solution for a very particular case in which $\theta_1 = \infty$ and $w_1(r) \equiv 1$. In other words we find, for all admissible values of the parameters p_1, p_2, and θ, necessary and sufficient conditions on w ensuring that the maximal operator is bounded from $L_{p_1}(\mathbb{R}^n) = LM_{p_1\infty,1}(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$.

2. Preliminary observations

We start with the following simple observations aimed at clarifying necessary assumptions on $0 < p_1, p_2, \theta \leqslant \infty$ for which for certain $w \in \Omega_\theta$ the operator M can be bounded from $L_{p_1}(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$.

REMARK 2. Let $0 < \theta \leqslant \infty$, $w \in \Omega_\theta$, and $0 < p_1 < p_2 \leqslant \infty$. Then there exists $r_0 > 0$ such that $\|w\|_{L_\theta(r_0,\infty)} > 0$. We can find $f \in L_{p_1}(\mathbb{R}^n)$ such that $f \notin L_{p_2}(B_{r_0})$. Then $Mf \notin L_{p_2}(B_{r_0})$ and therefore $Mf \notin LM_{p_2\theta,w}(\mathbb{R}^n)$, because

$$\|Mf\|_{LM_{p_2\theta,w}(\mathbb{R}^n)} \geqslant \|Mf\|_{L_{p_2}(B_{r_0})} \|w\|_{L_\theta(r_0,\infty)}.$$

Thus, in the problem of the boundedness of the maximal operator $M : L_{p_1}(\mathbb{R}^n) \to LM_{p_2\theta,w}(\mathbb{R}^n)$ one should assume that $p_2 \leqslant p_1$.

REMARK 3. Assume that $0 < \theta \leqslant \infty$, $w \in \Omega_\theta$, and $p_1 = p_2 > 1$. Then for each $\rho > 0$

$$\|M\|_{L_{p_1}(\mathbb{R}^n) \to LM_{p_1\theta,w}(\mathbb{R}^n)} \geqslant \frac{\|w(r)\|M\chi_{B_\rho}\|_{L_{p_1}(B_r)}\|_{L_\theta(\rho,\infty)}}{\|\chi_{B_\rho}\|_{L_{p_1}(B_\rho)}}.$$

Since $\|M\chi_{B_\rho}\|_{L_{p_1}(B_r)} \geqslant \|\chi_{B_\rho}\|_{L_{p_1}(B_\rho)}$ for all $r \geqslant \rho$, it follows that

$$\|M\|_{L_{p_1}(\mathbb{R}^n) \to LM_{p_1\theta,w}(\mathbb{R}^n)} \geqslant \|w\|_{L_\theta(\rho,\infty)}$$

for all $\rho > 0$. Hence

$$\|M\|_{L_{p_1}(\mathbb{R}^n) \to LM_{p_1\theta,w}(\mathbb{R}^n)} \geqslant \|w\|_{L_\theta(0,\infty)}.$$

On the other hand, by applying the classical L_{p_1}-estimate for the maximal function, it follows that

$$\begin{aligned}\|M\|_{L_{p_1}(\mathbb{R}^n) \to LM_{p_1\theta,w}(\mathbb{R}^n)} &= \sup_{\substack{f \in L_{p_1}(\mathbb{R}^n) \\ f \not\sim 0}} \frac{\|w(r)\|Mf\|_{L_{p_1}(B_r)}\|_{L_\theta(0,\infty)}}{\|f\|_{L_{p_1}(\mathbb{R}^n)}} \\ &\lesssim \sup_{\substack{f \in L_{p_1}(\mathbb{R}^n) \\ f \not\sim 0}} \frac{\|w(r)\|f\|_{L_{p_1}(\mathbb{R}^n)}\|_{L_\theta(0,\infty)}}{\|f\|_{L_{p_1}(\mathbb{R}^n)}} = \|w\|_{L_\theta(0,\infty)}\end{aligned}$$

uniformly in $w \in \Omega_\theta$. Thus

$$\|M\|_{L_{p_1}(\mathbb{R}^n) \to LM_{p_1\theta,w}(\mathbb{R}^n)} \approx \|w\|_{L_\theta(0,\infty)} \tag{6}$$

uniformly in $w \in \Omega_\theta$.

For similar reasons by the equality $\|\chi_{B_\rho}\|_{WL_1(B_\rho)} = \|\chi_{B_\rho}\|_{L_1(B_\rho)}$ and the boundedness of M from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ it follows that

$$\|M\|_{L_1(\mathbb{R}^n)\to WLM_{1\theta,w}(\mathbb{R}^n)} \approx \|w\|_{L_\theta(0,\infty)} \tag{7}$$

uniformly in $w \in \Omega_\theta$.

Equivalences (6) and (7) also follow by equivalences (4) and (5) with $w_1 \equiv 1$, $w_2 = w$, $\theta_1 = \infty$, $\theta_2 = \theta$, because

$$\sup_{0<t<\infty} \left\| w(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p}} \right\|_{L_\theta(0,\infty)} = \|w\|_{L_\theta(0,\infty)}.$$

If $p_1 = p_2 = 1$, then $\|M\|_{L_1(\mathbb{R}^n)\to LM_{1\theta,w}(\mathbb{R}^n)} = \infty$ for all $0 < \theta \leqslant \infty$ and $w \in \Omega_\theta$. This follows if one considers the test-functions χ_{B_ε} and passes to the limit as $\varepsilon \to 0^+$.

Summarizing, if one investigates the boundedness of M from $L_{p_1}(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$, then one should always assume that

$$0 < \theta \leqslant \infty, \quad 1 \leqslant p_1 \leqslant \infty, \quad 0 < p_2 \leqslant p_1 \text{ if } p_1 > 1, \quad 0 < p_2 < 1 \text{ if } p_1 = 1,$$

and $w \in \Omega_\theta$.

REMARK 4. What happens if $0 < p_2 < p_1$? If $p_1 > 1$, then by applying Hölder's inequality and the boundedness of M from $L_{p_1}(\mathbb{R}^n)$ to $L_{p_1}(\mathbb{R}^n)$ it immediately follows that

$$\|Mf\|_{L_{p_2}(B_r)} \leqslant (v_n r^n)^{\frac{1}{p_2}-\frac{1}{p_1}} \|Mf\|_{L_{p_1}(\mathbb{R}^n)} \lesssim r^{n(\frac{1}{p_2}-\frac{1}{p_1})} \|f\|_{L_{p_1}(\mathbb{R}^n)}$$

uniformly in $r > 0$, where v_n is the volume of the unit ball in $\mathbb{R}^n$, and

$$\|M\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} = \sup_{\substack{f \in L_{p_1}(\mathbb{R}^n) \\ f \not\sim 0}} \frac{\big\|w(r)\|Mf\|_{L_{p_2}(B_r)}\big\|_{L_\theta(0,\infty)}}{\|f\|_{L_{p_1}(\mathbb{R}^n)}}$$
$$\lesssim \left\| r^{n(\frac{1}{p_2}-\frac{1}{p_1})} w(r) \right\|_{L_\theta(0,\infty)}$$

uniformly in $w \in \Omega_\theta$. Hence the condition

$$\left\| r^{n(\frac{1}{p_2}-\frac{1}{p_1})} w(r) \right\|_{L_\theta(0,\infty)} < \infty \tag{8}$$

is sufficient for boundedness of the maximal operator M from $L_{p_1}(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$.

However, in spite of the fact that Hölder's inequality is sharp, it appears that this simple sufficient condition is also necessary if and only if $\theta = \infty$. If $\theta < \infty$ it is not necessary (though is rather close to being necessary). In this case necessary and sufficient conditions are much more sophisticated (see Theorem 4.)

If $p_2 < 1$, then condition (8) with $p_1 = 1$ is also sufficient for the boundedness of M from $L_1(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$. This follows since by the boundedness of M from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$

$$\begin{aligned}\|Mf\|_{L_{p_2}(B_r)} &= \|(Mf)\chi_{B_r}\|_{L_{p_2}(B_r)} \leqslant \|((Mf)\chi_{B_r})^*\|_{L_{p_2}(0,|B_r|)} \\ &\leqslant \Big(\sup_{0<t\leqslant|B_r|} t((Mf)\chi_{B_r})^*(t)\Big)\|t^{-1}\|_{L_{p_2}(0,|B_r|)} \\ &= (1-p_2)^{\frac{1}{p_2}}(v_n r^n)^{\frac{1}{p_2}-1}\|Mf\|_{WL_1(B_r)} \lesssim r^{n(\frac{1}{p_2}-1)}\|f\|_{L_1(\mathbb{R}^n)}\end{aligned}$$

uniformly in $r > 0$.

REMARK 5. Equivalence (6) and the way how it is obtained mean that the usage of the spaces $LM_{p_1\theta,w}(\mathbb{R}^n)$ does not contribute much to the study of the properties of Mf for $f \in L_{p_1}(\mathbb{R}^n)$ because (6) follows directly by the classical estimate for $\|Mf\|_{L_{p_1}(\mathbb{R}^n)}$. The same refers to the case in which the space $LM_{p_1\theta,w}(\mathbb{R}^n)$ is replaced by the space $LM_{p_2\theta,w}(\mathbb{R}^n)$ with $p_2 < p_1$ and $\theta = \infty$. The situation is different if $p_2 < p_1$ and $\theta < \infty$ as Theorem 4 and Remark 8 below show.

3. Main results

Let $f_*(x) = f^*(v_n|x|^n)$, $x \in \mathbb{R}^n$, be the radially symmetric non-increasing rearrangement of f. Recall that for all $0 < p \leqslant \infty$ and $0 < r \leqslant \infty$

$$\|f\|_{L_p(B_r)} \leqslant \|f^*\|_{L_p(0,|B_r|)} = \|f_*\|_{L_p(B_r)} \tag{9}$$

and

$$\|f\|_{L_p(\mathbb{R}^n)} = \|f^*\|_{L_p(0,\infty)} = \|f_*\|_{L_p(\mathbb{R}^n)}. \tag{10}$$

Also, for all $0 < p \leqslant \infty$ and $0 < r \leqslant \infty$,

$$\|f\|_{WL_p(B_r)} \leqslant \|f^*\|_{WL_p(0,|B_r|)} = \|f_*\|_{WL_p(B_r)} \tag{11}$$

and

$$\|f\|_{WL_p(\mathbb{R}^n)} = \|f^*\|_{WL_p(0,\infty)} = \|f_*\|_{WL_p(\mathbb{R}^n)}. \tag{12}$$

Denote by $\mathfrak{M}(\mathbb{R}^n)$ the space of all functions measurable on $\mathbb{R}^n$ and by $K^{\downarrow}(\mathbb{R}^n)$ the cone of all functions f of the form $f(x) = g(|x|)$, $x \in \mathbb{R}^n$, where g is non-negative and non-increasing on $[0,\infty)$.

Note that for all $0 < p \leqslant \infty$ and for all $0 < r \leqslant \infty$

$$\begin{aligned}\|f\|_{WL_p(B_r)} &= \sup_{0<t\leqslant|B_r|} t^{\frac{1}{p}}(f_*\chi_{B_r})^*(t) = \sup_{t>0} t^{\frac{1}{p}}(f^*\chi_{(0,|B_r|)})^*(t) \\ &= \sup_{t>0} t^{\frac{1}{p}} f^*(t)\chi_{(0,|B_r|)}(t) = \sup_{0<t\leqslant|B_r|} t^{\frac{1}{p}} f^*(t), \quad f \in K^{\downarrow}(\mathbb{R}^n).\end{aligned} \tag{13}$$

In particular

$$\|f_*\|_{WL_p(B_r)} = \sup_{0<t\leqslant|B_r|} t^{\frac{1}{p}} f^*(t), \quad f \in \mathfrak{M}(\mathbb{R}^n).$$

LEMMA 2. *Let* $n \in \mathbb{N}$ *and* $1 \leqslant p \leqslant \infty$.
If $p > 1$, *then*

$$\|Mf\|_{L_p(B_r)} \lesssim \|f_*\|_{L_p(B_r)} \tag{14}$$

uniformly in $f \in L_p^{loc}(\mathbb{R}^n)$ *and in* $r \in (0,\infty]$, *and if* $p = 1$, *then*

$$\|Mf\|_{WL_1(B_r)} \lesssim \|f_*\|_{L_1(B_r)} \tag{15}$$

uniformly in $f \in L_1^{loc}(\mathbb{R}^n)$ *and in* $r \in (0,\infty]$.

Proof. The proofs of this and the next lemma are based on the following inequality: there exist a_1, $a_2 > 0$ such that for all $f \in \mathfrak{M}(\mathbb{R}^n)$

$$\frac{a_1}{t}\int_0^t f^*(\tau)\,d\tau \leqslant (Mf)^*(t) \leqslant \frac{a_2}{t}\int_0^t f^*(\tau)\,d\tau \tag{16}$$

for all $t > 0$. (See, for example, [1].)

Let $p > 1$. By (9), (16) and the Hardy inequality

$$\|Mf\|_{L_p(B_r)} \leqslant \|(Mf)^*\|_{L_p(0,|B_r|)} \leqslant a_2 \left\| \frac{1}{t}\int_0^t f^*(\tau)\,d\tau \right\|_{L_p(0,|B_r|)}$$
$$\leqslant p' a_2 \|f^*\|_{L_p(0,|B_r|)} = p' a_2 \|f_*\|_{L_p(B_r)}.$$

If $p = 1$, then by (11), (13) and (16)

$$\|Mf\|_{WL_1(B_r)} \leqslant \|(Mf)^*\|_{WL_1(0,|B_r|)} = \sup_{0<t\leqslant |B_r|} t(Mf)^*(t)$$
$$\leqslant a_2 \sup_{0<t\leqslant |B_r|} \int_0^t f^*(\tau)\,d\tau = a_2 \|f^*\|_{L_1(0,|B_r|)} = a_2 \|f_*\|_{L_1(B_r)}. \quad \square$$

LEMMA 3. *Let* $n \in \mathbb{N}$ *and* $1 \leqslant p \leqslant \infty$. *Then*

$$\|f\|_{L_p(B_r)} \approx \begin{cases} \|Mf\|_{L_p(B_r)} & \text{if } 1 < p \leqslant \infty, \\ \|Mf\|_{WL_1(B_r)} & \text{if } p = 1 \end{cases} \tag{17}$$

uniformly in $f \in K^{\downarrow}(\mathbb{R}^n) \cap L_p^{loc}(\mathbb{R}^n)$ *and in* $r \in (0,\infty]$.

Proof. Since $|f| \leqslant Mf$ almost everywhere on $\mathbb{R}^n$, for $p > 1$ by (14)

$$\|f\|_{L_p(B_r)} \leqslant \|Mf\|_{L_p(B_r)} \lesssim \|f_*\|_{L_p(B_r)} = \|f\|_{L_p(B_r)}$$

uniformly in $f \in K^{\downarrow}(\mathbb{R}^n) \cap L_p^{loc}(\mathbb{R}^n)$ and in $r \in (0,\infty)$.

If $p=1$ then, taking into account that for $f \in K^{\downarrow}(\mathbb{R}^n) \cap L_1^{loc}(\mathbb{R}^n)$ one has $Mf \in K^{\downarrow}(\mathbb{R}^n)$, hence by (13) and (16) it follows that

$$\|Mf\|_{WL_1(B_r)} = \sup_{0<t\leqslant |B_r|} t(Mf)^*(t)$$
$$\geqslant a_1 \sup_{0<t\leqslant |B_r|} \int_0^t f^*(\tau)\,d\tau = a_1\,\|f^*\|_{L_1(0,|B_r|)} = a_1\,\|f_*\|_{L_1(B_r)} = a_1\|f\|_{L_1(B_r)}$$

for all $f \in K^{\downarrow}(\mathbb{R}^n) \cap L_1^{loc}(\mathbb{R}^n)$ and for all $r \in (0,\infty)$. This inequality together with inequality (15) imply equivalence (17) for $p=1$. $\square$

Denote $L_p^{\downarrow}(\mathbb{R}^n) = L_p(\mathbb{R}^n) \cap K^{\downarrow}(\mathbb{R}^n)$.

THEOREM 2. *Let $n \in \mathbb{N}$, $0<p_2 \leqslant p_1 \leqslant \infty$, $0<\theta\leqslant\infty$, and $w \in \Omega_\theta$.*
1. *If $p_1>1$, then M is bounded from $L_{p_1}(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$ if and only if*

$$L_{p_1}^{\downarrow}(\mathbb{R}^n) \subset LM_{p_2\theta,w}(\mathbb{R}^n),$$

and

$$\|M\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \approx \|M\|_{L^{\downarrow}_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \tag{18}$$

$$\approx \|I\|_{L^{\downarrow}_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} = \|I\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \tag{19}$$

uniformly in $w \in \Omega_\theta$, where I is the corresponding embedding operator.
2. *If $p_1=p_2=1$, then M is bounded from $L_1(\mathbb{R}^n)$ to $WLM_{1\theta,w}(\mathbb{R}^n)$ if and only if*

$$L_1^{\downarrow}(\mathbb{R}^n) \subset LM_{1\theta,w}(\mathbb{R}^n),$$

and

$$\|M\|_{L_1(\mathbb{R}^n)\to WLM_{1\theta,w}(\mathbb{R}^n)} \approx \|M\|_{L^{\downarrow}_1(\mathbb{R}^n)\to WLM_{1\theta,w}(\mathbb{R}^n)} \tag{20}$$

$$\approx \|I\|_{L^{\downarrow}_1(\mathbb{R}^n)\to LM_{1\theta,w}(\mathbb{R}^n)} = \|I\|_{L_1(\mathbb{R}^n)\to LM_{1\theta,w}(\mathbb{R}^n)} \tag{21}$$

uniformly in $w \in \Omega_\theta$.

Proof.
1. *Proof of equivalence* (18) *for all* $0<p_2\leqslant p_1\leqslant\infty$. Note that by taking the spherical coordinates inequality (16) takes the form

$$\frac{a_1}{|B_{|x|}|}\int_{B_{|x|}} f_*(y)\,dy \leqslant (Mf)_*(x) \leqslant \frac{a_2}{|B_{|x|}|}\int_{B_{|x|}} f_*(y)\,dy.$$

Therefore

$$(Mf)_*(x) \leqslant \frac{a_2}{|B_{|x|}|}\int_{B_{|x|}} (f_*)_*(y)\,dy \leqslant \frac{a_2}{a_1}(Mf_*)_*(x).$$

Since $(Mf_*)_* = Mf_*$ we have

$$(Mf)_*(x) \leqslant \frac{a_2}{a_1}(Mf_*)(x), \quad x \in \mathbb{R}^n. \tag{22}$$

Hence, taking into account inequality (9) and equality (10), we get that

$$\begin{aligned}
&\|M\|_{L^{\downarrow}_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \leqslant \|M\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \\
&= \sup_{\substack{f\in L_{p_1}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|Mf\|_{L_{p_2}(B_r)}\|_{L_\theta(0,\infty)}}{\|f\|_{L_{p_1}(\mathbb{R}^n)}} \leqslant \sup_{\substack{f\in L_{p_1}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|(Mf)_*\|_{L_{p_2}(B_r)}\|_{L_\theta(0,\infty)}}{\|f\|_{L_{p_1}(\mathbb{R}^n)}} \\
&\leqslant \frac{a_2}{a_1} \sup_{\substack{f\in L_{p_1}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|Mf_*\|_{L_{p_2}(B_r)}\|_{L_\theta(0,\infty)}}{\|f_*\|_{L_{p_1}(\mathbb{R}^n)}} \leqslant \frac{a_2}{a_1}\|M\|_{L^{\downarrow}_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)},
\end{aligned}$$

which proves equivalence (18).

2. *Proof of the equality in* (19) *for all* $0 < p_2 \leqslant p_1 \leqslant \infty$. By (9) and (10)

$$\begin{aligned}
&\|I\|_{L^{\downarrow}_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \leqslant \|I\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \\
&= \sup_{\substack{f\in L_{p_1}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|f\|_{L_{p_2}(B_r)}\|_{L_\theta(0,\infty)}}{\|f\|_{L_{p_1}(\mathbb{R}^n)}} \leqslant \sup_{\substack{f\in L_{p_1}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|f_*\|_{L_{p_2}(B_r)}\|_{L_\theta(0,\infty)}}{\|f_*\|_{L_{p_1}(\mathbb{R}^n)}} \\
&\leqslant \sup_{\substack{g\in L^{\downarrow}_{p_1}(\mathbb{R}^n)\\ g\not\sim 0}} \frac{\|w(r)\|g\|_{L_{p_2}(B_r)}\|_{L_\theta(0,\infty)}}{\|g\|_{L_{p_1}(\mathbb{R}^n)}} = \|I\|_{L^{\downarrow}_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)}.
\end{aligned}$$

3. *Proof of the equivalence in* (19) *for* $0 < p_2 \leqslant p_1 \leqslant \infty$ *and* $p_1 > 1$. Since $|f| \leqslant Mf$ almost everywhere on $\mathbb{R}^n$

$$\begin{aligned}
\|I\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} &= \sup_{\substack{f\in L_{p_1}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|f\|_{L_{p_2}(B_r)}\|_{L_\theta(0,\infty)}}{\|f\|_{L_{p_1}(\mathbb{R}^n)}} \\
&\leqslant \sup_{\substack{f\in L_{p_1}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|Mf\|_{L_{p_2}(B_r)}\|_{L_\theta(0,\infty)}}{\|f\|_{L_{p_1}(\mathbb{R}^n)}} = \|M\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)}.
\end{aligned}$$

Since $1 < p_1 \leqslant \infty$, by the boundedness of M from $L_{p_1}(\mathbb{R}^n)$ to $L_{p_1}(\mathbb{R}^n)$

$$\begin{aligned}
\|M\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} &= \sup_{\substack{f\in L_{p_1}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|Mf\|_{L_{p_2}(B_r)}\|_{L_\theta(0,\infty)}}{\|f\|_{L_{p_1}(\mathbb{R}^n)}} \\
&\lesssim \sup_{\substack{f\in L_{p_1}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|Mf\|_{L_{p_2}(B_r)}\|_{L_\theta(0,\infty)}}{\|Mf\|_{L_{p_1}(\mathbb{R}^n)}} \leqslant \|I\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)}
\end{aligned}$$

uniformly in $w \in \Omega_\theta$, which proves, by taking into account Steps 1 and 2, the equivalence in (19).

4. *Proof of equivalence* (20). Follows similarly to Step 1 since by inequalities (11) and (22)

$$\|Mf\|_{WL_1(B_r)} \leqslant \|(Mf)_*\|_{WL_1(B_r)} \leqslant \frac{a_2}{a_1}\|Mf_*\|_{WL_1(B_r)}.$$

5. *Proof of the equality and the equivalence in* (21). The equality in (21) is proved in Step 2. As for the equivalence in (21) we note that, since for $p = 1$ equivalence (17) holds uniformly in $f \in L_1^{\downarrow}(\mathbb{R}^n)$ and in $r \in (0,\infty]$, we have

$$\|I\|_{L_1^{\downarrow}(\mathbb{R}^n)\to LM_{1\theta,w}(\mathbb{R}^n)} = \sup_{\substack{f\in L_1^{\downarrow}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|f\|_{L_1(B_r)}\|_{L_\theta(0,\infty)}}{\|f\|_{L_1(\mathbb{R}^n)}}$$

$$\approx \sup_{\substack{f\in L_1^{\downarrow}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|w(r)\|Mf\|_{WL_1(B_r)}\|_{L_\theta(0,\infty)}}{\|f\|_{L_1(\mathbb{R}^n)}} = \|M\|_{L_1^{\downarrow}(\mathbb{R}^n)\to WLM_{1\theta,w}(\mathbb{R}^n)}. \qquad \square$$

LEMMA 4. *Let $n\in\mathbb{N}$, $0<p_2\leqslant p_1\leqslant\infty$, $0<\theta\leqslant\infty$, and $w\in\Omega_\theta$. Then for each $0<\varepsilon<\infty$*

$$\|I\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \gtrsim t^{-\varepsilon}\left\|r^{n(\frac{1}{p_2}-\frac{1}{p_1})+\varepsilon}w(r)\right\|_{L_\theta(0,t)} \tag{23}$$

and

$$\|I\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \gtrsim t^{\varepsilon}\left\|r^{n(\frac{1}{p_2}-\frac{1}{p_1})-\varepsilon}w(r)\right\|_{L_\theta(t,\infty)} \tag{24}$$

uniformly in $0<t<\infty$ and $w\in\Omega_\theta$.

Proof. First

$$\|I\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \geqslant \frac{\|w(r)\|\,|x|^{-\frac{n}{p_1}+\varepsilon}\chi_{B_t}(x)\|_{L_{p_2}(B_r)}\|_{L_\theta(0,t)}}{\|\,|x|^{-\frac{n}{p_1}+\varepsilon}\chi_{B_t}(x)\|_{L_{p_1}(\mathbb{R}^n)}}$$

$$= (nv_n)^{\frac{1}{p_2}-\frac{1}{p_1}} p_1^{\frac{1}{p_1}} p_2^{-\frac{1}{p_2}} \varepsilon^{\frac{1}{p_1}} \left(n\left(\frac{1}{p_2}-\frac{1}{p_1}\right)+\varepsilon\right)^{-\frac{1}{p_2}} t^{-\varepsilon}\left\|r^{n(\frac{1}{p_2}-\frac{1}{p_1})+\varepsilon}w(r)\right\|_{L_\theta(0,t)}$$

which implies inequality in (23).

Next, let $g_\varepsilon(x) = t^{-\frac{n}{p_1}-\varepsilon}$ if $|x|\leqslant t$ and $g_\varepsilon(x)=|x|^{-\frac{n}{p_1}-\varepsilon}$ if $|x|>t$. If $r\geqslant t$, then $g_\varepsilon(x)\geqslant r^{-\frac{n}{p_1}-\varepsilon}$ for all $x\in B_r$, hence

$$\|g_\varepsilon\|_{L_{p_2}(B_r)} \geqslant v_n^{\frac{1}{p_2}} r^{n(\frac{1}{p_2}-\frac{1}{p_1})-\varepsilon}.$$

Since also

$$\|g_\varepsilon\|_{L_{p_1}(\mathbb{R}^n)} = v_n^{\frac{1}{p_1}}\left(1+\frac{n}{\varepsilon p_1}\right)^{\frac{1}{p_1}} t^{-\varepsilon},$$

it follows that

$$\|I\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \geqslant \frac{\|w(r)\|g_\varepsilon\|_{L_{p_2}(B_r)}\|_{L_\theta(t,\infty)}}{\|g_\varepsilon\|_{L_{p_1}(\mathbb{R}^n)}}$$

$$\geqslant v_n^{\frac{1}{p_2}-\frac{1}{p_1}}\left(1+\frac{n}{\varepsilon p_1}\right)^{-\frac{1}{p_1}} t^\varepsilon \left\|r^{n(\frac{1}{p_2}-\frac{1}{p_1})-\varepsilon} w(r)\right\|_{L_\theta(t,\infty)}$$

which implies inequality (24). □

COROLLARY 1. *Let* $n\in\mathbb{N}$, $0<p_2<p_1\leqslant\infty$, $p_1>1$, $0<\theta\leqslant\infty$, *and* $w\in\Omega_\theta$. *Then*

$$\|M\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \gtrsim \left\| t^{n(\frac{1}{p_2}-\frac{1}{p_1})} \left\| \left(\frac{r}{r+t}\right)^{\frac{n}{p_2}} w(r)\right\|_{L_\theta(0,\infty)}\right\|_{L_\infty(0,\infty)} \tag{25}$$

uniformly in $w\in\Omega_\theta$, *where the semi-norm* $\|\cdot\|_{L_\theta(0,\infty)}$ *is taken in the variable* r *and the semi-norm* $\|\cdot\|_{L_\infty(0,\infty)}$ *in the variable* t.

Proof. By Theorem 2, inequality (23) with $\varepsilon=\frac{n}{p_1}$ and inequality (24) with $\varepsilon = n(\frac{1}{p_2}-\frac{1}{p_1})$

$$\|M\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \gtrsim t^{-\frac{n}{p_1}} \left\|r^{\frac{n}{p_2}} w(r)\right\|_{L_\theta(0,t)} + t^{n(\frac{1}{p_2}-\frac{1}{p_1})}\left\|w(r)\right\|_{L_\theta(t,\infty)}$$

uniformly in $w\in\Omega_\theta$ and $0<t<\infty$. Hence

$$\|M\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)}$$

$$\gtrsim \sup_{0<t<\infty}\left(t^{-\frac{n}{p_1}} \left\|r^{\frac{n}{p_2}} w(r)\right\|_{L_\theta(0,t)} + t^{n(\frac{1}{p_2}-\frac{1}{p_1})}\left\|w(r)\right\|_{L_\theta(t,\infty)}\right)$$

uniformly in $w\in\Omega_\theta$.

This inequality implies inequality (25), because its right-hand side is equivalent to the right-hand side of inequality (25) uniformly in $w\in\Omega_\theta$. □

COROLLARY 2. *Let* $n\in\mathbb{N}$, $0<p_2<p_1\leqslant\infty$, $p_1>1$, *and* $w\in\Omega_\infty$. *Then*

$$\|M\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\infty,w}(\mathbb{R}^n)} \approx \|r^{n(\frac{1}{p_2}-\frac{1}{p_1})} w(r)\|_{L_\infty(0,\infty)} \tag{26}$$

uniformly in $w\in\Omega_\infty$.

Proof. It suffices to notice that

$$\left\| t^{n(\frac{1}{p_2}-\frac{1}{p_1})} \left\| \left(\frac{r}{r+t}\right)^{\frac{n}{p_2}} w(r)\right\|_{L_\infty(0,\infty)}\right\|_{L_\infty(0,\infty)}$$

$$= \left\|\left\| t^{n(\frac{1}{p_2}-\frac{1}{p_1})} \left(\frac{r}{r+t}\right)^{\frac{n}{p_2}} \right\|_{L_\infty(0,\infty)} w(r)\right\|_{L_\infty(0,\infty)}$$

$$= \left\|\xi^{n(\frac{1}{p_2}-\frac{1}{p_1})}(1+\xi)^{-\frac{n}{p_2}}\right\|_{L_\infty(0,\infty)}\cdot \left\|r^{n(\frac{1}{p_2}-\frac{1}{p_1})} w(r)\right\|_{L_\infty(0,\infty)}.$$

(In the first line the first $L_\infty(0,\infty)$-norm is taken in r and the second one in t. In the second line the order of these norms is changed.) $\square$

Let H be the Hardy operator

$$(Hg)(r) = \int_0^r g(t)dt, \quad 0 < r < \infty.$$

THEOREM 3. *Let* $n \in \mathbb{N}$, $0 < p_2 \leqslant p_1 \leqslant \infty$, $p_2 < \infty$, $0 < \theta \leqslant \infty$, *and* $w \in \Omega_\theta$.
1. *If* $p_1 > 1$, *then*

$$\|M\|_{L_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} \approx \|H\|^{\frac{1}{p_2}}_{L^{\downarrow}_{\frac{p_1}{p_2}}(0,\infty)\to L_{\frac{\theta}{p_2},v}(0,\infty)} = \|H\|^{\frac{1}{p_2}}_{L_{\frac{p_1}{p_2}}(0,\infty)\to L_{\frac{\theta}{p_2},v}(0,\infty)} \tag{27}$$

uniformly in $w \in \Omega_\theta$, *where*

$$v(r) = \left(w(r^{\frac{1}{n}})r^{\frac{1}{\theta}(\frac{1}{n}-1)}\right)^{p_2}, \quad 0 < r < \infty. \tag{28}$$

2. *If* $p_1 = p_2 = 1$, *then*

$$\|M\|_{L_1(\mathbb{R}^n)\to WLM_{1\theta,w}(\mathbb{R}^n)} \approx \|H\|_{L_1^{\downarrow}(0,\infty)\to L_{\theta,v}(0,\infty)} = \|H\|_{L_1(0,\infty)\to L_{\theta,v}(0,\infty)} \tag{29}$$

uniformly in $w \in \Omega_\theta$.

Proof. **1.** We apply Theorem 2. First we prove that for all the parameters under consideration

$$\|I\|_{L^{\downarrow}_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} = c\,\|H\|^{\frac{1}{p_2}}_{L^{\downarrow}_{\frac{p_1}{p_2}}(0,\infty)\to L_{\frac{\theta}{p_2},v}(0,\infty)},$$

where $c > 0$ depends only on n, p_1, p_2, and θ.

Let $f \in L^{\downarrow}_{p_1}(\mathbb{R}^n)$, hence $f(x) = g(|x|)$ where g is a non-negative non-increasing function on $(0,\infty)$. By taking the spherical coordinates it follows that

$$\begin{aligned}\|f\|_{L_{p_2}(B_r)} &= \left(\int\limits_{B_r} g(|x|)^{p_2}\,dx\right)^{\frac{1}{p_2}} = (nv_n)^{\frac{1}{p_2}}\left(\int\limits_0^r g(\rho)^{p_2}\rho^{n-1}\,d\rho\right)^{\frac{1}{p_2}}\\ &= v_n^{\frac{1}{p_2}}\left(\int\limits_0^{r^n} g(\tau^{\frac{1}{n}})^{p_2}\,d\tau\right)^{\frac{1}{p_2}} = v_n^{\frac{1}{p_2}}\left((H\varphi)(r^n)\right)^{\frac{1}{p_2}},\end{aligned}$$

where

$$\varphi(\tau) = g\left(\tau^{\frac{1}{n}}\right)^{p_2}, \quad \tau \in (0,\infty),$$

is a non-negative non-increasing function on $(0,\infty)$.

Hence

$$\|f\|_{LM_{p_2\theta,w}(\mathbb{R}^n)} = v_n^{\frac{1}{p_2}} \|w(r)\big((H\varphi)(r^n)\big)^{\frac{1}{p_2}}\|_{L_\theta(0,\infty)}$$
$$= v_n^{\frac{1}{p_2}} \|w(r)^{p_2}(H\varphi)(r^n)\|^{\frac{1}{p_2}}_{L_{\frac{\theta}{p_2}}(0,\infty)}$$
$$= v_n^{\frac{1}{p_2}} n^{-\frac{1}{\theta}} \|vH\varphi\|^{\frac{1}{p_2}}_{L_{\frac{\theta}{p_2}}(0,\infty)} = v_n^{\frac{1}{p_2}} n^{-\frac{1}{\theta}} \|H\varphi\|^{\frac{1}{p_2}}_{L_{\frac{\theta}{p_2},v}(0,\infty)}.$$

Also

$$\|f\|_{L_{p_1}(\mathbb{R}^n)} = v_n^{\frac{1}{p_1}} \left(\int_0^\infty g(\tau^{\frac{1}{n}})^{p_1}\, d\tau\right)^{\frac{1}{p_1}} = v_n^{\frac{1}{p_1}} \|\varphi\|^{\frac{1}{p_2}}_{L_{\frac{p_1}{p_2}}(0,\infty)}.$$

Therefore, since $f \in L^{\downarrow}_{p_1}(\mathbb{R}^n) \Longleftrightarrow \varphi \in L^{\downarrow}_{\frac{p_1}{p_2}}(0,\infty)$,

$$\|I\|_{L^{\downarrow}_{p_1}(\mathbb{R}^n)\to LM_{p_2\theta,w}(\mathbb{R}^n)} = \sup_{\substack{f\in L^{\downarrow}_{p_1}(\mathbb{R}^n)\\ f\not\sim 0}} \frac{\|f\|_{LM_{p_2\theta,w}(\mathbb{R}^n)}}{\|f\|_{L_{p_1}(\mathbb{R}^n)}}$$
$$= n^{-\frac{1}{\theta}} v_n^{\frac{1}{p_2}-\frac{1}{p_1}} \left(\sup_{\substack{\varphi\in L^{\downarrow}_{\frac{p_1}{p_2}}(0,\infty)\\ \varphi\not\sim 0}} \frac{\|H\varphi\|_{L_{\frac{\theta}{p_2},v}(0,\infty)}}{\|\varphi\|_{L_{\frac{p_1}{p_2}}(0,\infty)}}\right)^{\frac{1}{p_2}}$$
$$= n^{-\frac{1}{\theta}} v_n^{\frac{1}{p_2}-\frac{1}{p_1}} \|H\|^{\frac{1}{p_2}}_{L^{\downarrow}_{\frac{p_1}{p_2}}(0,\infty)\to L_{\frac{\theta}{p_2},v}(0,\infty)}.$$

The equivalences in (27) and (29) follow by Theorem 2.

2. The equalities in (27) and (29) easily follow, similarly to Step 2 of the proof of Theorem 2, since for all $r > 0$

$$|(Hf)(r)| \leqslant \int_0^r |f(t)|\,dt \leqslant \int_0^r f^*(t)\,dt = (Hf^*)(r), \quad r \in (0,\infty).$$

Indeed, for all the parameters under consideration,

$$\|H\|_{L^{\downarrow}_{\frac{p_1}{p_2}}(0,\infty)\to L_{\frac{\theta}{p_2},v}(0,\infty)} \leqslant \|H\|_{L_{\frac{p_1}{p_2}}(0,\infty)} \to L_{\frac{\theta}{p_2},v}(0,\infty)$$
$$= \sup_{\substack{f\in L_{\frac{p_1}{p_2}}(0,\infty)\\ f\not\sim 0}} \frac{\|Hf\|_{L_{\frac{\theta}{p_2},v}(0,\infty)}}{\|f\|_{L_{\frac{p_1}{p_2}}(0,\infty)}} \leqslant \sup_{\substack{f\in L_{\frac{p_1}{p_2}}(0,\infty)\\ f\not\sim 0}} \frac{\|Hf^*\|_{L_{\frac{\theta}{p_2},v}(0,\infty)}}{\|f^*\|_{L_{\frac{p_1}{p_2}}(0,\infty)}}$$
$$\leqslant \sup_{\substack{g\in L^{\downarrow}_{\frac{p_1}{p_2}}(0,\infty)\\ g\not\sim 0}} \frac{\|Hg\|_{L_{\frac{\theta}{p_2},v}(0,\infty)}}{\|g\|_{L_{\frac{p_1}{p_2}}(0,\infty)}} = \|H\|_{L^{\downarrow}_{\frac{p_1}{p_2}}(0,\infty)\to L_{\frac{\theta}{p_2},v}(0,\infty)}. \quad \square$$

In the proof of the next theorem we shall use the following corollary of the Hardy inequality.

LEMMA 5. *Let $\alpha > \beta > 0$ if $1 \leqslant s < \infty$ and $\alpha \geqslant \beta > 0$ if $s = \infty$. Then for each function ψ non-negative and measurable on $(0,\infty)$*

$$\left\| t^{-\beta-\frac{1}{s}} \int_0^t \tau^\alpha \psi(\tau)\, d\tau \right\|_{L_s(0,\infty)} \leqslant \frac{\alpha}{\beta} \left\| t^{\alpha-\beta-\frac{1}{s}} \int_t^\infty \psi(\tau)\, d\tau \right\|_{L_s(0,\infty)}. \tag{30}$$

Proof. Note that for all $t > 0$

$$\int_0^t \left(\int_\tau^\infty \psi(\xi)\, d\xi \right) \tau^{\alpha-1}\, d\tau \geqslant \int_0^t \left(\int_0^\xi \tau^{\alpha-1}\, d\tau \right) \psi(\xi)\, d\xi = \frac{1}{\alpha} \int_0^t \xi^\alpha \psi(\xi)\, d\xi .$$

Hence, by applying the Hardy inequality, we get

$$\left\| t^{-\beta-\frac{1}{s}} \int_0^t \tau^\alpha \psi(\tau)\, d\tau \right\|_{L_s(0,\infty)}$$

$$\leqslant \alpha \left\| t^{-\beta-\frac{1}{s}} \int_0^t \left(\int_\tau^\infty \psi(\xi)\, d\xi \right) \tau^{\alpha-1}\, d\tau \right\|_{L_s(0,\infty)} \leqslant \frac{\alpha}{\beta} \left\| t^{\alpha-\beta-\frac{1}{s}} \int_t^\infty \psi(\tau)\, d\tau \right\|_{L_s(0,\infty)}. \qquad \square$$

THEOREM 4. *Let $n \in \mathbb{N}$, $0 < p_2 \leqslant p_1 \leqslant \infty$, $0 < \theta \leqslant \infty$, and $w \in \Omega_\theta$.*

1. *If $1 < p_2 = p_1$, $0 < \theta \leqslant \infty$ or $0 < p_2 < p_1$, $p_1 > 1$, $\theta = \infty$, then*

$$\|M\|_{L_{p_1}(\mathbb{R}^n) \to LM_{p_2\theta,w}(\mathbb{R}^n)} \approx \left\| r^{n(\frac{1}{p_2}-\frac{1}{p_1})} w(r) \right\|_{L_\theta(0,\infty)} \tag{31}$$

uniformly in $w \in \Omega_\theta$.

In particular, if $1 < p \leqslant \infty$, $0 < \theta \leqslant \infty$, then

$$\|M\|_{L_p(\mathbb{R}^n) \to LM_{p\theta,w}(\mathbb{R}^n)} \approx \left\| w(r) \right\|_{L_\theta(0,\infty)} \tag{32}$$

uniformly in $w \in \Omega_\theta$. Also for all $0 < \theta \leqslant \infty$

$$\|M\|_{L_1(\mathbb{R}^n) \to WLM_{1\theta,w}(\mathbb{R}^n)} \approx \left\| w(r) \right\|_{L_\theta(0,\infty)} \tag{33}$$

uniformly in $w \in \Omega_\infty$.

2. *If $0 < p_2 < p_1$, $p_1 > 1$, and $\theta < \infty$, then*

$$\|M\|_{L_{p_1}(\mathbb{R}^n) \to LM_{p_2\theta,w}(\mathbb{R}^n)} \approx \| t^{n(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{s}} \|w(r)\|_{L_\theta(t,\infty)} \|_{L_s(0,\infty)} \tag{34}$$

$$\approx \left\| t^{n(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{s}} \left\| \left(\frac{r}{r+t} \right)^{\frac{n}{p_2}} w(r) \right\|_{L_\theta(0,\infty)} \right\|_{L_s(0,\infty)} \tag{35}$$

uniformly in $w \in \Omega_\theta$, *where*

$$s = \begin{cases} \frac{p_1\theta}{p_1-\theta} & \text{if } \theta < p_1\,, \\ \infty & \text{if } \theta \geqslant p_1\,. \end{cases} \tag{36}$$

(*Here the semi-norm* $\|\cdot\|_{L_\theta(0,\infty)}$ *is taken in the variable* r *and the semi-norm* $\|\cdot\|_{L_s(0,\infty)}$ *in the variable* t.)

Proof. **1.** If $p_1 = p_2 \geqslant 1$, equivalences (32) and (33) are proved in Remark 3. If $0 < p_2 < p_1$ and $\theta = \infty$, then equivalence (31) is proved in Corollary 2.

2. In the rest of the proof v is the function defined by formula (28). If $\theta \geqslant p_1$ (hence $\frac{p_1}{p_2} \leqslant \frac{\theta}{p_2}$), by Theorem 2 on page 42 of [8] it follows that uniformly in $w \in \Omega_\theta$

$$\|H\|^{\frac{1}{p_2}}_{L_{\frac{p_1}{p_2}}(0,\infty)\to L_{\frac{\theta}{p_2},v}(0,\infty)} \approx \sup_{t>0} t^{\frac{1}{p_2}-\frac{1}{p_1}} \left(\int_t^\infty v(\rho)^{\frac{\theta}{p_2}}\, d\rho \right)^{\frac{1}{\theta}}$$

$$= \sup_{t>0} t^{\frac{1}{p_2}-\frac{1}{p_1}} \left(\int_t^\infty w(\rho^{\frac{1}{n}})^\theta \rho^{\frac{1}{n}-1}\, d\rho \right)^{\frac{1}{\theta}} = n^{-\frac{1}{\theta}} \sup_{t>0} t^{\frac{1}{p_2}-\frac{1}{p_1}} \left(\int_{t^{\frac{1}{n}}}^\infty w(r)^\theta\, dr \right)^{\frac{1}{\theta}}$$

$$= n^{-\frac{1}{\theta}} \|t^{n(\frac{1}{p_2}-\frac{1}{p_1})} \|w(r)\|_{L_\theta(t,\infty)}\|_{L_\infty(0,\infty)}\,. \tag{37}$$

3. Let $\theta < p_1$ (hence $\frac{p_1}{p_2} > \frac{\theta}{p_2}$). Since $\frac{p_1}{p_2} > 1$ by Theorem 1 on page 47 of [8] if $\frac{\theta}{p_2} \geqslant 1$ and by Theorem 2.4 of [11] if $\frac{\theta}{p_2} < 1$ it follows that uniformly in $w \in \Omega_\theta$

$$\|H\|^{\frac{1}{p_2}}_{L_{\frac{p_1}{p_2}}(0,\infty)\to L_{\frac{\theta}{p_2},v}(0,\infty)} \approx \|t^{n(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{s}} \|w(r)\|_{L_\theta(t,\infty)}\|_{L_s(0,\infty)}\,. \tag{38}$$

By Theorem 3 equivalences (37) and (38) imply equivalence (34).

4. Equivalence (35) follows by Lemma 5 since

$$\left\| t^{n(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{s}} \left\| \left(\frac{r}{r+t}\right)^{\frac{n}{p_2}} w(r) \right\|_{L_\theta(0,\infty)} \right\|_{L_s(0,\infty)}$$

$$\approx \|t^{-\frac{n}{p_1}-\frac{1}{s}} \|r^{\frac{n}{p_2}} w(r)\|_{L_\theta(0,t)}\|_{L_s(0,\infty)} + \|t^{n(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{s}} \|w(r)\|_{L_\theta(t,\infty)}\|_{L_s(0,\infty)}$$

uniformly in $w \in \Omega_\theta$ and by inequality (30)

$$\|t^{-\frac{n}{p_1}-\frac{1}{s}} \|r^{\frac{n}{p_2}} w(r)\|_{L_\theta(0,t)}\|_{L_s(0,\infty)} = \left\| t^{-\frac{n\theta}{p_1}-\frac{\theta}{s}} \int_0^t r^{\frac{n\theta}{p_2}} w(r)^\theta\, dr \right\|^{\frac{1}{\theta}}_{L_{\frac{s}{\theta}}(0,\infty)}$$

$$\leqslant \left(\frac{p_1}{p_2}\right)^{\frac{1}{\theta}} \left\| t^{n\theta(\frac{1}{p_2}-\frac{1}{p_1})-\frac{\theta}{s}} \int_t^\infty w(r)^\theta\, dr \right\|^{\frac{1}{\theta}}_{L_{\frac{s}{\theta}}(0,\infty)}$$

$$= \left(\frac{p_1}{p_2}\right)^{\frac{1}{\theta}} \|t^{n(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{s}} \|w(r)\|_{L_\theta(t,\infty)}\|_{L_s(0,\infty)}\,.$$

(Note that $\frac{s}{\theta} = \frac{p_1}{p_1-\theta} > 1$.) $\square$

REMARK 6. Since $\theta \leqslant s$, the right-hand side of equivalence (35) does not exceed the right-hand side of equivalence (31) which conforms with Remark 4. Indeed by the appropriate inequality for the mixed semi-norms

$$\begin{aligned}
&\left\| t^{n(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{s}} \left\| \left(\frac{r}{r+t}\right)^{\frac{n}{p_2}} w(r) \right\|_{L_{\theta,r}(0,\infty)} \right\|_{L_{s,t}(0,\infty)} \\
\leqslant &\left\| \left\| t^{n(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{s}} \left(\frac{r}{r+t}\right)^{\frac{n}{p_2}} w(r) \right\|_{L_{s,t}(0,\infty)} \right\|_{L_{\theta,r}(0,\infty)} \\
= &\left\| \xi^{n(\frac{1}{p_2}-\frac{1}{p_1})-\frac{1}{s}} (1+\xi)^{-\frac{n}{p_2}} \right\|_{L_s(0,\infty)} \cdot \left\| r^{n(\frac{1}{p_2}-\frac{1}{p_1})} w(r) \right\|_{L_\theta(0,\infty)}.
\end{aligned} \tag{39}$$

EXAMPLE 1. Let $n \in \mathbb{N}$, $0 < p_2 \leqslant p_1 \leqslant \infty$, $p_1 > 1$, $0 < \theta \leqslant \infty$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and

$$w(r) = \begin{cases} r^{-\lambda_1} & \text{if } 0 < r \leqslant 1, \\ r^{-\lambda_2} & \text{if } 1 \leqslant r < \infty. \end{cases} \tag{40}$$

Then $w \in \Omega_\theta$ if and only if $\lambda_2 > \frac{1}{\theta}$ for $\theta < \infty$ and $\lambda_2 \geqslant 0$ for $\theta = \infty$.

Under this assumption M is bounded from $L_{p_1}(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$ if and only if

1) for $p_2 < p_1 \leqslant \theta \leqslant \infty$

$$\lambda_1 \leqslant n\Big(\frac{1}{p_2} - \frac{1}{p_1}\Big) + \frac{1}{\theta}, \quad \lambda_2 \geqslant n\Big(\frac{1}{p_2} - \frac{1}{p_1}\Big) + \frac{1}{\theta}, \tag{41}$$

2) for $p_2 < p_1$, $\theta < p_1$

$$\lambda_1 < n\Big(\frac{1}{p_2} - \frac{1}{p_1}\Big) + \frac{1}{\theta}, \quad \lambda_2 > n\Big(\frac{1}{p_2} - \frac{1}{p_1}\Big) + \frac{1}{\theta}, \tag{42}$$

3) for $p_2 = p_1$

$$\lambda_1 \leqslant 0 \text{ if } \theta = \infty, \quad \lambda_1 < \frac{1}{\theta} \text{ if } \theta < \infty \tag{43}$$

(if $p_2 = p_1 = 1$, this condition is necessary and sufficient for the boundedness of M from $L_1(\mathbb{R}^n)$ to $WLM_{1\theta,w}(\mathbb{R}^n)$).

EXAMPLE 2. (Particular case of Example 1.) Let $n \in \mathbb{N}$, $0 < p_2 \leqslant p_1 \leqslant \infty$, $p_1 > 1$, $0 < \theta \leqslant \infty$, $\lambda > 0$ for $\theta < \infty$ and $\lambda \geqslant 0$ for $\theta = \infty$.

Then M is bounded from $L_{p_1}(\mathbb{R}^n)$ to $LM^\lambda_{p_2\theta}(\mathbb{R}^n) \equiv LM_{p_2\theta, r^{-\lambda-\frac{1}{\theta}}}(\mathbb{R}^n)$ if and only if

$$p_1 \leqslant \theta \quad \text{and} \quad \lambda = n\Big(\frac{1}{p_2} - \frac{1}{p_1}\Big).$$

(The necessity of the above equality also easily follows by the dilation argument.)

If $p_1 = p_2 = p > 1$, then M is bounded from $L_p(\mathbb{R}^n)$ to $LM^\lambda_{p\theta}(\mathbb{R}^n)$ only in the trivial case $\theta = \infty$ and $\lambda = 0$. Similarly, if $p_1 = p_2 = 1$, then M is bounded from $L_1(\mathbb{R}^n)$ to $WLM^\lambda_{1\theta}(\mathbb{R}^n)$ only in this trivial case.

EXAMPLE 3. Let $n \in \mathbb{N}$, $0 < p_2 < p_1$, $p_1 > 1$, $0 < \theta < p_1$, $\gamma \in \mathbb{R}$, and

$$w(r) = r^{-n(\frac{1}{p_2} - \frac{1}{p_1}) + \frac{1}{\theta}} (1 + |\ln r|)^{-\gamma}.$$

Then M is bounded from $L_{p_1}(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$ if and only if $\gamma > \frac{1}{\theta} - \frac{1}{p_1}$.

REMARK 7. Examples 1 and 3 imply, in particular, that the right-hand side of equivalence (34) is not equivalent to the right-hand side of equivalence (31) for all $0 < p_2 < p_1 \leqslant \infty$, $p_1 > 1$, and $0 < \theta < \infty$.

Open problem. *Find necessary and sufficient conditions on a function $w \in \Omega_\theta$ ensuring that the maximal operator M is bounded from $L_1(\mathbb{R}^n)$ to $LM_{p_2\theta,w}(\mathbb{R}^n)$ where $0 < p_2 < 1$ and $0 < \theta \leqslant \infty$.*

Some comments. Condition (8) with $p_1 = 1$ is still a sufficient condition for the boundedness of M in this case. However, the technique used in this paper does not allow obtaining necessary and sufficient conditions on w ensuring the boundedness. The reason for that is that in this case

$$\|M\|_{L_1(\mathbb{R}^n) \to LM_{p_2\theta,w}(\mathbb{R}^n)} \approx \|I\|_{WL_1^{\downarrow\downarrow}(\mathbb{R}^n) \to LM_{p_2\theta,w}(\mathbb{R}^n)}$$

uniformly in $w \in \Omega_\theta$, where $WL_1^{\downarrow\downarrow}(\mathbb{R}^n)$ is the subspace of $WL_1(\mathbb{R}^n)$ consisting of all functions of the form Mf with $f \in L_1^{\downarrow}(\mathbb{R}^n)$. (This follows by Theorem 2 and by an argument similar to the one of Step 3 of the proof of that theorem based on the equivalence $\|f\|_{L_1(\mathbb{R}^n)} \approx \|Mf\|_{WL_1(\mathbb{R}^n)}$.)

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