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ON THE p–MIXED AFFINE SURFACE AREA

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Abstract. Some new inequalities for i-th mixed p-affine surface area are established. The results in special cases yield some of the recent results on inequalities of this type.

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$ $(n > 2)$. Let $\mathscr{K}^n$ denote the subset of all convex bodies (compact, convex subsets with the origin in its interiors) in $\mathbb{R}^n$. Let $\mathscr{K}_s^n$ denote the set of origin-symmetric convex bodies in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n. We use $V(K)$ for the n-dimensional volume of convex body K. Let $h(K,\cdot): S^{n-1} \to \mathbb{R}$, denote the support function of $K \in \mathscr{K}^n$; i.e. for $u \in S^{n-1}$

$$h(K,u) = Max\{u \cdot x : x \in K\},$$

where $u \cdot x$ denotes the usual inner product u and x in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathscr{K}^n$, i.e., for $K, L \in \mathscr{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

1. Notation and preliminaries

1.1. The i-th mixed affine surface area

A convex body K is said to have a positive continuous curvature function, $f(K,\cdot): S^{n-1} \to [0,\infty)$, if for each $L \in \mathscr{K}^n$, the mixed volume $V_1(K,L) = (K,\ldots,K,L)$ has the integral representation (see [1]).

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} f(K,u)h(L,u)dS(u),$$

where dS is the $(n-1)$-dimensional volume element on S^{n-1}.

The subset of $\mathscr{K}^n$ consisting of bodies which have a positive continuous curvature function will be denoted by $\mathscr{F}^n$. Let $\mathscr{F}_s^n$ denote the set of all bodies in $\mathscr{K}_s^n$, and have

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a positive continuous curvature function. The ith mixed affine surface area of $K \in \mathscr{F}^n$, $\Omega_i(K)$, is defined by

$$\Omega_i(K) = \frac{1}{n}\int_{S^{n-1}} f(K,u)^{\frac{n-i}{n+1}} dS(u), \quad i \in \mathbb{R}. \tag{1.1}$$

In 1987, Lutwak [2] established the following circle inequality and Brunn-Minkowski inequality for i-th mixed affine surface area, respectively.

THEOREM 1.1. *Let $K \in \mathscr{F}^n$, $i, j, k \in \mathbb{R}$, and $j < i < k$, then*

$$\Omega_i(K) \leqslant \Omega_j(K)^{\frac{k-i}{k-j}} \Omega_k(K)^{\frac{i-j}{k-j}}, \tag{1.2}$$

with inequality if and only if K is a ball.

THEOREM 1.2. *Let $K, L \in \mathscr{F}^n$, $i, j, k \in \mathbb{R}$.*
(i) *If $i < -1$, then*

$$\Omega_i(K \breve{+} L)^{\frac{n+1}{n-i}} \leqslant \Omega_i(K)^{\frac{n+1}{n-i}} + \Omega_i(K)^{\frac{n+1}{n-i}}, \tag{1.3}$$

with inequality if and only if K and L are homothetic.
(ii) *If $i > -1$, then*

$$\Omega_i(K \breve{+} L)^{\frac{n+1}{n-i}} \geqslant \Omega_i(K)^{\frac{n+1}{n-i}} + \Omega_i(K)^{\frac{n+1}{n-i}}, \tag{1.4}$$

with inequality if and only if K and L are homothetic. The sum $\breve{+}$ is Blaschke sum.

The definition of Blaschke combination for convex bodies can be stated that (see e.g. [3]): For $K, L \in \mathscr{K}^n$, the Blaschke combination of K and L, $K \breve{+} L \in \mathscr{K}^n$, defined by

$$S(K \breve{+} L, \cdot) = S(K, \cdot) + S(L, \cdot),$$

where $S(K, \cdot)$ is a positive Borel measure on S^{n-1}, called the surface area measure of convex body K (see e.g. [4]).

1.2. The p-mixed affine surface area

In 1996, p-mixed affine surface area of $K \in \mathscr{F}^n$, $\Omega_{p,i}(K)$ $(p \geqslant 1)$, is defined by (see [5])

$$\Omega_{p,i}(K) = \frac{1}{n}\int_{S^{n-1}} f_p(K,u)^{\frac{n-i}{n+p}} dS(u), \quad i \in \mathbb{R}, \tag{1.5}$$

where $f_p(K,u)$ is L_p-curvature function. A convex body $K \in \mathscr{K}^n$ is said to have a L_p-curvature function $f_p(K, \cdot) : S^{n-1} \to \mathbb{R}$, if its L_p-surface area measure $S_p(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS_p(K, \cdot)}{dS} = f_p(K, \cdot). \tag{1.6}$$

Moreover, L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure $S(K,\cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} = h(K,\cdot)^{1-p}.$$

In 2007, inequalities (1.2), (1.3) and (1.4) were extended to the following inequalities, respectively (see [6]).

THEOREM 1.3. *Let $K \in \mathscr{F}^n$, $p \geqslant 1$, $i,j,k \in \mathbb{R}$ and $j < i < k$, then*

$$\Omega_{p,i}(K) \leqslant \Omega_{p,j}(K)^{\frac{k-i}{k-j}} \Omega_{p,k}(K)^{\frac{i-j}{k-j}}, \tag{1.7}$$

with inequality if and only if K is a ball.

THEOREM 1.4. *Let $K,L \in \mathscr{F}_s^n$, $p \geqslant 1$, $i,j,k \in \mathbb{R}$.*
(i) *if $i < -p$, then*

$$\Omega_{p,i}(K \breve{+}_p L)^{\frac{n+p}{n-i}} \leqslant \Omega_{p,j}(K)^{\frac{n+p}{n-i}} + \Omega_{p,k}(K)^{\frac{n+p}{n-i}}, \tag{1.8}$$

with inequality if and only if K and L are homothetic.
(ii) *If $i > -p$, then*

$$\Omega_{p,i}(K \breve{+}_p L)^{\frac{n+p}{n-i}} \geqslant \Omega_{p,j}(K)^{\frac{n+p}{n-i}} + \Omega_{p,k}(K)^{\frac{n+p}{n-i}}, \tag{1.9}$$

with inequality if and only if K and L are homothetic. The sum $\breve{+}_p$ is L_p-Blaschke sum.

The definition of L_p-Blaschke combination for convex bodies was given by Lutwak (see [7]). For $K,L \in \mathscr{K}_s^n$ and $n \neq p \geqslant 1$, L_p-Blaschke combination of K and L, $K \breve{+}_p L \in \mathscr{K}_s^n$, defined by

$$S_p(K \breve{+}_p L,\cdot) = S_p(K,\cdot) + S_p(L,\cdot). \tag{1.10}$$

2. Statement of results

The aim of the present paper is to establish the following new inequalities for mixed p-affine surface area. Our results in special cases yield (1.7), (1.8) and (1.9), respectively.

THEOREM 2.1. *Let $K,L \in \mathscr{F}_s^n$ and let $i \neq j$, $p \geqslant 1$ and $i,j,p \in \mathbb{R}$.*
(i) *If $i \leqslant -p \leqslant j \leqslant n$, then*

$$\left(\frac{\Omega_{p,i}(K \breve{+}_p L)}{\Omega_{p,j}(K \breve{+}_p L)}\right)^{\frac{n+p}{j-i}} \leqslant \left(\frac{\Omega_{p,i}(K)}{\Omega_{p,j}(K)}\right)^{\frac{n+p}{j-i}} + \left(\frac{\Omega_{p,i}(L)}{\Omega_{p,j}(L)}\right)^{\frac{n+p}{j-i}}, \tag{2.1}$$

with equality if and only if K and L are homothetic.

(ii) *If* $-p \leqslant i \leqslant n \leqslant j$, *then*

$$\left(\frac{\Omega_{p,i}(K\breve{+}_pL)}{\Omega_{p,j}(K\breve{+}_pL)}\right)^{\frac{n+p}{j-i}} \geqslant \left(\frac{\Omega_{p,i}(K)}{\Omega_{p,j}(K)}\right)^{\frac{n+p}{j-i}} + \left(\frac{\Omega_{p,i}(L)}{\Omega_{p,j}(L)}\right)^{\frac{n+p}{j-i}}, \tag{2.2}$$

with equality if and only if K *and* L *are homothetic.*

REMARK 2.2. Taking for $j=n$ in (2.1) and (2.2), and from (1.1) and in view of $\int_{S^{n-1}} dS(u) = n\omega_n$ is a constant, then (2.1) and (2.2) change to (1.8) and (1.9), respectively.

REMARK 2.3. Taking for $p=1$ in Theorem 2.1, we get the following result.
If $K,\, L \in \mathscr{F}^n$ and let $i \neq j,\ i,j \in \mathbb{R}$.
(i) If $i \leqslant -1 \leqslant j \leqslant n$, then

$$\left(\frac{\Omega_i(K\breve{+}L)}{\Omega_j(K\breve{+}L)}\right)^{\frac{n+1}{j-i}} \leqslant \left(\frac{\Omega_i(K)}{\Omega_j(K)}\right)^{\frac{n+1}{j-i}} + \left(\frac{\Omega_i(L)}{\Omega_j(L)}\right)^{\frac{n+1}{j-i}}, \tag{2.3}$$

with equality if and only if K and L are homothetic.
(i) If $-1 \leqslant i \leqslant n \leqslant j$, then

$$\left(\frac{\Omega_i(K\breve{+}L)}{\Omega_j(K\breve{+}L)}\right)^{\frac{n+1}{j-i}} \geqslant \left(\frac{\Omega_i(K)}{\Omega_j(K)}\right)^{\frac{n+1}{j-i}} + \left(\frac{\Omega_i(L)}{\Omega_j(L)}\right)^{\frac{n+1}{j-i}}, \tag{2.4}$$

with equality if and only if K and L are homothetic.
Taking for $j=n$ in (2.3) and (2.4), (2.3) and (2.4) change to (1.3) and (1.4), respectively.

REMARK 2.4. Taking for $i=-p$ and $j=0$ in (2.1), (2.1) changes to the following interesting result.

$$\left(\frac{\Omega_{p,-p}(K\breve{+}_pL)}{\Omega_p(K\breve{+}_pL)}\right)^{\frac{n+p}{p}} \leqslant \left(\frac{\Omega_{p,-p}(K)}{\Omega_p(K)}\right)^{\frac{n+p}{p}} + \left(\frac{\Omega_{p,-p}(L)}{\Omega_p(L)}\right)^{\frac{n+p}{p}},\ \ p \geqslant 1,$$

with equality if and only if K and L are homothetic.
Taking for $i=-n$ and $j=0$ in (2.1), (2.1) changes to the following interesting result.

$$\left(\frac{\Omega_{p,-n}(K\breve{+}_pL)}{\Omega_p(K\breve{+}_pL)}\right)^{\frac{n+p}{n}} \leqslant \left(\frac{\Omega_{p,-n}(K)}{\Omega_p(K)}\right)^{\frac{n+p}{n}} + \left(\frac{\Omega_{p,-n}(L)}{\Omega_p(L)}\right)^{\frac{n+p}{n}},\ \ 1 \leqslant p \leqslant n,$$

with equality if and only if K and L are homothetic.
On the other hand, taking for $i=0$ and $j=n$ in (2.2), (2.2) changes to the following interesting result.

$$\Omega_p(K\breve{+}_pL)^{\frac{n+p}{n}} \geqslant \Omega_p(K)^{\frac{n+p}{n}} + \Omega_p(L)^{\frac{n+p}{n}},\ \ p \geqslant 1 \tag{2.5}$$

with equality if and only if K and L are homothetic.

Taking for $p=1$ in (2.5), (2.5) becomes the following inequality established by Lutwak[8].

$$\Omega(K\breve{+}L)^{\frac{n+1}{n}} \geqslant \Omega(K)^{\frac{n+1}{n}} + \Omega(L)^{\frac{n+1}{n}},$$

with equality if and only if K and L are homothetic.

REMARK 2.5. Taking for $i=-p$ and $j=n$ in (2.1) and (2.2), then (2.1) and (2.2) change to the following result, respectively.

$$\Omega_{p,-p}(K\breve{+}_pL) \leqslant \Omega_{p,-p}(K)+\Omega_{p,-p}(L), \ \ p\geqslant 1, \tag{2.6}$$

with equality if and only if K and L are homothetic.

$$\Omega_{p,-p}(K\breve{+}_pL) \geqslant \Omega_{p,-p}(K)+\Omega_{p,-p}(L), \ \ p\geqslant 1, \tag{2.7}$$

with equality if and only if K and L are homothetic.

From (2.6) and (2.7), we get the following interesting result.

$$\Omega_{p,-p}(K\breve{+}_pL) = \Omega_{p,-p}(K)+\Omega_{p,-p}(L), \ \ p\geqslant 1.$$

3. Proof of results

LEMMA 3.1. ([9], p. 28) *If* $p\geqslant 1\geqslant r\geqslant 0,\ f,g\geqslant 0,$ *and* ϕ *is a distribution function, then*

$$\left(\frac{\int (f+g)^p d\phi}{\int (f+g)^r d\phi}\right)^{\frac{1}{p-r}} \leqslant \left(\frac{\int f^p d\phi}{\int f^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int g^p d\phi}{\int g^r d\phi}\right)^{\frac{1}{p-r}}, \tag{3.1}$$

with equality if and only if the functions f *and* g *are proportional.*

In order to prove main result, we yet need to derive a new inequality below by a similar way in [9].

LEMMA 3.2. *If* $1\geqslant p\geqslant 0\geqslant r,\ f,g\geqslant 0,$ *and* ϕ *is a distribution function, then*

$$\left(\frac{\int (f+g)^p d\phi}{\int (f+g)^r d\phi}\right)^{1/(p-r)} \geqslant \left(\frac{\int f^p d\phi}{\int f^r d\phi}\right)^{1/(p-r)} + \left(\frac{\int g^p d\phi}{\int g^r d\phi}\right)^{1/(p-r)}, \tag{3.2}$$

with equality if and only if the functions f *and* g *are proportional.*

Proof. If $\alpha_1\geqslant 0,\ \alpha_1\geqslant 0,\ \beta_1>0$ and $\beta_2>0$, and $-1<\lambda<0$, from Radon's inequality (see [10], p. 61), we have

$$\frac{\alpha_1^{\lambda+1}}{\beta_1^{\lambda}} + \frac{\alpha_2^{\lambda+1}}{\beta_2^{\lambda}} \leqslant \frac{(\alpha_1+\alpha_2)^{\lambda+1}}{(\beta_1+\beta_2)^{\lambda}}, \tag{3.3}$$

with equality if and only if (α) and (β) are proportional.

Let $\alpha_1 = (\int f^p d\phi)^{1/p}$, $\beta_1 = (\int f^r d\phi)^{1/r}$, $\alpha_2 = (\int g^p d\phi)^{1/p}$, $\beta_2 = (\int g^r d\phi)^{1/r}$, and let $\lambda = \frac{r}{p-r}$, we obtain

$$\left(\frac{\int f^p d\phi}{\int f^r d\phi}\right)^{1/(p-r)} + \left(\frac{\int g^p d\phi}{\int g^r d\phi}\right)^{1/(p-r)} \leqslant \frac{\left[(\int f^p d\phi)^{1/p} + (\int g^p d\phi)^{1/p}\right]^{p/(p-r)}}{\left[(\int f^r d\phi)^{1/r} + (\int g^r d\phi)^{1/r}\right]^{r/(p-r)}}. \tag{3.4}$$

We have assumed $p > 0 > r$, since $-1 < \lambda = \frac{r}{p-r} < 0$.

On the other hand, by Minkowski inequality with $1 \geqslant p > 0$ and $r < 0$ respectively, both

$$\left[\left(\int f^p d\phi\right)^{1/p} + \left(\int g^p d\phi\right)^{1/p}\right]^p \leqslant \int (f+g)^p d\phi, \tag{3.5}$$

with equality if and only if f and g are proportional, and

$$\left[\left(\int f^r d\phi\right)^{1/r} + \left(\int g^r d\phi\right)^{1/r}\right]^r \geqslant \int (f+g)^r d\phi. \tag{3.6}$$

with equality if and only if f and g are proportional.

From (3.4), (3.5) and (3.6), (3.2) follows. □

We will need combine inequalities (3.1) and (3.2) to prove the following Theorem.

THEOREM 3.3. *Let $K, L \in \mathscr{F}_s^n$ and let $i \neq j$, $p \geqslant 1$ and $i, j, p \in \mathbb{R}$.*
(i) *If $i \leqslant -p \leqslant j \leqslant n$, then*

$$\left(\frac{\Omega_{p,i}(K \breve{+}_p L)}{\Omega_{p,j}(K \breve{+}_p L)}\right)^{\frac{n+p}{j-i}} \leqslant \left(\frac{\Omega_{p,i}(K)}{\Omega_{p,j}(K)}\right)^{\frac{n+p}{j-i}} + \left(\frac{\Omega_{p,i}(L)}{\Omega_{p,j}(L)}\right)^{\frac{n+p}{j-i}}, \tag{3.7}$$

with equality if and only if K and L are homothetic.
(ii) *If $-p \leqslant i \leqslant n \leqslant j$, then*

$$\left(\frac{\Omega_{p,i}(K \breve{+}_p L)}{\Omega_{p,j}(K \breve{+}_p L)}\right)^{\frac{n+p}{j-i}} \geqslant \left(\frac{\Omega_{p,i}(K)}{\Omega_{p,j}(K)}\right)^{\frac{n+p}{j-i}} + \left(\frac{\Omega_{p,i}(L)}{\Omega_{p,j}(L)}\right)^{\frac{n+p}{j-i}}, \tag{3.8}$$

with equality if and only if K and L are homothetic.

Proof. (i) From (1.5), (1.6) and (1.10), we have

$$\Omega_{p,i}(K \breve{+}_p L) = \int_{S^{n-1}} f_p(K \breve{+}_p L, u)^{\frac{n-i}{n+p}} dS(u) = \int_{S^{n-1}} (f_p(K,u) + f_p(L,u))^{\frac{n-i}{n+p}} dS(u) \tag{3.9}$$

and

$$\Omega_{p,j}(K \breve{+}_p L) = \int_{S^{n-1}} (f_p(K,u) + f_p(L,u))^{\frac{n-j}{n+p}} dS(u). \tag{3.10}$$

Since $i \leqslant -p \leqslant j \leqslant n$, we have

$$0 \leqslant \frac{n-j}{n+p} \leqslant 1 \leqslant \frac{n-i}{n+p}. \tag{3.11}$$

From (3.9), (3.10) and (3.11) and in view of the in Lemma 3.1, we obtain that

$$\begin{aligned}
\left(\frac{\Omega_{p,i}(K\breve{+}_pL)}{\Omega_{p,j}(K\breve{+}_pL)}\right)^{\frac{n+p}{j-i}} &= \left(\frac{\int_{S^{n-1}}(f_p(K,u)+f_p(L,u))^{\frac{n-i}{n+p}}dS(u)}{\int_{S^{n-1}}(f_p(K,u)+f_p(L,u))^{\frac{n-j}{n+p}}dS(u)}\right)^{\frac{1}{\frac{n-i}{n+p}-\frac{n-j}{n+p}}} \\
&\leqslant \left(\frac{\int_{S^{n-1}}f_p(K,u)^{\frac{n-i}{n+p}}dS(u)}{\int_{S^{n-1}}f_p(K,u)^{\frac{n-j}{n+p}}dS(u)}\right)^{\frac{n+p}{j-i}} + \left(\frac{\int_{S^{n-1}}f_p(L,u)^{\frac{p}{n+1}}dS(u)}{\int_{S^{n-1}}f_p(L,u)^{\frac{r}{n+1}}dS(u)}\right)^{\frac{n+p}{j-i}} \\
&= \left(\frac{\Omega_{p,i}(K)}{\Omega_{p,j}(K)}\right)^{\frac{n+p}{j-i}} + \left(\frac{\Omega_{p,i}(L)}{\Omega_{p,j}(L)}\right)^{\frac{n+p}{j-i}}.
\end{aligned}$$

The sign of equality holds if and only if the functions $f_p(K,u)$ and $f_p(L,u)$ are proportional. Hence, the sign of equality holds if and only if K and L are homothetic.

(ii) Similarly above proof, in view of $-p \leqslant i \leqslant n \leqslant j \Rightarrow 1 \geqslant \frac{n-i}{n+p} \geqslant 0 \geqslant \frac{n-j}{n+p}$, and from (3.9) and (3.10) and by using the inequality in Lemma 3.2, inequality (3.8) easily follows.

We finally remark that inequalities for affine area and mixed surface area were established in [11–15] and et al. Some new inequalities for mixed p-surface area were recently established in [16–19] and et al. $\square$

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