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HIGHER ORDER DYNAMIC INEQUALITIES ON TIME SCALES

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Abstract. In this paper, we will prove some new dynamic inequalities of higher orders on time scales. The results contain some continuous and discrete inequalities as special cases. We will prove the results by making use of Hölder's inequality and Taylor monomials on time scales.

1. Introduction

In this paper, we study dynamic inequalities where the domain of the unknown function is a so-called time scale $\mathbb{T}$. The cases when the time scale equals to the reals or to the integers represent the classical theories of differential and of difference inequalities. A cover story article in New Scientist [24] discusses several other possible applications. Continuous and discrete inequalities are important in the analysis of qualitative properties of solutions of differential and difference equations [3, 18, 19] and as a result we believe that dynamic inequalities on time scales will be important in the analysis of qualitative properties of solutions of dynamic equations [20, 21]. In this paper, we will prove some new dynamic inequalities involving higher order on time scales.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [9]), i.e, when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$ where $q>1$. For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [6], [7] which summarize and organize much of the time scale calculus. The study of dynamic inequalities of Opial type on time scales was initiated by Bohner and Kaymakçalan [5] in 2001; see also the recent papers [10], [23] and [25] and the references cited therein.

In [5] the authors showed that if $x:[0,b]\cap\mathbb{T}\to\mathbb{R}$ is delta differentiable with $x(0)=0$, then

$$\int_0^b |x(t)+x^\sigma(t)|\left|x^\Delta(t)\right|\Delta t \leqslant b\int_0^b \left|x^\Delta(t)\right|^2\Delta t. \tag{1.1}$$

In addition they proved that if r and q are positive rd-continuous functions on $[0,b]$, $\int_a^b(\Delta t/r(t))<\infty$, q nonincreasing and $x:[0,b]\cap\mathbb{T}\to\mathbb{R}$ is delta differentiable with $x(0)=0$, then

$$\int_0^b q^\sigma(t)\left|(x(t)+x^\sigma(t))x^\Delta(t)\right|\Delta t \leqslant \int_0^b\frac{\Delta t}{r(t)}\int_0^b r(t)q(t)\left|x^\Delta(t)\right|^2\Delta t. \tag{1.2}$$

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In [10] Karpuz, Kaymakçalan and Öcalan proved an inequality similar to the inequality (1.2) where $q^{\sigma}(t)$ is replaced by $q(t)$, namely

$$\int_a^b q(t)\left|(x(t)+x^{\sigma}(t))x^{\Delta}(t)\right|\Delta t \leqslant K_q(a,b)\int_0^b \left|x^{\Delta}(t)\right|^2\Delta t, \tag{1.3}$$

where q is a positive rd-continuous function on $[a,b]$, and $x:[a,b]\cap\mathbb{T}\to\mathbb{R}$ is delta differentiable with $x(a)=0$ and

$$K_q(a,b)=\left(2\int_a^b q^2(u)(\sigma(u)-a)\Delta u\right)^{\frac{1}{2}}. \tag{1.4}$$

In [20] the author proved that if $y:[a,X]_{\mathbb{T}}\to\mathbb{R}$ is delta differentiable with $y(a)=0$, then

$$\int_a^X s(x)\left|y(x)+y^{\sigma}(x)\right|\left|y^{\Delta}(x)\right|\Delta x\leqslant K_1(a,X)\int_a^X r(x)\left|y^{\Delta}(x)\right|^2\Delta x,$$

where $s\in C_{rd}([a,X]_{\mathbb{T}},\mathbb{R})$ and r be a positive rd-continuous function on $(a,X)_{\mathbb{T}}$ such that $\int_a^X r^{-1}(t)\Delta t<\infty$, and

$$K_1(a,X)=\sqrt{2}\left(\int_a^X \frac{s^2(x)}{r(x)}\left(\int_a^x \frac{\Delta t}{r(t)}\right)\Delta x\right)^{\frac{1}{2}}+\sup_{a\leqslant x\leqslant X}\left(\mu(x)\frac{|s(x)|}{r(x)}\right).$$

For contributions of different types of inequalities on time scales, we refer the reader to the papers [2, 16, 17] and the references cited therein. The inequalities that we will prove in this paper are inequalities on higher order derivatives. The main results will be proved by making use of the Hölder inequality (see [6, Theorem 6.13])

$$\int_a^b |f(t)g(t)|\Delta t\leqslant\left[\int_a^b |f(t)|^{\gamma}\Delta t\right]^{\frac{1}{\gamma}}\left[\int_a^b |g(t)|^{\nu}\Delta t\right]^{\frac{1}{\nu}}, \tag{1.5}$$

where $a,\ b\in\mathbb{T}$ and $f;g\ \in C_{rd}(\mathbb{T})$, $\gamma>1$ and $\frac{1}{\nu}+\frac{1}{\gamma}=1$. In our analysis we will also make use of the well known inequality (see [12, page 500])

$$|a+b|^r\leqslant 2^{r-1}\left(|a|^r+|b|^r\right), \tag{1.6}$$

where $a,\ b$ are positive real numbers and $r\geqslant 1$, and also we will use the Taylor monomials on time scales. Some special cases on continuous and discrete spaces can be derived from our results.

2. Main Results

Before we state and prove the main results, for completeness, we recall the following concepts related to the notion of time scales [8]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by: $\sigma(t):=\inf\{s\in$

$\mathbb{T}: s > t\}$, and $\rho(t) := \sup\{s \in \mathbb{T}: s < t\}$, where $\sup\emptyset = \inf\mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left–dense if $\rho(t) = t$ and $t > \inf\mathbb{T}$, is right–dense if $\sigma(t) = t$, is left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$.

A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We will assume that $\sup\mathbb{T} = \infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$.

In this paper we will refer to the (delta) integral which we can define as follows: If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [6]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. Now, we define the Taylor monomials or generalized polynomials as defined originally by Agarwal and Bohner [1]. These monomials are important because they are related to Cauchy functions for certain dynamic equations which are important in variations of constants formulas. The Taylor monomials $h_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, are defined recursively as follows: The function h_0 is defined by $h_0(t,s) = 1$, for all $s,t \in \mathbb{T}$, and given h_k for $k \in \mathbb{N}_0$, the function h_{k+1} is defined by

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau, \text{ for all } s,t \in \mathbb{T}.$$

If we let $h_k^{\Delta}(t,s)$ denote for each fixed $s \in \mathbb{T}$, the derivative of $h(t,s)$ with respect to t, then

$$h_k^{\Delta}(t,s) = h_{k-1}(t,s), \quad k \in \mathbb{N}, \ t \in \mathbb{T},$$

for each fixed $s \in \mathbb{T}$. We also consider the function g_0 defined by $g_0(t,s) = 1$, for all $s,t \in \mathbb{T}$, and given g_k for $k \in \mathbb{N}_0$, the function g_{k+1} is defined by

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau),s)\Delta\tau, \text{ for all } s,t \in \mathbb{T}.$$

If we let $g_k^{\Delta}(t,s)$ denote for each fixed $s \in \mathbb{T}$, the derivative of $g(t,s)$ with respect to t, then

$$g_k^{\Delta}(t,s) = g_{k-1}(\sigma(t),s), \quad k \in \mathbb{N}, \ t \in \mathbb{T},$$

for each fixed $s \in \mathbb{T}$. By Theorem 1.112 in [6], we see that $h_k(t,s) = (-1)^k g_k(s,t)$. We denote by $C_{rd}^{(n)}(\mathbb{T})$ the space of all functions $f \in C_{rd}(\mathbb{T})$ such that $f^{\Delta_i} \in C_{rd}(\mathbb{T})$ for $i = 0,1,2,...,n$ for $n \in \mathbb{N}$. For the function $f : \mathbb{T} \to \mathbb{R}$ we consider the second derivative f^{Δ_2} provided f^{Δ} is delta differentiable on $\mathbb{T}$ with derivative $f^{\Delta_2} = (f^{\Delta})^{\Delta}$. Similarly, we define the $n^{th}-$order derivative $f^{\Delta_n} = (f^{\Delta_{n-1}})^{\Delta}$. Now, we are ready to state the Taylor formula that we will need to prove the main results in this paper. This formula as proved in [4] states that: Assume that $f \in C_{rd}^{(n)}(\mathbb{T})$ and $s \in \mathbb{T}$, then

$$f(t) = \sum_{k=0}^{n-1} f^{\Delta_k}(s)h_k(t,s) + \int_s^t h_{n-1}(t,(\sigma(\tau)))f^{\Delta_n}(\tau)\Delta\tau. \tag{2.1}$$

As a special case if $m < n$, then

$$f^{\Delta_m}(t) = \sum_{k=0}^{n-m-1} f^{\Delta_{k+m}}(s)h_k(t,s) + \int_s^t h_{n-m-1}(t,(\sigma(\tau))f^{\Delta_n}(\tau)\Delta\tau. \tag{2.2}$$

Now, we are ready to state and prove our main results.

THEOREM 2.1. *Let* $\mathbb{T}$ *be a time scale with* $a,\ \tau \in \mathbb{T}$ *and* $p,\ q$ *be positive real numbers such that* $p > 1,\ 1/p + 1/q = 1,$ *and let* $r,\ s$ *be nonnegative rd-continuous functions on* $(a,\tau)_{\mathbb{T}}$. *If* $y \in C_{rd}^{(n)}([a,\tau]\cap\mathbb{T})$ *with* $y^{\Delta_i}(a) = 0$, *for* $i = 0,1,2,...,n-1$, *then*

$$\int_a^\tau s(t)\,|y(t)+y^\sigma(t)|^p \left|y^{\Delta_n}(t)\right|^q \Delta t \leqslant K(a,\tau,p,q)\int_a^\tau r(t)\left|y^{\Delta_n}(t)\right|^{p+q}\Delta t, \tag{2.3}$$

where $K(a,\tau,p,q) = K_1(a,\tau,p,q) + K_2(a,\tau,p,q)$,

$$K_1(a,\tau,p,q) = 2^{2p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \times \left(\int_a^\tau \frac{(s(t))^{\frac{p+q}{p}}}{(r(t))^{\frac{q}{p}}}\left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{(p+q-1)}\Delta t\right)^{\frac{p}{p+q}},$$

and

$$K_2(a,\tau,p,q) = 2^{p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \times \left(\int_a^\tau \frac{\mu^{p+q}(t)(s(t))^{\frac{p+q}{p}}}{(r(t))^{\frac{q}{p}}}\left(\int_a^t \frac{h_{n-2}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{(p+q-1)}\Delta t\right)^{\frac{p}{p+q}}.$$

Proof. From Taylor's formula (2.1), since $y^{\Delta_i}(a) = 0$, for $i = 0,1,2,...,n-1$, we have

$$y(t) = \int_a^t h_{n-1}(t,\sigma(s))y^{\Delta_n}(s)\Delta s, \quad \text{for } t \in [a,\tau]_{\mathbb{T}}. \tag{2.4}$$

This implies that

$$|y(t)| \leqslant \int_a^t \frac{h_{n-1}(t,\sigma(s))}{(r(s))^{\frac{1}{p+q}}}(r(s))^{\frac{1}{p+q}}\left|y^{\Delta_n}(s)\right|\Delta s.$$

Applying the Hölder inequality (1.5) with

$$f(s) = \frac{h_{n-1}(t,\sigma(s))}{(r(s))^{\frac{1}{p+q}}},\ g(s) = (r(s))^{\frac{1}{p+q}}\left|y^{\Delta_n}(s)\right|,$$

$$\gamma = \frac{p+q}{p+q-1} \text{ and } \nu = p+q,$$

we have

$$\int_a^t h_{n-1}(t,\sigma(s))\left|y^{\Delta_n}(s)\right|\Delta s \leqslant \left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{(r(s))^{\frac{1}{p+q-1}}}\Delta s\right)^{\frac{p+q-1}{p+q}} \times\left(\int_a^t r(s)\left|y^{\Delta_n}(s)\right|^{p+q}\Delta s\right)^{\frac{1}{p+q}}.$$

Then, for $a \leqslant t \leqslant \tau$, we have

$$|y(t)|^p \leqslant \left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{(r(s))^{\frac{1}{p+q-1}}}\Delta s\right)^{p(\frac{p+q-1}{p+q})} \left(\int_a^t r(s)\left|y^{\Delta_n}(s)\right|^{p+q}\Delta s\right)^{\frac{p}{p+q}}. \tag{2.5}$$

Since $y^\sigma = y+\mu y^\Delta$, we have

$$y(t)+y^\sigma(t) = 2y(t)+\mu y^\Delta(t).$$

Applying the inequality (1.6), we get (where $p>1$) that

$$|y+y^\sigma|^p \leqslant 2^{p-1}(2^p|y|^p+\mu^p\left|y^\Delta\right|^p) = 2^{2p-1}|y|^p+2^{p-1}\mu^p\left|y^\Delta\right|^p. \tag{2.6}$$

From (2.5), we have

$$|y(t)|^p\left|y^{\Delta_n}(t)\right|^q \leqslant \left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{(r(s))^{\frac{1}{p+q-1}}}\Delta s\right)^{p(\frac{p+q-1}{p+q})} \times\left|y^{\Delta_n}(t)\right|^q\left(\int_a^t r(s)\left|y^{\Delta_n}(s)\right|^{p+q}\Delta s\right)^{\frac{p}{p+q}}. \tag{2.7}$$

Also, by using (2.2), we have

$$\left|y^{\Delta}(t)\right|^p\left|y^{\Delta_n}(t)\right|^q \leqslant \left(\int_a^t \frac{h_{n-2}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{(r(s))^{\frac{1}{p+q-1}}}\Delta s\right)^{p(\frac{p+q-1}{p+q})} \times\left|y^{\Delta_n}(t)\right|^q\left(\int_a^t r(s)\left|y^{\Delta_n}(s)\right|^{p+q}\Delta s\right)^{\frac{p}{p+q}}. \tag{2.8}$$

Substituting (2.8) and (2.7) into (2.6), we have

$$
\begin{aligned}
s(t)\left|y(t)+y^{\sigma}(t)\right|^{p}\left|y^{\Delta_n}(t)\right|^{q} \leqslant\ & 2^{2p-1}s(t)\left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{(r(s))^{\frac{1}{p+q-1}}}\Delta s\right)^{p\left(\frac{p+q-1}{p+q}\right)} \\
& \times\left|y^{\Delta_n}(t)\right|^{q}\left(\int_a^t r(s)\left|y^{\Delta_n}(s)\right|^{p+q}\Delta s\right)^{\frac{p}{p+q}} \\
& +2^{p-1}\mu^{p}(t)s(t)\left(\int_a^t \frac{h_{n-2}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{(r(s))^{\frac{1}{p+q-1}}}\Delta s\right)^{p\left(\frac{p+q-1}{p+q}\right)} \\
& \times\left|y^{\Delta_n}(t)\right|^{q}\left(\int_a^t r(s)\left|y^{\Delta_n}(s)\right|^{p+q}\Delta s\right)^{\frac{p}{p+q}}. \qquad (2.9)
\end{aligned}
$$

Setting

$$
z(t):=\int_a^t r(s)\left|y^{\Delta_n}(s)\right|^{p+q}\Delta s, \qquad (2.10)
$$

we see that $z(a)=0,$ and

$$
z^{\Delta}(t)=r(t)\left|y^{\Delta_n}(t)\right|^{p+q}>0. \qquad (2.11)
$$

From this, we have

$$
\left|y^{\Delta_n}(t)\right|^{p+q}=\frac{z^{\Delta}(t)}{r(t)}, \quad \text{and } \left|y^{\Delta_n}(t)\right|^{q}=\left(\frac{z^{\Delta}(t)}{r(t)}\right)^{\frac{q}{p+q}}. \qquad (2.12)
$$

Since s is nonnegative on $(a,\tau),$ we have from (2.7) and (2.12) that

$$
\begin{aligned}
& 2^{2p-1}s(t)\left|y(t)\right|^{p}\left|y^{\Delta_n}(t)\right|^{q} \\
& \leqslant 2^{2p-1}s(t)\left(\frac{1}{r(t)}\right)^{\frac{q}{p+q}}\times\left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{p\left(\frac{p+q-1}{p+q}\right)} \\
& \times(z(t))^{\frac{p}{p+q}}\left(z^{\Delta}(t)\right)^{\frac{q}{p+q}}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& 2^{2p-1}\int_a^{\tau} s(t)\left|y(t)\right|^{p}\left|y^{\Delta_n}(t)\right|^{q}\Delta t \\
& \leqslant 2^{2p-1}\int_a^{\tau} s(t)\left(\frac{1}{r(t)}\right)^{\frac{q}{p+q}}\times\left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{p\left(\frac{p+q-1}{p+q}\right)} \\
& \times(z(t))^{\frac{p}{p+q}}\left(z^{\Delta}(t)\right)^{\frac{q}{p+q}}\Delta t.
\end{aligned}
$$

Applying the Hölder inequality (1.5) with indices $(p+q)/p$ and $(p+q)/q$ on the right hand side, we have

$$\begin{aligned}
&2^{2p-1}\int_a^\tau s(t)\,|y(t)|^p\left|y^{\Delta_n}(t)\right|^q\Delta t\\
&\leqslant 2^{2p-1}\left(\int_a^\tau s^{\frac{p+q}{p}}(t)\left(\frac{1}{r(t)}\right)^{\frac{q}{p}}\left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{(p+q-1)}\Delta t\right)^{\frac{p}{p+q}}\\
&\times\left(\int_a^\tau z^{\frac{p}{q}}(t)z^\Delta(t)\Delta t\right)^{\frac{q}{p+q}}.
\end{aligned}\tag{2.13}$$

From (2.11), and the chain rule formula

$$(x^\gamma(t))^\Delta=\gamma\int_0^1[hx^\sigma+(1-h)x]^{\gamma-1}\,dhx^\Delta(t),\tag{2.14}$$

which is a simple consequence of Keller's chain rule [6, Theorem 1.90], we obtain

$$z^{\frac{p}{q}}(t)z^\Delta(t)\leqslant\frac{q}{p+q}\left(z^{\frac{p+q}{q}}(t)\right)^\Delta.\tag{2.15}$$

Substituting (2.15) into (2.13) and using the fact that $z(a)=0$, we have that

$$\begin{aligned}
&2^{2p-1}\int_a^\tau s(t)\,|y(t)|^p\left|y^{\Delta_n}(t)\right|^q\Delta t\\
&\leqslant 2^{2p-1}\left(\int_a^\tau s^{\frac{p+q}{p}}(t)\left(\frac{1}{r(t)}\right)^{\frac{q}{p}}\left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{(p+q-1)}\Delta t\right)^{\frac{p}{p+q}}\\
&\times\left(\frac{p}{p+q}\right)^{\frac{q}{p+q}}\left(\int_a^\tau\left(z^{\frac{p+q}{q}}(s)\right)^\Delta\Delta s\right)^{\frac{q}{p+q}}\\
&=\left(\int_a^\tau s^{\frac{p+q}{p}}(t)\left(\frac{1}{r(t)}\right)^{\frac{q}{p}}\left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{(p+q-1)}\Delta t\right)^{\frac{p}{p+q}}\\
&\times 2^{2p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}}z(\tau).
\end{aligned}$$

Using (2.10), we have from the last inequality that

$$2^{2p-1}\int_a^\tau s(t)\,|y(t)|^p\left|y^{\Delta_n}(t)\right|^q\Delta t\leqslant K_1(a,\tau,p,q)\int_a^\tau r(t)\left|y^{\Delta_n}(t)\right|^{p+q}\Delta t.\tag{2.16}$$

Proceeding as above, we also have

$$\begin{aligned}
&2^{p-1}\int_a^{\tau}\mu^p(t)s(t)\left(\int_a^t \frac{h_{n-2}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{(r(s))^{\frac{1}{p+q-1}}}\Delta s\right)^{p\left(\frac{p+q-1}{p+q}\right)} \\
&\quad\times\left|y^{\Delta_n}(t)\right|^q\left(\int_a^t r(s)\left|y^{\Delta_n}(s)\right|^{p+q}\Delta s\right)^{\frac{p}{p+q}}\Delta t \\
&\leqslant K_2(a,\tau,p,q)\int_a^{\tau} r(t)\left|y^{\Delta_n}(t)\right|^{p+q}\Delta t. \qquad (2.17)
\end{aligned}$$

Integrating (2.9) from a to τ and using (2.16) and (2.17), we have

$$\begin{aligned}
&\int_a^{\tau} s(t)\left|y(t)+y^{\sigma}(t)\right|^p\left|y^{\Delta_n}(t)\right|^q\Delta t \\
&\leqslant (K_1(a,\tau,p,q)+K_2(a,\tau,p,q))\int_a^{\tau} r(t)\left|y^{\Delta_n}(t)\right|^{p+q}\Delta t,
\end{aligned}$$

which is the desired inequality (2.3). The proof is complete. □

Here, we only state the following theorem, since its proof is the same as that of Theorem 2.1, with $[a,\tau]_{\mathbb{T}}$ replaced by $[\tau,b]_{\mathbb{T}}$ and $y(t)$ in (2.4) is replaced by

$$y(t)=(-1)^n\int_t^b g_{n-1}(\sigma(s),t)y^{\Delta_n}(s)\Delta s, \text{ for } t\in[\tau,b]_{\mathbb{T}}.$$

THEOREM 2.2. Let $\mathbb{T}$ be a time scale with $\tau,\ b\in\mathbb{T}$ and $p,\ q$ be positive real numbers such that $p>1,\ 1/p+1/q=1$, and let r,s be nonnegative rd-continuous functions on $(\tau,b)_{\mathbb{T}}$. If $y\in C_{rd}^{(n)}([\tau,b]\cap\mathbb{T})$ with $y^{\Delta_i}(b)=0$, for $0\leqslant i\leqslant n-1$, then

$$\int_{\tau}^{b} s(t)\left|y(t)+y^{\sigma}(t)\right|^p\left|y^{\Delta_n}(t)\right|^q\Delta t\leqslant K^*(\tau,b,p,q)\int_{\tau}^{b} r(t)\left|y^{\Delta_n}(t)\right|^{p+q}\Delta t, \qquad (2.18)$$

where $K^*(\tau,b,p,q)=K_1^*(\tau,b,p,q)+K_2^*(\tau,b,p,q)$,

$$\begin{aligned}
K_1^*(\tau,b,p,q) &= 2^{2p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\
&\quad\times\left(\int_{\tau}^{b}\frac{(s(t))^{\frac{p+q}{p}}}{(r(t))^{\frac{q}{p}}}\left(\int_t^b \frac{g_{n-1}^{\frac{p+q}{p+q-1}}(\sigma(s),t)}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{(p+q-1)}\Delta t\right)^{\frac{p}{p+q}}.
\end{aligned}$$

and

$$\begin{aligned}
K_2^*(\tau,b,p,q) &= 2^{2p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\
&\quad\times\left(\int_{\tau}^{b}\frac{\mu^{p+q}(t)(s(t))^{\frac{p+q}{p}}}{(r(t))^{\frac{q}{p}}}\left(\int_t^b \frac{g_{n-2}^{\frac{p+q}{p+q-1}}(\sigma(s),t)}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{(p+q-1)}\Delta t\right)^{\frac{p}{p+q}}.
\end{aligned}$$

In the following, we assume that there exists $\tau \in (a,b)_{\mathbb{T}}$ such that

$$K_1(p,q) = K_1(a,\tau,p,q) = K_1^*(\tau,b,p,q) < \infty,$$
$$K_2(p,q) = K_2(a,\tau,p,q) = K_2^*(\tau,b,p,q) < \infty,$$

where $K_1(a,\tau,p,q)$, $K_2(\tau,b,p,q)$, $K_1^*(a,\tau,p,q)$, and $K_2^*(\tau,b,p,q)$ are defined as in Theorems 2.1 and 2.2. Note that since

$$\int_a^b s(t)\,|y(t)+y^{\sigma}(t)|^p \left|y^{\Delta_n}(t)\right|^q \Delta t$$
$$= \int_a^{\tau} s(t)\,|y(t)+y^{\sigma}(t)|^p \left|y^{\Delta_n}(t)\right|^q \Delta t + \int_{\tau}^b s(t)\,|y(t)+y^{\sigma}(t)|^p \left|y^{\Delta_n}(t)\right|^q \Delta t,$$

then the proof of the following theorem will be a combination of Theorems 2.1 and 2.2.

THEOREM 2.3. *Let $\mathbb{T}$ be a time scale with $a,\ b \in \mathbb{T}$ and $p,\ q$ be positive real numbers such that $p>1,\ 1/p+1/q=1$, and let $r,\ s$ be nonnegative rd-continuous functions on $(a,b)_{\mathbb{T}}$. If $y \in C_{rd}^{(n)}([a,b]\cap\mathbb{T})$ with $y^{\Delta_i}(a)=y^{\Delta_i}(b)=0$, for $i=0,1,2,\ldots,n-1$, then*

$$\int_a^b s(t)\,|y(t)+y^{\sigma}(t)|^p \left|y^{\Delta_n}(t)\right|^q \Delta t \leqslant K(a,b)\int_a^b r(t)\left|y^{\Delta_n}(t)\right|^{p+q} \Delta t, \tag{2.19}$$

where $K(a,b) = K_1(p,q)+K_2(p,q)$.

For $r=s$ in Theorem 2.1, we obtain the following result.

COROLLARY 2.1. *Let $\mathbb{T}$ be a time scale with $a,\ \tau \in \mathbb{T}$ and $p,\ q$ be positive real numbers such that $p>1,\ 1/p+1/q=1$, and let r be a nonnegative rd-continuous function on $(a,\tau)_{\mathbb{T}}$. If $y \in C_{rd}^{(n)}([a,\tau]\cap\mathbb{T})$ with $y^{\Delta_i}(a)=0$, for $i=0,1,2,\ldots,n-1$, then*

$$\int_a^{\tau} r(t)\,|y(t)+y^{\sigma}(t)|^p \left|y^{\Delta_n}(t)\right|^q \Delta t \leqslant K^*(a,\tau,p,q)\int_a^{\tau} r(t)\left|y^{\Delta_n}(t)\right|^{p+q} \Delta t, \tag{2.20}$$

where

$$\begin{aligned}
K^*(a,\tau,p,q) &= 2^{2p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\
&\quad\times\left(\int_a^{\tau} r(t)\left(\int_a^t \frac{h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{(p+q-1)}\Delta t\right)^{\frac{p}{p+q}} \\
&\quad+2^{p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\
&\quad\times\left(\int_a^{\tau} \mu^{p+q}(t)r(t)\left(\int_a^t \frac{h_{n-2}^{\frac{p+q}{p+q-1}}(t,\sigma(s))}{r^{\frac{1}{p+q-1}}(s)}\Delta s\right)^{(p+q-1)}\Delta t\right)^{\frac{p}{p+q}}.
\end{aligned}$$

From Theorems 2.2 and 2.3 one can derive similar results by setting $r = s$. The details are left to the interested reader.

Setting $r = 1$ in (2.20), we have the following result.

COROLLARY 2.2. *Let $\mathbb{T}$ be a time scale with $a,\ \tau \in \mathbb{T}$ and $p,\ q$ be positive real numbers such that $p > 1,\ 1/p + 1/q = 1$. If $y \in C_{rd}^{(n)}([a,\tau] \cap \mathbb{T})$ is delta differentiable with $y^{\Delta_i}(a) = 0$, for $i = 0,1,2,...,n-1$, then*

$$\int_a^\tau |y(t) + y^\sigma(t)|^p \left|y^{\Delta_n}(t)\right|^q \Delta t \leqslant L(a,b,p,q) \int_a^\tau \left|y^{\Delta_n}(t)\right|^{p+q} \Delta t, \tag{2.21}$$

where

$$L(a,\tau,p,q) = 2^{2p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \left(\int_a^\tau \left(\int_a^t h_{n-1}^{\frac{p+q}{p+q-1}}(t,\sigma(s))\Delta s\right)^{(p+q-1)} \Delta t\right)^{\frac{p}{p+q}}$$
$$+2^{p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \left(\int_a^\tau \mu^{p+q}(t) \left(\int_a^t h_{n-2}^{\frac{p+q}{p+q-1}}(t,\sigma(s))\Delta s\right)^{(p+q-1)} \Delta t\right)^{\frac{p}{p+q}}.$$

Note that when $\mathbb{T} = \mathbb{R}$, we have $y^\sigma = y$ and $\mu(t) = 0$. Then from Theorems 2.1 and 2.2 we have the following differential inequalities.

COROLLARY 2.3. *Assume that $p,\ q$ be positive real numbers such that $p > 1$, $1/p + 1/q = 1$, and let $r,\ s$ be nonnegative continuous functions on (a,τ). If $y \in C^{(n)}([a,\tau] \cap \mathbb{R})$ with $y^{(i)} = 0$, for $i = 0,1,2,...,n-1$, then*

$$\int_a^\tau s(t)\,|y(t)|^p \left|y^{(n)}(t)\right|^q dt \leqslant C_1(a,\tau,p,q) \int_a^\tau r(t) \left|y^{(n)}(t)\right|^{p+q} dt,$$

where

$$C_1(a,\tau,p,q) = 2^{p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}}$$
$$\times \left(\int_a^\tau \frac{(s(t))^{\frac{p+q}{p}}}{(r(t))^{\frac{q}{p}}} \left(\int_a^t \frac{(t-s)^{\frac{p+q}{p+q-1}(n-1)}}{(n-1)!\,r^{\frac{1}{p+q-1}}(s)} ds\right)^{(p+q-1)} dt\right)^{\frac{p}{p+q}}.$$

COROLLARY 2.4. *Assume that $p,\ q$ be positive real numbers such that $p > 1$, $1/p + 1/q = 1$, and let r, s be nonnegative continuous functions on $(\tau,b)_{\mathbb{R}}$. If $y \in C^{(n)}([a,\tau] \cap \mathbb{R})$ with $y^{(i)}(b) = 0$, for $i = 0,1,2,...,n-1$, then*

$$\int_\tau^b s(t)\,|y(t)|^p \left|y^{(n)}(t)\right|^q dt \leqslant C_2(\tau,b,p,q) \int_\tau^b r(t) \left|y^{(n)}(t)\right|^{p+q} dt,$$

where

$$C_2(\tau,b,p,q) = 2^{p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \times\left(\int_\tau^b \frac{(s(t))^{\frac{p+q}{p}}}{(r(t))^{\frac{q}{p}}}\left(\int_t^b \frac{(s-t)^{\frac{p+q}{p+q-1}(n-1)}}{(n-1)!r^{\frac{1}{p+q-1}}(s)}ds\right)^{(p+q-1)} dt\right)^{\frac{p}{p+q}}.$$

Note that when $\mathbb{T}=\mathbb{R}$, we have $y^\sigma = y$, $\mu(t)=0$ and then Corollary 2.2 gives us the following result.

COROLLARY 2.5. *Assume that p, q be positive real numbers such that $p>1$, $1/p+1/q=1$. If $y\in C^{(n)}([a,\tau]\cap\mathbb{R})$ with $y^{(i)}(a)=0$, for $i=0,1,2,\dots,n-1$, then*

$$\int_a^\tau |y(t)|^p\left|y^{(n)}(t)\right|^q dt \leqslant 2^{p-1}\left(\frac{q}{(p+q)}\right)^{\frac{q}{p+q}} \times\left(\frac{(\tau-a)^{(p+q)(n-1)+(p+q)}}{M}\right)^{\frac{p}{p+q}}\int_a^\tau \left|y^{(n)}(t)\right|^{p+q} dt, \quad (2.22)$$

where $M=(n-1)!(\frac{p+q}{p+q-1}(n-1)+1)^{(p+q-1)})(p+q)(n-1)+(p+q)$.

Note also that when $p=1$, $q=1$ and $n=1$, we have the following result: If $y\in C^1([a,\tau]\cap\mathbb{R})$ with $y(a)=0$, then

$$\int_a^\tau |y(t)|\left|y'(t)\right| dt \leqslant \frac{1}{2}(\tau-a)\int_a^\tau \left|y'(t)\right|^2 dt,$$

which is classical Opial's inequality (see Opial [14] and Olech [13]).

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