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WEIGHTED ESTIMATES FOR ITERATED COMMUTATORS OF MULTILINEAR OPERATORS WITH NON–SMOOTH KERNELS

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Abstract. Let T be the multilinear Calderón-Zygmund operator with non-smooth kernels and T_* be its corresponding maximal function. In this paper, we give the weighted strong type estimates and weak end-point estimates for the iterated commutators of multilinear operator T_*. Similar results still hold for the operator T.

1. Introduction

Multilinear Calderón-Zygmund operators were introduced and first studied by Coifman and Meyer [6], [7], [8], and later by Grafakos and Torres [15], [16]. In analogy with the linear theory, the class of multilinear singular integrals with standard Calderón-Zygmund kernels provides a fundamental topic of investigation within the framework of the general theory. The study of this subject was recently enjoyed a resurgence of renewed interest and activity. In particular, the study of multilinear singular integral operators with non-standard kernels have recently received increasing attention.

First, we give some background for the multilinear singular integral operators with standard Calderón-Zygmund kernels. Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : \mathscr{S}(\mathbb{R}^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$$

Following [15], we say that T is an m-linear Calderón-Zygmund operator if, for some $1 \leqslant q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and if there exists a function K, defined off the diagonal

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$x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T\vec{f}(x) = T(f_1,\cdots,f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)dy_1\cdots dy_m, \quad (1.1)$$

for all $x \notin \bigcap_{j=1}^m \mathrm{supp} f_j$;

$$|K(y_0,y_1,\cdots,y_m)| \leqslant \frac{A}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}}; \quad (1.2)$$

for some $A > 0$ and all $(x,y_1,\cdots,y_m)$ with $x \neq y_j$ for some j. and

$$|K(y_0,y_1,...,y_j,...,y_m) - K(y_0,y_1,...,y'_j,...,y_m)| \leqslant \frac{A|y_j - y'_j|^{\varepsilon}}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\varepsilon}} \quad (1.3)$$

for some $\varepsilon > 0$ and all $0 \leqslant j \leqslant m$, whenever $|y_j - y'_j| \leqslant \frac{1}{2}\max_{0\leqslant k\leqslant m}|y_j - y_k|$. Such kernels are called $m-$linear Calderón-Zygmund kernels and the collection of such functions is denoted by $m - CZK(A,\varepsilon)$ in [15]. The maximal multilinear singular integral operator is defined by

$$T_*(\vec{f})(x) = \sup_{\delta>0}|T_\delta(f_1,\cdots,f_m)(x)|,$$

where T_δ are the smooth truncations of T given by

$$T_\delta(f_1,\cdots,f_m)(x) = \int_{|x-y_1|^2+...+|x-y_m|^2>\delta^2} K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)d\vec{y}.$$

Here, $d\vec{y} = dy_1\cdots dy_m$.

As is pointed in [16], $T_*(\vec{f})(x)$ is pointwise well-defined when $f_j \in L^{q_j}(\mathbb{R}^n)$ with $1 \leqslant q_j \leqslant \infty$.

Recently, the theory of weighted multilinear Calderón-Zygmund type operators was established in [18], [5], [20]. Two different kinds of commutators associated with multilinear Calderón-Zygmund singular integral type operators were studied in [18], [25] and [28], weighted strong and weak $L\log L$ type estimates were obtained.

DEFINITION 1.1. (Commutators in the j-th entry) ([18], [5]) Given a collection of locally integrable functions $\vec{b} = (b_1,\cdots,b_m)$, we define the commutators of the m-linear Calderón-Zygmund operator T to be

$$[\vec{b},T](\vec{f}) = T_{\vec{b}}(f_1,\ldots,f_m) = \sum_{j=1}^m T^j_{\vec{b}}(\vec{f}),$$

where each term is the commutator of b_j and T in the j-th entry of T, that is,

$$T^j_{\vec{b}}(\vec{f}) = b_jT(f_1,\ldots,f_j,\ldots,f_m) - T(f_1,\ldots,b_jf_j,\ldots,f_m).$$

In [25], the following more general iterated commutators of multilinear Calderón-Zygmund operators and pointwise multiplication with functions in BMO are defined and studied in products of Lebesgue spaces, including strong type and weak end-point estimates with multiple $A_{\vec{p}}$ weights.

$$T_{\Pi\vec{b}}(\vec{f})(x) = [b_1,[b_2,\cdots[b_{m-1},[b_m,T]_m]_{m-1}\cdots]_2]_1(\vec{f})(x)$$
$$= \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{m}(b_j(x)-b_j(y_j))K(x,y_1,\cdots,y_m)\prod_{i=1}^{m} f_i(y_i)\,d\vec{y}. \tag{1.4}$$

Still more recently, in [28] the author studied the iterated commutators of maximal multilinear singular integral operator defined by

$$T_{*,\Pi\vec{b}}(\vec{f})(x) = \sup_{\delta>0}\left|[b_1,[b_2,\cdots[b_{m-1},[b_m,T_\delta]_m]_{m-1}\cdots]_2]_1(\vec{f})(x)\right|$$
$$= \sup_{\delta>0}\left|\int_{|x-y_1|^2+\ldots+|x-y_m|^2>\delta^2} \prod_{j=1}^{m}(b_j(x)-b_j(y_j))K(x,y_1,\cdots,y_m)\prod_{i=1}^{m} f_i(y_i)\,d\vec{y}\right| \tag{1.5}$$

We list some results for T_* as follows:

THEOREM A. ([16]) *Let* $1 \leqslant q_i < \infty$, *and* q *be such that* $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, *and* $\omega \in A_{q_1} \cap \cdots \cap A_{q_m}$. *Let* T *be an* m*-linear Calderón-Zygmund operator. Then there exists a constant* $C_{q,n} < \infty$ *so that for all* $\vec{f} = (f_1,\cdots,f_m)$

$$\|T_*(\vec{f})\|_{L^q(\omega)} \leqslant C_{n,q}(A+W)\prod_{i=1}^{m}\|f_i\|_{L^{q_i}(\omega)},$$

where W *is the norm of* T *in the mapping* T*:* $L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$.

THEOREM B. ([4]) *Assume that* $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$ *and* $\vec{w} \in A_{\vec{p}}$ *(see section 2 for the definition of* $A_{\vec{p}}$*), then*

(i) If $1 < p_1, \ldots, p_m < \infty$, *then* T_* *is bounded from* $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ *to* $L^p(\vec{\omega})$;

(ii) If $1 \leqslant p_1, \ldots, p_m < \infty$, *then* T_* *is bounded from* $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ *to* $L^{p,\infty}(\vec{\omega})$.

Recently, many mathematicians are concerned to remove or replace the smoothness condition on the kernel [1], [10], [19], [13]. In order to state clearly, we first give some preparation, We will work with a class of integral operators $\{A_t\}_{t>0}$, that plays the role of an approximation to the identity as in [10]. We assume that the operators A_t are associated with kernels $a_t(x,y)$ in the sense that

$$A_t f(x) = \int_{\mathbb{R}^n} a_t(x,y)f(y)dy$$

for every function $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, and the kernels $a_t(x,y)$ satisfy the following size conditions

$$|a_t(x,y)| \leqslant h_t(x,y) := t^{-n/s} h\left(\frac{|x-y|}{t^{1/s}}\right), \tag{1.6}$$

where s is a positive fixed constant and h is a positive, bounded, decreasing function satisfying

$$\lim_{r\to 0} r^{n+\eta} h(r^s) = 0 \tag{1.7}$$

for some $\eta > 0$. These conditions imply that for some $C > 0$ and all $0 < \eta \leqslant \eta'$,the kernels $a_t(x,y)$ satisfy

$$|a_t(x,y)| \leqslant Ct^{-n/s}(1+t^{-1/s}|x-y|)^{-n-\eta'}.$$

It can be verified that

$$|A_t f(x)| \leqslant \int_{\mathbb{R}^n} |a_t(x,y)||f(y)|dy \leqslant CMf(x). \tag{1.8}$$

for every $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$(see for instance,[11]).

An m-linear operator $T : \mathscr{S}(R^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is linear in every entry and consequently it has m formally transpose. The jth transpose $T^{*,j}$ of T is defined via

$$\langle T^{*,j}(f_1,\cdots,f_j,\cdots,f_m),g\rangle = \langle T(f_1,\cdots,f_{j-1},g,f_{j+1},\cdots,f_m),f_j\rangle,$$

for all $f_1,\cdots,f_m,g$ in $\mathscr{S}(\mathbb{R}^n)$. It is easy to check that the kernel $K^{*,j}$ of $T^{*,j}$ is related to the kernel K of T via the identity

$$K^{*,j}(x,y_1,\cdots,y_{j-1},y_j,y_{j+1},\cdots,y_m) = K(y_j,y_1,\cdots,y_{j-1},x,y_{j+1},\cdots,y_m).$$

Note that if a multilinear operator T maps a product of Banach spaces $X_1 \times \cdots \times X_m$ into another Banach space X, then the transpose $T^{*,j}$ maps the product of Banach spaces $X_1 \times \cdots \times X_{j-1} \times X^* \times X_{j+1} \times \cdots \times X_m$ into X_j^*. Moreover, the norms of T and $T^{*,j}$ are equal. To maintain uniform notation, we may occasionally denote T by $T^{*,0}$ and K by $K^{*,0}$.

ASSUMPTION (H0). We always assume that there exists some $1 \leqslant q_1,\cdots,q_m < \infty$ and some $0 < q < \infty$ with $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, such that T_* and T both map $L^{q_1} \times \cdots \times L^{q_m}$ to $L^{q,\infty}$.

ASSUMPTION (H1). Assume that for each $i = 1,\cdots,m$ there exist operators $\{A_t^{(i)}\}_{t>0}$ with kernels $a_t^i(x,y)$ that satisfy condition (1.6) and (1.7) with constants s and η and that for every $j = 0,1,2,\cdots,m$, there exist kernels $K_t^{*,j,(i)}(x,y_1,\cdots,y_m)$ such that

$$\begin{aligned}&\langle T^{*,j}(f_1,\cdots,A_t^{(i)} f_i,\cdots,f_m),g\rangle \\ &\quad = \int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^m} K_t^{*,j,(i)}(x,y_1,\cdots,y_m) f_1(y_1)\cdots f_m(y_m) g(x) dy_1\cdots dy_m dx,\end{aligned} \tag{1.9}$$

for all $f_1,\cdots,f_m,g$ in $\mathscr{S}$ with $\cap_{k=1}^m \mathrm{supp} f_k \cap \mathrm{supp} g = \phi$. There exists a function $\phi \in C(\mathbb{R})$ with $supp\phi \in [-1,1]$ and a constant $\varepsilon > 0$ so that for every $j = 0,1,\cdots,m$ and every $i = 1,2,\cdots,m$, we have

$$|K^{*,j}(x,y_1,\cdots,y_m) - K_t^{*,j,(i)}(x,y_1,\cdots,y_m)| \leqslant \frac{A}{(|x-y_1|+\cdots+|x-y_m|)^{mn}} \sum_{k=1,k\neq i}^{m} \phi\left(\frac{|y_i-y_k|}{t^{1/s}}\right) + \frac{At^{\varepsilon/s}}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\varepsilon}} \tag{1.10}$$

whenever $t^{1/s} \leqslant |x-y_i|/2$.

Kernels K that satisfy the size estimate (1.2) and assumption (H1) with parameters $m, A, s, \eta, \varepsilon$ are called generalized Calderón-Zygmund kernels, and their collection is denoted by $m-GCZK(A,s,\eta,\varepsilon)$. We say that T is of class $m-GCZO(A,s,\eta,\varepsilon)$ if T has an associated kernel K in $m-GCZK(A,s,\eta,\varepsilon)$.

ASSUMPTION (H2). Assume that there exist operators $\{A_t\}_{t>0}$ with kernels $a_t(x,y)$ that satisfy condition (1.6) and (1.7) with constants s and η, and there exsit kernels $K_t^{(0)}(x,y_1,\cdots,y_m)$ such that for all $x,y_1,\cdots,y_m \in \mathbb{R}^n$ and $t > 0$ the representation is valid

$$K_t^{(0)}(x,y_1,\cdots,y_m) = \int_{\mathbb{R}^n} K(z,y_1,\cdots,y_m) a_t(x,z) dz. \tag{1.11}$$

Assume also that there exist a function $\phi \in \mathscr{C}(\mathbb{R})$ and $\phi \subset [-1,1]$ and a constant $\varepsilon > 0$ such that

$$|K(x,y_1,\cdots,y_m) - K_t^{(0)}(x,y_1,\cdots,y_m)| \leqslant \frac{A}{(\sum_{k=1}^m |x-y_k|)^{mn}} \sum_{\substack{k=1\\k\neq j}}^{m} \phi\left(\frac{|x-y_k|}{t^{1/s}}\right) + \frac{At^{\varepsilon/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\varepsilon}} \tag{1.12}$$

for some $A > 0$, whenever $2t^{1/s} \leqslant \max_{1\leqslant j\leqslant m} |x-y_j|$. Moreover, assume that for all $x,y_1,\cdots,y_m \in \mathbb{R}^n$,

$$|K_t^{(0)}(x,y_1,\cdots,y_m)| \leqslant \frac{A}{(\sum_{k=1}^m |x-y_k|)^{mn}}, \tag{1.13}$$

whenever $2t^{1/s} \leqslant \min_{1\leqslant j\leqslant m} |x-y_j|$, and for all $x,x',y_1,\cdots,y_m \in \mathbb{R}^n$,

$$|K_t^{(0)}(x,y_1,\cdots,y_m) - K_t^{(0)}(x',y_1,\cdots,y_m)| \leqslant \frac{At^{\varepsilon/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\varepsilon}}, \tag{1.14}$$

whenever $2t^{1/s} \leqslant \min_{1\leqslant j\leqslant m} |x-y_j|$ and $2|x-x'| \leqslant t^{1/s}$.

THEOREM C. *Assume that T be an m-linear operator in m-$GCZO(A,s,\eta,\varepsilon)$ and its kernel K satisfies assumption (H2). Moreover, for some $1 \leqslant q_1,\cdots,q_{m-1} \leqslant \infty$, $q_m \in (1,\infty)$ satisfying $\frac{1}{q_1}+\cdots+\frac{1}{q_m} = \frac{1}{q}$, and T maps $L^{q_1}\times\cdots\times L^{q_m}$ to L^q. Let $1 \leqslant p_1,\cdots,p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1}+\cdots+\frac{1}{p_m}$, $\vec{\omega} = (\omega_1,\cdots,\omega_m) \in A_{\vec{p}}$ with $\vec{p} = (p_1,\cdots,p_m)$, then* *(i)* T_* *can be extended to a bounded operator from* $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)$ *to* $L^p(\nu_{\vec{\omega}})$ *if all the exponents* p_j *are strictly greater than* 1;

(ii) T_* *can be extended to a bounded operator from* $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)$ *to* $L^{p,\infty}(\nu_{\vec{\omega}})$ *if some exponents* p_j *is equal to* 1.

Similar results hold for T.

In 2009, Gong and Li [19] obtained the following weighted estimate for $T_{\vec{b}}$:

THEOREM D. *Let* T *be an m-linear operator associated with a kernel* K *satisfying assumption (H1) and (H2), and let* $T_{\vec{b}}$ *be a multilinear commutator with* $\vec{b}\in BMO^m$. *Let* $1<p_1',\cdots,p_m',p'<\infty$ *be given numbers satisfying*

$$\frac{1}{p_1'}+\ldots+\frac{1}{p_m'}=\frac{1}{p'}. \tag{1.15}$$

Assume that T *maps* $L^{p_1'}(\mathbb{R}^n)\times\cdots\times L^{p_m'}(\mathbb{R}^n)$ *to* $L^{p'}(\mathbb{R}^n)$. *Let* p,p_j *be numbers satisfying* $1<p_j,p<\infty$ *and* $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$ *and let* $\omega\in A_p$, *then there is a constant* C *so that for all* $\vec{f}=(f_1,\cdots,f_m)$, *where each* f_i *is a smooth function with compact support,*

$$||T_{\vec{b}}(\vec{f})||_{L^p(\omega)}\leqslant C||\vec{b}||_{BMO^m}\prod_{j=1}^m||f_j||_{L^{p_j}(\omega)}$$

and hence, $T_{\vec{b}}$ *extends as a bounded operator from* $L^{p_1}(\omega)\times\cdots\times L^{p_m}(\omega)$ *into* $L^p(\omega)$, *where* $||\vec{b}||_{BMO^m}=\sup_i||b_i||_{BMO}$.

In 2010, Anh and Duong in [1] obtained the following estimates for $T_{\vec{b}}$.

THEOREM E. *Assume that* T *satisfies (H1) and (H2). Let* ω *be an* A_∞ *weight, function* $\Phi(t)=t(1+\log^+t)$ *and* $p>0$. *Suppose that* $\vec{b}\in BMO^m$ *with* $||\vec{b}||_{BMO}=1$. *Then, there exists a constant* $C>0$, *depending on the* A_∞ *constant of* ω, *such that*

$$\int_{\mathbb{R}^n}|T_{\vec{b}}(\vec{f})|^p\omega(x)dx\leqslant C\int_{\mathbb{R}^n}\prod_{i=1}^m M_{L(logL)}f_i(x)^p\omega(x)dx \tag{1.16}$$

and

$$\begin{aligned}&\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\omega(\{y\in\mathbb{R}^n:|T_{\vec{b}}(\vec{f})(y)|>t^m\})\\&\qquad\leqslant C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\omega(\{y\in\mathbb{R}^n:\prod_{i=1}^m M_{L(\log L)}f_i(y)>t^m\})\end{aligned} \tag{1.17}$$

for all bounded vector function $\vec{f}=(f_1,\cdots,f_m)$ *with compact support.*

THEOREM F. *Assume that T satisfies (H1) and (H2). Let $\vec{b}\in BMO^m$ with $||\vec{b}||_{BMO}=1$ and $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ with $1<p_j<\infty$, $j=1,\cdots,m$. Then we have*

(i) There exists a constant C such that

$$||T_{\vec{b}}(\vec{f})||_{L^p(\nu_{\vec{\omega}})}\leqslant C\prod_{i=1}^{m}||f_i||_{L^{p_i}(M\omega_i)}.$$

(ii) If each $\omega_i\in A_p$, then there exists a constant C such that

$$||T_{\vec{b}}(\vec{f})||_{L^p(\nu_{\vec{\omega}})}\leqslant C\prod_{i=1}^{m}||f_i||_{L^{p_i}(\omega_i)}.$$

Here $\nu_{\vec{\omega}}=\prod_{i=1}^{m}\omega_i^{\frac{p}{p_j}}$.

Comparing with the results for standard multilinear Calderón-Zygmund operators, one may naturally ask the following questions: Can we obtain the similar results for the iterated commutators of multilinear Calderón-Zygmund operators with non-smooth kernels? In particular, does the result hold for maximal operator T_* with non-smooth kernels? It is obviously that estimates involving the maximal function T_* lead to non-linear type analysis, which in turn adds the difficulty of dealing this type operators.

Inspired by the above results, we first establish the following estimates for iterated commutators with non-smooth kernels for $T_{*,\Pi\vec{b}}$, which give positive answers to the above questions. These results are new, even in the case for the commutators in the j-th entry.

THEOREM 1.2. *Assume that T is an operator in m-GCZO(A,s,η,ε) and its kernel satisfies assumption (H2). Let ω be an A_∞ weight, function $\Phi(t)=t(1+\log^+ t)$ and $p>0$. Suppose that $\vec{b}\in BMO^m$. Then, there exists a constant $C>0$, depending on the A_∞ constant of ω, such that*

$$\int_{\mathbb{R}^n}|T_{*,\Pi\vec{b}}(\vec{f})|^p\omega(x)dx\leqslant C\prod_{i=1}^{m}||b_i||_{BMO}\int_{\mathbb{R}^n}\prod_{i=1}^{m}M_{L(\log L)}f_i(x)^p\omega(x)dx; \tag{1.18}$$

At the endpoint, we have

$$\begin{aligned}&\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\omega(\{y\in\mathbb{R}^n:|T_{*,\Pi\vec{b}}(\vec{f})(y)|>t^m\})\\&\qquad\leqslant C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\omega(\{y\in\mathbb{R}^n:\prod_{i=1}^{m}M_{L(\log L)}f_i(y)>t^m\});\end{aligned} \tag{1.19}$$

for all bounded vector function $\vec{f}=(f_1,\cdots,f_m)$ with compact support.

Similar results still hold for $T_{*,\vec{b}}$.

As for the multiple weights $\vec{w}\in A_{\vec{p}}$, we have

THEOREM 1.3. *Assume that T is an operator in m-GCZO(A,s,η,ε) and its kernel satisfies assumption (H2). Let $\vec{b}\in BMO^m$, $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ with $1<p_j<\infty$, $j=1,\cdots,m$. Then we have*

(i) There exists a constant C such that

$$||T_{*,\Pi\vec{b}}(\vec{f})||_{L^p(\nu_{\vec{\omega}})}\leqslant C\prod_{i=1}^m||b_i||_{BMO}\prod_{i=1}^m||f_i||_{L^{p_i}(M\omega_i)};$$

(ii) If each $\omega_i\in A_{p_j}$, then there exists a constant C such that

$$||T_{*,\Pi\vec{b}}(\vec{f})||_{L^p(\nu_{\vec{\omega}})}\leqslant C\prod_{i=1}^m||b_i||_{BMO}\prod_{i=1}^m||f_i||_{L^{p_i}(\omega_i)},$$

where $\nu_{\vec{\omega}}=\prod_{i=1}^m\omega_i^{p/p_j}$.

Similar results still hold for $T_{*,\vec{b}}$.

COROLLARY 1.4. *The results in Theorem 1.1 and 1.2 still hold for $T_{\Pi\vec{b}}$, which extend the results in* [1] *significantly.*

As for one weight $w\in A_p$, we get

THEOREM 1.5. *Assume that T is an operator in m-GCZO(A,s,η,ε) and its kernel satisfies assumption (H2), and $\vec{b}\in BMO^m$. Let p,p_j be numbers satisfying $1<p_j,p<\infty$ and $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$ and let $\omega\in A_p$. Then there is a constant C so that for all $\vec{f}=(f_1,\cdots,f_m)$, where each f_i is a smooth function with compact support,*

$$||T_{\Pi\vec{b}}(\vec{f})||_{L^p(\omega)}\leqslant C\prod_{i=1}^m||b_i||_{BMO}\prod_{j=1}^m||f_j||_{L^{p_j}(\omega)}$$

and hence, $T_{\Pi\vec{b}}$ extends as a bounded operator from $L^{p_1}(\omega)\times\cdots\times L^{p_m}(\omega)$ into $L^p(\omega)$.

REMARK. The above theorem gives the results for the iterated commutators of T with the weight $w\in A_p$, thus it improves the results in Theorem D for the commutators in j-entry significantly.

The article is organized as follows. Initially, some definitions and preliminaries are given in section 2. Next, we will focus on the main results Theorem 1.1–1.4 after established some main lemmas in section 3.

2. Preliminaries

DEFINITION 2.1. A locally integral function b on $\mathbb{R}^n$ is said to be in $BMO(\mathbb{R}^n)$ if and only if

$$\sup_B\frac{1}{|B|}\int_B|b(y)-b_B|dy<\infty,$$

where $b_B = \frac{1}{|B|}\int_B b(y)dy$. The BMO norm of b is defined by

$$||b||_* = \sup_B \frac{1}{|B|}\int_B |b(y) - b_B|dy < \infty.$$

The classical John-Nirenberg inequality shows that functions in BMO are locally exponentially integrable. This implies that, for any $1 \leqslant q < \infty$, the function in BMO can be described by means of the condition

$$\sup\left\{\frac{1}{|B|}\int_B |b(x) - b_B|^q \, dx\right\}^{\frac{1}{q}} < C < \infty.$$

Given a Young function Φ, we define the Φ-average of a function f over a ball B by

$$\|f\|_{\Phi,B} = \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, dx \leqslant 1\right\}$$

$$M_{L(\log L)}(f_i)(x) = \sup_{B \ni x} \|f_i\|_{L(\log L),B},$$

$$\mathscr{M}_{L(\log L)}(\vec{f})(x) = \sup_{B \ni x} \prod_{i=1}^m \|f_i\|_{L\log L,B},$$

We prepare some lemmas which will be used later. The following Hölder's inequality on Orlicz spaces can be seen in [27, p. 58].

LEMMA 2.2. (Generalized Hölder's inequality) ([27]) *Let* $\phi(t) = t(1 + \log^+ t)$ *and* $\psi(t) = e^t - 1$ *and suppose that*

$$\|f\|_\phi \triangleq \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \phi\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu \leqslant 1\right\} < \infty$$

$$\|g\|_\psi \triangleq \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \psi\Big(\frac{|g(x)|}{\lambda}\Big)\, d\mu \leqslant 1\right\} < \infty$$

with respect to some measure μ*, then for any ball* B

$$\frac{1}{|B|}\int_B |fg| \leqslant 2\|f\|_{L(\log L),B}\|g\|_{\exp L,B}. \tag{2.1}$$

Some other inequalities are also necessary.

LEMMA 2.3. ([5]) *Suppose that* $r > 1$ *and* $b \in BMO$*, then for any* f *satisfying the condition of generalized Hölder's inequality there is a* $C > 0$ *independent of* $\vec{f}$ *and* b *such that*

$$\frac{1}{|B|}\int_B |f| \leqslant C\|f\|_{L(\log L),B}; \tag{2.2}$$

$$\|f\|_{L(\log L),B} \leqslant C\Big(\frac{1}{|B|}\int_B |f|^r\Big)^{\frac{1}{r}}; \tag{2.3}$$

$$\frac{1}{|B|}\int_B |(b-b_B)f| \leqslant C\|b\|_{BMO}\|f\|_{L(\log L),B}; \tag{2.4}$$

$$\left(\sup_B \frac{1}{|B|}\int_B |b-b_B|^{r-1}\right)^{\frac{1}{r-1}} \leqslant C\|b\|_{BMO}. \tag{2.5}$$

Similar inequalities still hold for $M_{L(\log L)}(f_i)$ (see p. 84, [1]).

DEFINITION 2.4. The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|dy.$$

For $r>0$, we define the non-centered maximal operator

$$M_r f(x) = \sup_{x\in B}\left(\frac{1}{|B|}\int_B |f(y)|^r dy\right)^{\frac{1}{r}}$$

M_1 is the standard Hardy-Littlewood maximal operator M.

For any $f\in L^p(\mathbb{R}^n)$, $p\geqslant 1$, the sharp maximal function $M_A^\sharp f$ associated the generalized approximations to the identity $\{e^{-tA}, t>0\}$ is given by

$$M_A^\sharp f(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)-A_{t_B}f(y)|dy$$

where $t_B = r_B^2$ and r_B is the radius of the ball B.

PROPOSITION 2.5. [19] *Assume that there exist operator $\{A_t\}_{t>0}$ with kernels $a_t(x,y)$ satisfying conditions (1.6) and (1.7) with constants s and η. Let $\omega\in A_\infty$, $\lambda>0$ and $f\in L^p(\mathbb{R}^n)$ for some $1<p<\infty$. Then for every $0<\sigma<1$, we can find $\gamma>0$ independent of λ, f in such a way that*

$$\omega\left(\{x\in\mathbb{R}^n : Mf(x) > C\lambda, M_A^\sharp f(x)\leqslant\gamma\lambda\}\right) \leqslant \sigma\omega\left(\{x\in\mathbb{R}^n : Mf(x)>\lambda\}\right),$$

where $C>1$ is a fixed constant depending only on n.

As a consequence, we have the following estimate:

$$||f||_{L^p(\omega dx)} \leqslant ||Mf||_{L^p(\omega dx)} \leqslant C||M_A^\sharp f||_{L^p(\omega dx)}$$

for every $f\in L^p(\mathbb{R}^n), 1<p<\infty$.

DEFINITION 2.6. [18] (Multilinear $A_{\vec{p}}$ condition) Let $1\leqslant p_1,\cdots,p_m<\infty$. Given $\vec{\omega}=(\omega_1,\cdots,\omega_m)$, set

$$\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$$

We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_B \left(\frac{1}{|B|}\int_B \prod_{i=1}^m \omega_i^{\frac{p}{p_i}}\right)^{\frac{1}{p}} \prod_{i=1}^m \left(\frac{1}{|B|}\int_B \omega_i^{1-p_i'}\right)^{\frac{1}{p_i'}} < \infty,$$

when $p_i = 1$, $\left(\frac{1}{|B|}\int_B \omega_i^{1-p_i'}\right)^{\frac{1}{p_i'}}$ is understood as $(\inf_B \omega_i)^{-1}$.

DEFINITION 2.7. For $\delta > 0$, M_δ is the maximal function defined by

$$M_\delta f(x) = M(|f|^\delta)^{\frac{1}{\delta}}(x) = \left(\sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)|^\delta dy\right)^{\frac{1}{\delta}}.$$

In addition, $M^\sharp$ is the sharp maximal function of Feffeman and Stein,

$$M^\sharp f(x) = \sup_{B\ni x}\inf_c \frac{1}{|B|}\int_B |f(y)-c|dy \approx \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)-f_B|dy.$$

and

$$M_\delta^\sharp f(x) = M^\sharp(|f|^\delta)^{\frac{1}{\delta}}(x).$$

We will use the Fefferman-Stein inequality just the same as in [12].

Let $0 < p, \delta < \infty$ and ω be any Mackenhoupt A_∞ weight. Then there exists a constant C independent of f such that the inequality

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p \omega(x)dx \leqslant C\int_{\mathbb{R}^n} (M_\delta^\sharp f(x))^p \omega(x)dx, \tag{2.6}$$

holds for any function f for which the left-hand side is finite.

3. Proof of Theorem 1.1–1.4

To proof the main results, we need the following lemmas. We just consider the case $m = 2$ for simplicity, our method still hold for general m with little modifications on Lemma 3.1 and Lemma 3.2.

LEMMA 3.1. *Assume that T is a bilinear operator in 2-GCZO(A,s,η,ε) and its kernel satisfies assumption (H2). Let $b_i \in BMO$, $i = 1,2$, $1 < r, q_1, q_2 < \infty$. Then there exists a constant $C > 0$ independent of f_1 and f_2 such that*

$$\begin{aligned} M_A^\sharp T_{\Pi\vec{b}}(f_1,f_2)(x) \leqslant C\{&\prod_{i=1}^2 ||b_i||_{BMO} M_r(T(f_1,f_2))(x) \\ &+ ||b_1||_{BMO} M_r([b_2,T](f_1,f_2))(x) \\ &+ ||b_2||_{BMO} M_r([b_1,T](f_1,f_2))(x) \\ &+ \prod_{i=1}^2 ||b_i||_{BMO}\prod_{i=1}^2 M_{q_i}f_i(x)\}, \end{aligned} \tag{3.1}$$

and

$$M_A^\sharp[b_1,T](f_1,f_2)(x) \leqslant C||b_1||_{BMO}\big\{M_r(T(f_1,f_2))(x)+\prod_{i=1}^{2}M_{q_i}f_i\big\}, \tag{3.2}$$

and

$$M_A^\sharp[b_2,T](f_1,f_2)(x) \leqslant C||b_2||_{BMO}\big\{M_r(T(f_1,f_2))(x)+\prod_{i=1}^{2}M_{q_i}f_i\big\}, \tag{3.3}$$

where

$$[b_1,T](f_1,f_2)(x) = b_1(x)T(f_1,f_2)(x) - T(b_1f_1,f_2)(x),$$
$$[b_2,T](f_1,f_2)(x) = b_2(x)T(f_1,f_2)(x) - T(f_1,b_2f_2)(x),$$

for any function f_1, f_2 belong to L_c^∞ and for every $x \in \mathbb{R}^n$.

Proof. We should point out that $[b_1,T](f_1,f_2)(x) = T_{\vec{b}}^1(f_1,f_2)(x)$ and $[b_2,T](f_1,f_2)(x) = T_{\vec{b}}^2(f_1,f_2)(x)$. So we can apply Theorem D to obtain (3.2) and (3.3) respectively. For any $x \in R^n$ and a ball B containing x, to prove (3.1), it suffices to prove

$$\begin{aligned}
&\frac{1}{|B|}\int_B |T_{\Pi\vec{b}}(f_1,f_2)(z) - A_{t_B}(T_{\Pi\vec{b}}(f_1,f_2))(z)|dz \\
&\leqslant C\big\{\prod_{i=1}^{2}||b_i||_{BMO}M_r(T(f_1,f_2))(x) + ||b_1||_{BMO}M_r([b_2,T](f_1,f_2))(x) \\
&\quad + ||b_2||_{BMO}M_r([b_1,T](f_1,f_2))(x) + \prod_{i=1}^{2}||b_i||_{BMO}\prod_{i=1}^{2}M_{q_i}f_i(x)\big\}.
\end{aligned} \tag{3.4}$$

Note for any constant λ_i, $i = 1,2.$ we have

$$\begin{aligned}
&T_{\Pi\vec{b}}(f_1,f_2)(z) \\
&= \int_{\mathbb{R}^{2n}}(b_1(z)-b_1(y_1))(b_2(z)-b_2(y_2))K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 \\
&= \int_{\mathbb{R}^{2n}}(b_1(z)-\lambda_1+\lambda_1-b_1(y_1))(b_2(z)-\lambda_2+\lambda_2-b_2(y_2)) \\
&\quad \times K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 \\
&= (b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z) \\
&\quad -(b_1(z)-\lambda_1)\int_{\mathbb{R}^{2n}}(b_2(y_2)-\lambda_2)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 \\
&\quad -(b_2(z)-\lambda_2)\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 \\
&\quad +\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)(b_2(y_2)-\lambda_2)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 \\
&= (b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z) \\
&\quad -(b_1(z)-\lambda_1)\int_{\mathbb{R}^{2n}}(b_2(y_2)-b_2(z)+b_2(z)-\lambda_2)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2
\end{aligned}$$

$$
\begin{aligned}
&-(b_2(z)-\lambda_2)\int_{\mathbb{R}^{2n}}(b_1(y_1)-b_1(z)+b_1(z)-\lambda_1)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2\\
&+\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)(b_2(y_2)-\lambda_2)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2\\
=&-(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)+(b_1(z)-\lambda_1)[b_2,T](f_1,f_2)(z)\\
&+(b_2(z)-\lambda_2)[b_1,T](f_1,f_2)(z)+T((b_1(\cdot)-\lambda_1)f_1,(b_2(\cdot)-\lambda_2)f_2)(z).
\end{aligned}
$$

By multilinearity, we can write

$$
\begin{aligned}
&T_{\Pi\vec{b}}(f_1,f_2)(z)-A_{t_B}(T_{\Pi\vec{b}}(f_1,f_2))(z)\\
&=[-(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)]+A_{t_B}[(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)]\\
&\quad+[(b_1(z)-\lambda_1)[b_2,T](f_1,f_2)(z)]-A_{t_B}[(b_1(z)-\lambda_1)[b_2,T](f_1,f_2)(z)]\\
&\quad+[(b_2(z)-\lambda_2)[b_1,T](f_1,f_2)(z)]-A_{t_B}[(b_2(z)-\lambda_2)[b_1,T](f_1,f_2)(z)]\\
&\quad+\{T((b_1(\cdot)-\lambda_1)f_1,(b_2(\cdot)-\lambda_2)f_2)(z)-A_{t_B}[T((b_1(\cdot)-\lambda_1)f_1,(b_2(\cdot)-\lambda_2)f_2)(z)]\}\\
&=I(z)+II(z)+III(z)+V(z)+IV(z)+VI(z)+VII(z).
\end{aligned}
\tag{3.5}
$$

Thus, we have

$$
\begin{aligned}
&\frac{1}{|B|}\int_B|T_{\Pi\vec{b}}(f_1,f_2)(z)-A_{t_B}(T_{\Pi\vec{b}}(f_1,f_2))(z)|dz\\
&\leqslant C(I+II+III+IV+V+VI+VII),
\end{aligned}
\tag{3.6}
$$

where $I=\frac{1}{|B|}\int_B|I(z)|dz$, and II, III, IV, V, VI, VII are similar.

By the Hölder's inequality, we get

$$
\begin{aligned}
I&\leqslant\left(\frac{1}{|B|}\int_B|b_1(z)-\lambda_1|^{r_1}dx\right)^{\frac{1}{r_1}}\left(\frac{1}{|B|}\int_B|b_2(z)-\lambda_2|^{r_2}dx\right)^{\frac{1}{r_2}}\\
&\quad\times\left(\frac{1}{|B|}\int_B|T(f_1,f_2)(z)|^{r}dz\right)^{\frac{1}{r}}\\
&\leqslant C\prod_{i=1}^{2}||b_i||_{BMO}M_r(T(f_1,f_2))(x),
\end{aligned}
\tag{3.7}
$$

where r, r_1, $r_2>1$, such that $\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}=1$.

For III, we have

$$
\begin{aligned}
III&\leqslant\left(\frac{1}{|B|}\int_B|b_1(z)-\lambda_1|^{s}dx\right)^{\frac{1}{s}}\left(\frac{1}{|B|}\int_B|[b_2,T](f_1,f_2)(z)|^{r}dz\right)^{\frac{1}{r}}\\
&\leqslant C||b_1||_{BMO}M_r([b_2,T](f_1,f_2))(x),
\end{aligned}
\tag{3.8}
$$

where $s>1$ such that $\frac{1}{r}+\frac{1}{s}=1$.

Similarly, we have

$$\begin{aligned} V &\leqslant \left(\frac{1}{|B|}\int_B |b_2(z)-\lambda_2|^s dx\right)^{\frac{1}{s}} \left(\frac{1}{|B|}\int_B |[b_1,T](f_1,f_2)(z)|^r dz\right)^{\frac{1}{r}} \\ &\leqslant C||b_2||_{BMO} M_r([b_1,T](f_1,f_2))(x). \end{aligned} \tag{3.9}$$

For II, we have

$$\begin{aligned} II &\leqslant \frac{1}{|B|}\int_B |A_{t_B}[(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)]|dz \\ &\leqslant C\frac{1}{|B|}\int_B |M[(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)]|dz \\ &\leqslant C\left(\frac{1}{|B|}\int_B |M[(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)]|^{\delta} dz\right)^{\frac{1}{\delta}} \\ &\leqslant C\left(\frac{1}{|B|}\int_B |[(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)]|^{\delta} dz\right)^{\frac{1}{\delta}} \\ &\leqslant C\left(\frac{1}{|B|}\int_B |(b_1(z)-\lambda_1)|^{r_1'} dz\right)^{\frac{1}{r_1'}} \left(\frac{1}{|B|}\int_B |(b_2(z)-\lambda_2)|^{r_2'} dz\right)^{\frac{1}{r_2'}} \\ &\quad \times \left(\frac{1}{|B|}\int_B |T(f_1,f_2)(z)|^r dz\right)^{\frac{1}{r}} \\ &\leqslant C\prod_{i=1}^{2} ||b_i||_{BMO} M_r T(\vec{f})(x), \end{aligned} \tag{3.10}$$

where $\frac{1}{\delta}=\frac{1}{r}+\frac{1}{r_1'}+\frac{1}{r_2'}$ and $\delta,\ r_1',\ r_2',\ r>1$.

For IV, we have

$$\begin{aligned} IV &\leqslant \frac{1}{|B|}\int_B |A_{t_B}[(b_1(z)-\lambda_1)[b_2,T](f_1,f_2)(z)]|dz \\ &\leqslant C\frac{1}{|B|}\int_B |M[(b_1(z)-\lambda_1)[b_2,T](f_1,f_2)(z)]|dz \\ &\leqslant C\left(\frac{1}{|B|}\int_B |M[(b_1(z)-\lambda_1)[b_2,T](f_1,f_2)(z)]|^{\delta} dz\right)^{\frac{1}{\delta}} \\ &\leqslant C\left(\frac{1}{|B|}\int_B |[(b_1(z)-\lambda_1)[b_2,T](f_1,f_2)(z)]|^{\delta} dz\right)^{\frac{1}{\delta}} \\ &\leqslant C\left(\frac{1}{|B|}\int_B |(b_1(z)-\lambda_1)|^{r_1''} dz\right)^{\frac{1}{r_1''}} \left(\frac{1}{|B|}\int_B |[b_2,T](f_1,f_2)|^r dz\right)^{\frac{1}{r}} \\ &\leqslant C||b_1||_{BMO} M_r([b_2,T](f_1,f_2))(x), \end{aligned} \tag{3.11}$$

where $\frac{1}{\delta}=\frac{1}{r}+\frac{1}{r_1''}$ and $\delta,\ r_1'',\ r>1$.

Similarly, we have

$$VI \leqslant C||b_2||_{BMO}M_r([b_1,T](f_1,f_2))(x). \tag{3.12}$$

To estimate the last term VII, we split function $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i\chi_{B^*}$, $i = 1,2$. By multilinearity, we can write

$$\begin{aligned} VII \leqslant \quad & \sum_{k_1,k_2}\Big(T((b_1(\cdot)-\lambda_1)f_1^{k_1},(b_2(\cdot)-\lambda_2)f_2^{k_2})(z) \\ & -A_{t_B}[T((b_1(\cdot)-\lambda_1)f_1^{k_1},(b_2(\cdot)-\lambda_2)f_2^{k_2})(z)]\Big), \end{aligned} \tag{3.13}$$

where each $k_i = 0$ or ∞ in each term.

Case (A). If $k_1 = k_2 = \infty$, we claim for $z \in B$

$$\begin{aligned} & (T((b_1(\cdot)-\lambda_1)f_1^\infty,(b_2(\cdot)-\lambda_2)f_2^\infty)(z) - A_{t_B}[T((b_1(\cdot)-\lambda_1)f_1^\infty,(b_2(\cdot)-\lambda_2)f_2^\infty)(z)]) \\ & \leqslant C\prod_{i=1}^{2}||b_i||_{BMO}\prod_{i=1}^{2}M_{q_i}f_i(x). \end{aligned} \tag{3.14}$$

Let us prove (3.14), First note that $|z-y_j| \geqslant 2t_B^{1/s}$ for $y_j \in (B^*)^c$ and $x \in B$. Using Assumption (H2) and Holder's inequality, we have

$$\begin{aligned} & (T((b_1(\cdot)-\lambda_1)f_1^\infty,(b_2(\cdot)-\lambda_2)f_2^\infty)(z) - A_{t_B}[T((b_1(\cdot)-\lambda_1)f_1^\infty,(b_2(\cdot)-\lambda_2)f_2^\infty)(z)]) \\ & \leqslant C\int_{(B^{*c})^2}\frac{t_B^{\frac{\varepsilon}{s}}}{(|z-y_1|+|z-y_2|)^{2n+\varepsilon}}|b(y_1)-\lambda_1||b(y_2)-\lambda_2|f_1(y_1)f_2(y_2)dy_1dy_2 \\ & \leqslant C\int_{B^{*c}}\frac{t_B^{\frac{\varepsilon}{2s}}}{|z-y_1|^{n+\frac{\varepsilon}{2}}}|b(y_1)-\lambda_1|f_1(y_1)dy_1\int_{B^{*c}}\frac{t_B^{\frac{\varepsilon}{2s}}}{|z-y_2|^{n+\frac{\varepsilon}{2}}}|b(y_2)-\lambda_2|f_1(y_2)dy_2 \\ & \leqslant C\sum_{k_1=1}^{\infty}\int_{3^{k_1+1}B\backslash 3^{k_1}B}\frac{r_B^{\frac{\varepsilon}{2s}}}{|z-y_1|^{n+\frac{\varepsilon}{2}}}|b(y_1)-\lambda_1|f_1(y_1)dy_1 \\ & \quad \times\sum_{k_2=1}^{\infty}\int_{3^{k_2+1}B\backslash 3^{k_2}B}\frac{r_B^{\frac{\varepsilon}{2s}}}{|z-y_2|^{n+\frac{\varepsilon}{2}}}|b(y_2)-\lambda_2|f_1(y_2)dy_2 \\ & \leqslant C\sum_{k_1=1}^{\infty}3^{-k_1\varepsilon/2}\frac{1}{|3^{k_1+1}B|}\int_{3^{k_1+1}B}|b(y_1)-b_{3^{k_1+1}B}+b_{3^{k_1+1}B}-b_{3B}||f_1(y_1)|dy_1 \\ & \quad \times\sum_{k_2=1}^{\infty}3^{-k_2\varepsilon/2}\frac{1}{|3^{k_2+1}B|}\int_{3^{k_2+1}B}|b(y_2)-b_{3^{k_2+1}B}+b_{3^{k_2+1}B}-b_{3B}||f_2(y_2)|dy_2 \\ & \leqslant C\sum_{k_1=1}^{\infty}3^{-k_1\varepsilon/2}\Bigg(\frac{1}{|3^{k_1+1}B|}\int_{3^{k_1+1}B}|b(y_1)-b_{3^{k_1+1}B}||f_1(y_1)|dy_1 \\ & \quad +\frac{|b_{3^{k_1+1}B}-b_{3B}|}{|3^{k_1+1}B|}\int_{3^{k_1+1}B}|f_1(y_1)|dy_1\Bigg) \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{k_2=1}^{\infty} 3^{-k_2\varepsilon/2}\left(\frac{1}{|3^{k_2+1}B|}\int_{3^{k_2+1}B}|b(y_2)-b_{3^{k_2+1}B}||f_2(y_2)|dy_2\right.\\
&\left.+\frac{|b_{3^{k_2+1}B}-b_{3B}|}{|3^{k_2+1}B|}\int_{3^{k_2+1}B}|f_2(y_2)|dy_2\right)\\
&\leqslant C\prod_{i=1}^{2}||b_i||_{BMO}\prod_{i=1}^{2}M_{q_i}f_i(x)\sum_{k_1=1}^{\infty}3^{-k_1\varepsilon/2}\sum_{k_2=1}^{\infty}3^{-k_2\varepsilon/2}\\
&\leqslant C\prod_{i=1}^{2}||b_i||_{BMO}\prod_{i=1}^{2}M_{q_i}f_i(x), \qquad (3.15)
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\frac{1}{|B|}\int_B|T((b_1(\cdot)-\lambda_1)f_1^{\infty},(b_2(\cdot)-\lambda_2)f_2^{\infty})(z)\\
&-A_{t_B}[T((b_1(\cdot)-\lambda_1)f_1^{\infty},(b_2(\cdot)-\lambda_2)f_2^{\infty})(z)]|dz\\
&\leqslant C\prod_{i=1}^{2}||b_i||_{BMO}\prod_{i=1}^{2}M_{q_i}f_i(x). \qquad (3.16)
\end{aligned}
$$

Case (B). If $k_1=k_2=0$, by Hölder's inequality and the boundedness of T, we get

$$
\begin{aligned}
&\frac{1}{|B|}\int_B|T((b_1(\cdot)-\lambda_1)f_1^{0},(b_2(\cdot)-\lambda_2)f_2^{0})(z)|dz\\
&\leqslant\left(\frac{1}{|B|}\int_B|T((b_1(\cdot)-\lambda_1)f_1^{0},(b_2(\cdot)-\lambda_2)f_2^{0})(z)|^{\mu}dz\right)^{\frac{1}{\mu}}\\
&\leqslant\left(\frac{1}{|B|}\int_B|(b_1(y_1)-\lambda_1)f_1^{0}(y_1)|^{\mu_1}dy_1\right)^{\frac{1}{\mu_1}}\left(\frac{1}{|B|}\int_B|(b_1(y_2)-\lambda_2)f_2^{0}(y_2)|^{\mu_2}dy_2\right)^{\frac{1}{\mu_2}}\\
&\leqslant C\prod_{i=1}^{2}||b_i||_{BMO}\prod_{i=1}^{2}M_{q_i}f_i(x), \qquad (3.17)
\end{aligned}
$$

where $1<\mu$, μ_1, μ_2 such that $\frac{1}{\mu}=\frac{1}{\mu_1}+\frac{1}{\mu_2}$.

Case (C). If $k_1=0$, $k_2=\infty$ or $k_1=\infty$, $k_2=0$, we just consider the case $k_1=0$, $k_2=\infty$. For $z\in B$, we have

$$
\begin{aligned}
&|T((b_1(\cdot)-\lambda_1)f_1^{0},(b_2(\cdot)-\lambda_2)f_2^{\infty})(z)-A_{t_B}[T((b_1(\cdot)-\lambda_1)f_1^{0},(b_2(\cdot)-\lambda_2)f_2^{\infty})(z)]|\\
&\leqslant C\int_{(B^*)^c}\int_{B^*}\frac{1}{(|z-y_1|+|z-y_2|)^{2n}}|b(y_1)-\lambda_1||b(y_2)-\lambda_2|f_1(y_1)f_2(y_2)dy_1dy_2\\
&\leqslant C\int_{B^*}|b(y_1)-\lambda_1|f_1(y_1)dy_1\int_{(B^*)^c}\frac{1}{|z-y_2|^{2n}}|b(y_2)-\lambda_2|f_2(y_2)dy_2\\
&\leqslant C\frac{1}{|B|}\int_{B^*}|b(y_1)-\lambda_1|f_1(y_1)dy_1|B|\int_{(B^*)^c}\frac{1}{|z-y_2|^{2n}}|b(y_2)-\lambda_2|f_2(y_2)dy_2
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C||b_1||_{BMO}\left(\frac{1}{|B|}\int_{B^*}|f_1(y_1)|^{q_1}dy_1\right)^{\frac{1}{q_1}}|B| \\
&\quad\times\sum_{k=1}^{\infty}\int_{3^{k+1}B\setminus 3^kB}\frac{1}{|z-y_2|^{2n}}|b(y_2)-\lambda_2|f_2(y_2)dy_2 \\
&\leqslant C||b_1||_{BMO}M_{q_1}f_1(x)r^n\sum_{k=1}^{\infty}\frac{1}{(3^kr)^{2n}}\int_{3^{k+1}B}|b(y_2)-\lambda_2|f_2(y_2)dy_2 \\
&\leqslant C||b_1||_{BMO}M_{q_1}f_1(x)\sum_{k=1}^{\infty}\frac{1}{3^{kn}}\frac{1}{|3^{k+1}B|}\int_{3^{k+1}B}|b(y_2)-\lambda_2|f_2(y_2)dy_2 \\
&\leqslant C\prod_{i=1}^{2}||b_i||_{BMO}\prod_{i=1}^{2}M_{q_i}f_i(x). \quad \square
\end{aligned} \tag{3.18}
$$

To prove Theorem 1.1 and Theorem 1.2, we need the following estimates for $T^*_{\Pi\vec{b}}(f_1,f_2)$ and $T^*_{\vec{b}}(f_1,f_2)$.

LEMMA 3.2. *Assume that T is a bilinear operator in 2-GCZO(A,s,η,ε) and its kernel satisfies assumption (H2). Let $T^*_{\Pi\vec{b}}$ be a multilinear commutator with $\vec{b}\in BMO^2$ and let $0<\delta<\min\{r,\frac{1}{2}\}$. Then, there exists a constant $C>0$, depending on δ and r, such that*

$$
\begin{aligned}
&M^{\sharp}_{\delta}T^*_{\Pi\vec{b}}(f_1,f_2)(x) \\
&\leqslant C\Big\{\prod_{i=1}^{2}||b_i||_{BMO}M_r(T(f_1,f_2))(x)+||b_1||_{BMO}M_r([b_2,T](f_1,f_2))(x) \\
&\quad+||b_2||_{BMO}M_r([b_1,T](f_1,f_2))(x)+\prod_{i=1}^{2}||b_i||_{BMO}M_{L(\log L)}(f_i)(x)\Big\};
\end{aligned} \tag{3.19}
$$

$$
M^{\sharp}_{\delta}T^*_{\vec{b}}(f_1,f_2)(x)\leqslant C||\vec{b}||_{BMO^m}\Big\{\prod_{i=1}^{2}M_{L(\log L)}(f_i)(x)+M_r(T(f_1,f_2))(x)\Big\}. \tag{3.20}
$$

Proof. Similarly as in [28], we control the iterated commutators of T_* by another two operators.

Let $u,v\in C^\infty([0,\infty))$ such that $|u'(t)|\leqslant Ct^{-1}$, $|v'(t)|\leqslant Ct^{-1}$ and satisfy

$$
\chi_{[2,\infty)}(t)\leqslant u(t)\leqslant\chi_{[1,\infty)}(t),\quad \chi_{[1,2]}(t)\leqslant v(t)\leqslant\chi_{[1/2,3]}(t).
$$

We define the maximal operators

$$
U^*(\vec{f})(x)=\sup_{\eta>0}\left|\int_{(\mathbb{R}^n)^m}K(x,y_1,\cdots,y_m)u(\sqrt{|x-y_1|+,\cdots,+|x-y_m|}/\eta)\prod_{i=1}^{m}f_i(y_i)d\vec{y}\right|,
$$

$$
V^*(\vec{f})(x)=\sup_{\eta>0}\left|\int_{(\mathbb{R}^n)^m}K(x,y_1,\cdots,y_m)v(\sqrt{|x-y_1|+,\cdots,+|x-y_m|}/\eta)\prod_{i=1}^{m}f_i(y_i)d\vec{y}\right|.
$$

For simplicity, we denote

$$K_{u,\eta}(x,y_1,\cdots,y_m)=K(x,y_1,\cdots,y_m)u(\sqrt{|x-y_1|+,\cdots,+|x-y_m|}/\eta),$$
$$K_{v,\eta}(x,y_1,\cdots,y_m)=K(x,y_1,\cdots,y_m)v(\sqrt{|x-y_1|+,\cdots,+|x-y_m|}/\eta),$$
$$U_\eta(\vec{f})=\int_{(\mathbb{R}^n)^m}K_{u,\eta}(x,y_1,\cdots,y_m)\prod_{i=1}^m f_i(y_i)d\vec{y}$$

and

$$V_\eta(\vec{f})=\int_{(\mathbb{R}^n)^m}K_{v,\eta}(x,y_1,\cdots,y_m)\prod_{i=1}^m f_i(y_i)d\vec{y}.$$

It is easy to see that $T_*(\vec{f})\leqslant U^*(\vec{f})(x)+V^*(\vec{f})(x)$.

Moreover, $T_{*,\Pi b}(\vec{f})\leqslant U^*_{\Pi b}(\vec{f})(x)+V^*_{\Pi b}(\vec{f})(x)$, where

$$\begin{aligned}U^*_{\Pi b}(\vec{f})(x)&=\sup_{\eta>0}\Big|[b_1,[b_2,\cdots[b_{m-1},[b_m,U_\eta]_m]_{m-1}\cdots]_2]_1(\vec{f})(x)\Big|\\&=\sup_{\eta>0}\Big|\int_{(\mathbb{R}^n)^m}K_{u,\eta}(x,y_1,\cdots,y_m)\prod_{j=1}^m(b_j(x)-b_j(y_j))\prod_{i=1}^m f_i(y_i)d\vec{y}\Big|,\end{aligned}$$

$$\begin{aligned}V^*_{\Pi b}(\vec{f})(x))&=\sup_{\eta>0}\Big|[b_1,[b_2,\cdots[b_{m-1},[b_m,V_\eta]_m]_{m-1}\cdots]_2]_1(\vec{f})(x)\Big|\\&=\sup_{\eta>0}\Big|\int_{(\mathbb{R}^n)^m}K_{v,\eta}(x,y_1,\cdots,y_m)\prod_{j=1}^m(b_j(x)-b_j(y_j))\prod_{i=1}^m f_i(y_i)d\vec{y}\Big|.\end{aligned}$$

We just need to prove that Lemma 3.2 holds for $U^*_{\prod\vec{b}}$ and $U^*_{\vec{b}}$. Let $c=\sup_{\eta>0}|\sum_{j=1}^3 c_{\eta,j}|$. As in Lemma 3.1, it is easy to see

$$\begin{aligned}|U^*_{\prod\vec{b}}(f_1,f_2)(z)-c|\leqslant&\ |(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)U^*(f_1,f_2)(z)|\\&+\sup_\eta|(b_1(z)-\lambda_1)[b_2,U_\eta](f_1,f_2)(z)|\\&+\sup_\eta|(b_2(z)-\lambda_2)[b_1,U_\eta](f_1,f_2)(z)|\\&+|U^*((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(z)-\sup_{\eta>0}|\sum_{j=1}^3 c_{\eta,j}||\\=&\ T_1(z)+T_2(z)+T_3(z)+T_4(z).\end{aligned}$$

Thus, we have

$$
\begin{aligned}
&\left(\frac{1}{|B|}\int_B \big||U^*_{\Pi\vec{b}}(f_1,f_2)(z)|^\delta - |c_\eta|\big|^\delta dz\right)^{\frac{1}{\delta}}\\
&\leqslant \left(\frac{1}{|B|}\int_B |U^*_{\Pi\vec{b}}(f_1,f_2)(z) - c_\eta|^\delta dz\right)^{\frac{1}{\delta}}\\
&\leqslant \left(\frac{1}{|B|}\int_B |(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)U^*(f_1,f_2)(z)|^\delta dz\right)^{\frac{1}{\delta}}\\
&\quad + \left(\frac{1}{|B|}\int_B |(b_1(z)-\lambda_1)[b_2,U^*](f_1,f_2)(z)|^\delta dz\right)^{\frac{1}{\delta}}\\
&\quad + \left(\frac{1}{|B|}\int_B |(b_2(z)-\lambda_2)[b_1,U^*](f_1,f_2)(z)|^\delta dz\right)^{\frac{1}{\delta}}\\
&\quad + \left(\frac{1}{|B|}\int_B \sup_{\eta>0}|U_\eta((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(z) - \sum_{j=1}^{3} c_{\eta,j}|^\delta dz\right)^{\frac{1}{\delta}}\\
&\doteq T_1+T_2+T_3+T_4.
\end{aligned}
$$

For T_1, by Hölder's inequality, we get

$$
\begin{aligned}
T_1 &\leqslant \left(\frac{1}{|B|}\int_B |b_1(z)-\lambda_1|^{r_1} dz\right)^{\frac{1}{r_1}} \left(\frac{1}{|B|}\int_B |b_2(z)-\lambda_2|^{r_2} dz\right)^{\frac{1}{r_2}}\\
&\quad \times \left(\frac{1}{|B|}\int_B |U^*(f_1,f_2)(z)|^r dz\right)^{\frac{1}{r}} \qquad (3.21)\\
&\leqslant C\prod_{i=1}^{2}||b_i||_{BMO} M_r(U^*(f_1,f_2))(x),
\end{aligned}
$$

where r_1, $r_2 > 1$, such that $\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{\delta}$.

For T_2, we have

$$
\begin{aligned}
T_2 &\leqslant \left(\frac{1}{|B|}\int_B |b_1(z)-\lambda_1|^s dz\right)^{\frac{1}{s}} \left(\frac{1}{|B|}\int_B |[b_2,U^*](f_1,f_2)(z)|^r dz\right)^{\frac{1}{r}} \qquad (3.22)\\
&\leqslant C||b_1||_{BMO} M_r([b_2,U^*](f_1,f_2))(x),
\end{aligned}
$$

where $s>1$ such that $\frac{1}{r}+\frac{1}{s}=\frac{1}{\delta}$.

Similarly, we have

$$
\begin{aligned}
T_3 &\leqslant \left(\frac{1}{|B|}\int_B |b_2(z)-\lambda_2|^s dz\right)^{\frac{1}{s}} \left(\frac{1}{|B|}\int_B |[b_1,U^*](f_1,f_2)(z)|^r dz\right)^{\frac{1}{r}} \qquad (3.23)\\
&\leqslant C||b_2||_{BMO} M_r([b_1,U^*](f_1,f_2))(x).
\end{aligned}
$$

For T_4, choose

$$c_{\eta,1} = U_\eta((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(x),$$

$$c_{\eta,2} = U_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0)(x),$$

$$c_{\eta,3} = U_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x).$$

We may split it in the following way

$$T_4 \leqslant T_{41}+T_{42}+T_{43}+T_{44},$$

where

$$T_{41} = \left(\frac{1}{|B|}\int_B |U^*((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)(z)|^\delta dx\right)^{\frac{1}{\delta}};$$

$$T_{42}=\left(\frac{1}{|B|}\int_B \sup_\eta |U_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(z)-U_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x)|^\delta dz\right)^{\frac{1}{\delta}};$$

$$T_{43}=\left(\frac{1}{|B|}\int_B \sup_\eta |U_\eta((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(z)-U_\eta((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(x)|^\delta dz\right)^{\frac{1}{\delta}};$$

and

$$T_{44}=\left(\frac{1}{|B|}\int_B \sup_\eta |U_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0)(z)-U_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0)(x)|^\delta dz\right)^{\frac{1}{\delta}}.$$

For T_{41}, take $\delta < \frac{1}{2}$, we get

$$\begin{aligned} T_{41} &\leqslant C||U^*((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)||_{L^{1/2,\infty}(B,\frac{dx}{|B|})} \\ &\leqslant \frac{1}{|B|}\int_B |(b_1-\lambda_1)f_1^0(z)|dz \frac{1}{|B|}\int_B |(b_2-\lambda_2)f_2^0(z)|dz \\ &\leqslant \prod_{i=1}^2 ||b_i||_{BMO} M_{L(\log L)}(f_i)(x). \end{aligned} \tag{3.24}$$

For T_{42}, since $x,z\in B$, $y_j \in \mathbb{R}^n \setminus (8\sqrt{n}+4)B$, we get $|z-x| \leqslant \sqrt{n}l(B) \leqslant \frac{1}{2}t^{\frac{1}{s}}$; $|y_j - z| > (4\sqrt{n}+1)l(B) > 2t^{\frac{1}{s}}$, hence $\phi(\frac{|y_j-z|}{t^{\frac{1}{s}}}) = 0$, for $j=1,2$. We can use Assumption (H2) to get

$$\begin{aligned} T_{42} &\leqslant \frac{1}{|B|}\int_B \sup_\eta |U_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(z)-U_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x)|dz \\ &\leqslant \frac{1}{|B|}\int_B\int_{(\mathbb{R}^n\setminus B^*)^2} (|K(z,\vec{y})-K_t^0(z,\vec{y})|+|K_t^0(z,\vec{y})-K(x,\vec{y})|)\prod_{i=1}^2 (b_i(y_i)-\lambda_i)f_i(y_i)d\vec{y}dz \\ &=: T_{42}^1+T_{42}^2. \end{aligned} \tag{3.25}$$

Let us estimate T_{42}^1 first.

$$
\begin{aligned}
T_{42}^1 &\leqslant \frac{1}{|B|}\int_B \sum_{k=1}^{\infty} \frac{|B^*|^{\frac{\varepsilon}{n}}}{(2^k|B^*|^{1/n})^{2n+\varepsilon}} \int_{(2^{k+1}B)^2} (b_1(y_1)-\lambda_1)f_1(y_1)(b_2(y_2)-\lambda_2)f_2(y_2)d\vec{y}dz \\
&\leqslant \sum_{k=1}^{\infty} \frac{|B^*|^{\frac{\varepsilon}{n}}}{(2^k|B^*|^{1/n})^{2n+\varepsilon}} \int_{2^{k+1}B} (b_1(y_1)-\lambda_1)f_1(y_1)dy_1 \int_{2^{k+1}B} (b_2(y_2)-\lambda_2)f_2(y_2)dy_2 \\
&\leqslant C\sum_{k=1}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{|2^{k+1}B^*|^{2n}} \int_{2^{k+1}B^*} (b_i(y_i)-\lambda_i)f_i(y_i)dy_i \\
&\leqslant C\prod_{i=1}^{2}\sum_{k=1}^{\infty} \frac{k}{2^{k\varepsilon}} ||b_i||_{BMO}||f_i||_{L(\log L),2^{k+1}B^*} \\
&\leqslant C\prod_{i=1}^{2} ||b_i||_{BMO} M_{L(\log L)}(f_i)(x).
\end{aligned}
$$

T_{42}^2 can be estimated in the same way. For T_{43}, we have

$$
\begin{aligned}
T_{43} &\leqslant \frac{1}{|B|}\int_B \sup_{\eta} |U_\eta((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(z) - U_\eta((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(x)|dz \\
&\leqslant \int_{B^*} |(b_1(y_1)-\lambda_1)f_1(y_1)|dy_1 \Bigg(\int_{\mathbb{R}^n\setminus B^*} \frac{t^{\varepsilon/s}|f_2(y_2)(b_2-\lambda_2)|dy_2}{|z-y_2|^{2n+\varepsilon}} \\
&\qquad + \int_{\mathbb{R}^n\setminus B^*} \frac{|f_2(y_2)(b_2-\lambda_2)|dy_2}{|z-y_2|^{2n}} \Bigg) \\
&\leqslant \int_{B^*} |(b_1(y_1)-\lambda_1)f_1(y_1)|dy_1 \int_{\mathbb{R}^n\setminus B^*} \frac{|f_2(y_2)(b_2-\lambda_2)|dy_2}{|z-y_2|^{2n}} \\
&\leqslant \frac{1}{|B^*|}\int_{B^*} |(b_1(y_1)-\lambda_1)f_1(y_1)|dy_1 |B^*| \int_{\mathbb{R}^n\setminus B^*} \frac{|f_2(y_2)(b_2-\lambda_2)|dy_2}{|z-y_2|^{2n}} \\
&\leqslant C||b_1||_{BMO}||f_1||_{L(\log L),B^*}|B^*| \sum_{k=1}^{\infty} \int_{2^{k+1}B^*\setminus 2^kB^*} \frac{|f_2(y_2)(b_2-\lambda_2)|dy_2}{|z-y_2|^{2n}} \\
&\leqslant C||b_1||_{BMO}||f_1||_{L(\log L),B^*}|B^*| \sum_{k=1}^{\infty} \frac{1}{(2^kr)^{2n}} \int_{2^{k+1}B^*\setminus 2^kB^*} |f_2(y_2)(b_2-\lambda_2)|dy_2 \\
&\leqslant C||b_1||_{BMO}||f_1||_{L(\log L),B^*} \sum_{k=1}^{\infty} \frac{1}{2^{kn}} \frac{1}{|2^{k+1}B^*|} \int_{2^{k+1}B^*} |f_2(y_2)(b_2-\lambda_2)|dy_2 \\
&\leqslant C\prod_{i=1}^{2} ||b_i||_{BMO}||f_1||_{L(\log L),B^*}||f_2||_{L(\log L),2^{k+1}B^*} \\
&\leqslant C\prod_{i=1}^{2} ||b_i||_{BMO} M_{L(\log L)}(f_i)(x).
\end{aligned}
$$

Similarly as T_{43}, we can get the estimates for T_{44}. Thus we proved (3.19).

We use the method in [1]. By linearity it is sufficient to consider the operator with only one symbol. Fix $b \in BMO$ and we consider the operator

$$U_b^*(\vec{f})(x) = \sup_{\eta>0} |b(x)U_\eta(f_1,\cdots,f_m) - U_\eta(bf_1,\cdots,f_m)(x)|.$$

Fix $x \in \mathbb{R}^n$. For any ball B with center at x, set $\lambda = b_{B^*}$ where $B^* = (8\sqrt{n}+4)B$. We have

$$U_b^*(\vec{f})(x) = \sup_{\eta>0} |(b(x)-\lambda)U_\eta(f_1,\cdots,f_m) - U_\eta((b-\lambda)f_1,\cdots,f_m)(x)|.$$

Let $c = \sup_{\eta>0} |\sum_{j=1}^3 c_{\eta,j}|$.

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B ||U_b^*(f_1,f_2)(z)|^\delta - |c|^\delta| dz\right)^{\frac{1}{\delta}} \\
&\leqslant \left(\frac{1}{|B|}\int_B |U_b^*(f_1,f_2)(z) - c|^\delta dz\right)^{\frac{1}{\delta}} \\
&\leqslant \left(\frac{1}{|B|}\int_B |(b(z)-\lambda)U^*(f_1,f_2)(z)|^\delta dz\right)^{\frac{1}{\delta}} \\
&\quad + \left(\frac{1}{|B|}\int_B \sup_{\eta>0} |U_\eta((b-\lambda)f_1,f_2)(z) - \sum_{j=1}^3 c_{\eta,j}|^\delta dz\right)^{\frac{1}{\delta}} =: (P_1+P_2).
\end{aligned}$$

For P_1, by Hölder's inequality, we get

$$\begin{aligned}
P_1 &\leqslant \left(\frac{1}{|B|}\int_B |b(z)-\lambda|^{r_1} dz\right)^{\frac{1}{r_1}} \left(\frac{1}{|B|}\int_B |U^*(f_1,f_2)(z)|^r dz\right)^{\frac{1}{r}} \\
&\leqslant C||b||_{BMO} M_\tau(U^*(f_1,f_2))(x),
\end{aligned} \tag{3.26}$$

where r, $r_1 > 1$, such that $\frac{1}{r} + \frac{1}{r_1} = \frac{1}{\delta}$. choose

$$\begin{aligned}
c_{\eta,1} &= U_\eta((b-\lambda)f_1^0, f_2^\infty)(x), \\
c_{\eta,2} &= U_\eta((b-\lambda)f_1^\infty, f_2^0)(x), \\
c_{\eta,3} &= U_\eta((b-\lambda)f_1^\infty, f_2^\infty)(x).
\end{aligned}$$

For P_2, we may split it in the following way

$$P_2 \leqslant P_{21} + P_{22} + P_{23} + P_{24},$$

where

$$\begin{aligned}
P_{21} &= \left(\frac{1}{|B|}\int_B |U^*((b-\lambda)f_1^0, f_2^0)(z)|^\delta dx\right)^{\frac{1}{\delta}}; \\
P_{22} &= \left(\frac{1}{|B|}\int_B \sup_\eta |U_\eta((b-\lambda)f_1^\infty, f_2^\infty)(z) - U_\eta((b-\lambda)f_1^\infty, f_2^\infty)(x)|^\delta dz\right)^{\frac{1}{\delta}}; \\
P_{23} &= \left(\frac{1}{|B|}\int_B \sup_\eta |U_\eta((b-\lambda)f_1^0, f_2^\infty)(z) - U_\eta((b-\lambda)f_1^0, f_2^\infty)(x)|^\delta dz\right)^{\frac{1}{\delta}}
\end{aligned}$$

and

$$P_{24}=\left(\frac{1}{|B|}\int_B \sup_\eta |U_\eta((b-\lambda)f_1^\infty,f_2^0)(z)-U_\eta((b-\lambda)f_1^\infty,f_2^0)(x)|^\delta dz\right)^{\frac{1}{\delta}}.$$

Noting that $\delta<\frac{1}{2}$, we get

$$\begin{aligned}P_{21}&\leqslant C||U^*((b-\lambda)f_1^0,f_2^0)||_{L^{1/2,\infty}(B,\frac{dx}{|B|})}\\&\leqslant \frac{1}{|B|}\int_B|(b-\lambda)f_1^0(z)|dz\frac{1}{|B|}\int_B|f_2^0(z)|dz\\&\leqslant ||b||_{BMO}\prod_{i=1}^2 M_{L(\log L)}(f_i)(x).\end{aligned}\tag{3.27}$$

Since $x,z\in B,\ y_j\in\mathbb{R}^n\setminus(8\sqrt{n}+4)B$, we get $|z-x|\leqslant\sqrt{n}l(B)\leqslant\frac{1}{2}t^{\frac{1}{s}};\ |y_j-z|>(4\sqrt{n}+1)l(B)>2t^{\frac{1}{s}}$, hence $\phi(\frac{|y_j-z|}{t^{\frac{1}{s}}})=0$, for $j=1,2$. We can use Assumption (H2) to get

$$\begin{aligned}P_{22}&=\left(\frac{1}{|B|}\int_B\sup_\eta|U_\eta((b-\lambda)f_1^\infty,f_2^\infty)(z)-U_\eta((b-\lambda)f_1^\infty,f_2^\infty)(x)|^\delta dz\right)^{\frac{1}{\delta}}\\&\leqslant\frac{1}{|B|}\int_B\sup_\eta|U_\eta((b-\lambda)f_1^\infty,f_2^\infty)(z)-U_\eta((b-\lambda)f_1^\infty,f_2^\infty)(x)|dz\\&\leqslant\frac{1}{|B|}\int_B\int_{(\mathbb{R}^n\setminus B^*)^m}(|K(z,\vec{y})-K_t^0(z,\vec{y})|+|K_t^0(z,\vec{y})-K(x,\vec{y})|)\\&\quad\times(b(y_1)-\lambda)f_1(y_1)f_2(y_2)dy_1dy_2dz\\&\doteq P_{22}^1+P_{22}^2.\end{aligned}\tag{3.28}$$

The estimate of P_{22} is similar to T_{42}, we have

$$\begin{aligned}P_{22}^1&\leqslant\frac{1}{|B|}\int_B\sum_{k=1}^\infty\frac{|B^*|^{\frac{\varepsilon}{n}}}{(2^k|B^*|^{1/n})^{2n+\varepsilon}}\int_{(2^{k+1}B)^2}(b(y_1)-\lambda)f_1(y_1)f_2(y_2)dy_1dy_2dz\\&\leqslant\sum_{k=1}^\infty\frac{|B^*|^{\frac{\varepsilon}{n}}}{(2^k|B^*|^{1/n})^{2n+\varepsilon}}\int_{2^{k+1}B}(b(y_1)-\lambda)f_1(y_1)dy_1\int_{2^{k+1}B}f_2(y_2)dy_2\\&\leqslant C\sum_{k=1}^\infty\frac{1}{2^{k\varepsilon}}\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}(b(y_1)-\lambda)f_1(y_1)dy_1\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B}f_2(y_2)dy_2\\&\leqslant C\sum_{k=1}^\infty\frac{k}{2^{k\varepsilon}}||b||_{BMO}||f_1||_{L(\log L),2^{k+1}B^*}|f_2|_{2^{k+1}B}\\&\leqslant C||b||_{BMO}\prod_{i=1}^2M_{L(\log L)}(f_i)(x).\end{aligned}\tag{3.29}$$

Similarly,

$$P_{22}^2\leqslant C||b||_{BMO}\prod_{i=1}^2M_{L(\log L)}(f_i)(x).\tag{3.30}$$

$$
\begin{aligned}
P_{23} &= \left(\frac{1}{|B|}\int_B \sup_\eta |U_\eta((b-\lambda)f_1^0,f_2^\infty)(z)-U_\eta((b-\lambda)f_1^0,f_2^\infty)(x)|^\delta dz\right)^{\frac{1}{\delta}}\\
&\leqslant \frac{1}{|B|}\int_B \sup_\eta |U_\eta((b-\lambda)f_1^0,f_2^\infty)(z)-U_\eta((b-\lambda)f_1^0,f_2^\infty)(x)|dz\\
&\leqslant \int_{B^*}|(b(y_1)-\lambda)f_1(y_1)|dy_1\left(\int_{\mathbb{R}^n\setminus B^*}\frac{t^{\varepsilon/s}|f_2(y_2))|dy_2}{|z-y_2|^{2n+\varepsilon}}+\int_{\mathbb{R}^n\setminus B^*}\frac{|f_2(y_2)|dy_2}{|z-y_2|^{2n}}\right)\\
&\leqslant \int_{B^*}|(b(y_1)-\lambda)f_1(y_1)|dy_1\int_{\mathbb{R}^n\setminus B^*}\frac{|f_2(y_2)|dy_2}{|z-y_2|^{2n}}\\
&\leqslant \frac{1}{|B^*|}\int_{B^*}|(b(y_1)-\lambda)f_1(y_1)|dy_1|B^*|\int_{\mathbb{R}^n\setminus B^*}\frac{|f_2(y_2)|dy_2}{|z-y_2|^{2n}}\\
&\leqslant C||b||_{BMO}||f_1||_{L(\log L),B^*}|B^*|\sum_{k=1}^\infty\int_{2^{k+1}B^*\setminus 2^kB^*}\frac{|f_2(y_2)|dy_2}{|z-y_2|^{2n}}\\
&\leqslant C||b||_{BMO}||f_1||_{L(\log L),B^*}|B^*|\sum_{k=1}^\infty\frac{1}{(2^kr)^{2n}}\int_{2^{k+1}B^*\setminus 2^kB^*}|f_2(y_2)|dy_2\\
&\leqslant C||b||_{BMO}||f_1||_{L(\log L),B^*}\sum_{k=1}^\infty\frac{1}{2^{kn}}\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}|f_2(y_2)|dy_2\\
&\leqslant C||b||_{BMO}\prod_{i=1}^2||f_1||_{L(\log L),B^*}||f_2||_{L(\log L),2^{k+1}B^*}\\
&\leqslant C||b||_{BMO}\prod_{i=1}^2 M_{L(\log L),B}(f_i)(x). \qquad (3.31)
\end{aligned}
$$

We can estimate P_{24} in the same way. Thus we finish the proof of (3.20). Then Lemma 3.2 is proved. □

Now we prove Theorem 1.4 first.

Proof. By Lemma 3.1, we have

$$
\begin{aligned}
&||T_{\prod\vec{b}}\vec{f}||_{L^p(\omega)} \leqslant ||M_A^\sharp T_{\prod\vec{b}}\vec{f}||_{L^p(\omega)}\\
&\leqslant C[\prod_{i=1}^2||b_i||_{BMO}||M_rT(f_1,f_2)||_{L^p(\omega)}+||b_1||_{BMO}||M_r([b_2,T](f_1,f_2))||_{L^p(\omega)}\\
&\quad +||b_2||_{BMO}||M_r([b_1,T](f_1,f_2))||_{L^p(\omega)}+\prod_{i=1}^2||b_i||_{BMO}||M_{q_1}f_1M_{q_2}f_2||_{L^p(\omega)}].
\end{aligned}
$$

We can choose $1<r<p$, $1<q_1<p_1$, $1<q_2<p_2$, by Theorem 1.1 in [19], we get

$$
||M_rT(f_1,f_2)||_{L^p(\omega)} \leqslant C||(T(f_1,f_2))||_{L^p(\omega)} \leqslant C\prod_{i=1}^2||f_i||_{L^{p_i}(\omega)}.
$$

$$
||M_{q_1}f_1M_{q_2}f_2||_{L^p(\omega)} \leqslant C||M_{q_1}f_1||_{L^{p_1}(\omega)}||M_{q_1}f_1||_{L^{p_2}(\omega)} \leqslant C\prod_{i=1}^2||f_i||_{L^{p_i}(\omega)}.
$$

Similarly,

$$||M_r([b_2,T](f_1,f_2))||_{L^p(\omega)} \leqslant ||[b_2,T](f_1,f_2)||_{L^p(\omega)} \leqslant C||b_2||_{BMO}\prod_{i=1}^{2}||f_i||_{L^{p_i}(\omega)}.$$

$$||M_r([b_1,T](f_1,f_2))||_{L^p(\omega)} \leqslant C||b_1||_{BMO}\prod_{i=1}^{2}||f_i||_{L^{p_i}(\omega)}.$$

Therefore, we obtain

$$||T_{\prod\vec{b}}\vec{f}||_{L^p(\omega)} \leqslant C\prod_{i=1}^{2}||b_i||_{BMO}\prod_{j=1}^{2}||f_j||_{L^{p_j}(\omega)}.$$

Let $b_1 \in BMO$ and $b_2 \in BMO$, as the argument as in [24], b_1 and b_2 can be approximated by bounded functions. Therefore, by taking limit there exists a constant C independent of f_1 and f_2 such that

$$||T_{\prod\vec{b}}\vec{f}||_{L^p(\omega)} \leqslant C\prod_{i=1}^{2}||b_i||_{BMO}\prod_{j=1}^{2}||f_j||_{L^{p_j}(\omega)}.$$

hold for all $f_1, f_2 \in L_c^\infty$. The density of L_c^∞ in $L^p(\omega)$ together with a standard argument implies Theorem 1.4. □

Proof of Theorem 1.1 and Theorem 1.2.

Proof. By Lemma 3.2, we can use the same argument as in [1] to finish the proof of Theorem 1.1 and Theorem 1.2 without any difficulty. We omit the proof. □

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