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SHARP BOUNDS FOR SEIFFERT MEAN IN TERMS OF WEIGHTED POWER MEANS OF ARITHMETIC MEAN AND GEOMETRIC MEAN

ZHEN-HANG YANG

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Abstract. For $a,b>0$ with $a\neq b$, let $P=(a-b)/(4\arctan\sqrt{a/b}-\pi)$, $A=(a+b)/2$, $G=\sqrt{ab}$ denote the Seiffert mean, arithmetic mean, geometric mean of a and b, respectively. In this paper, we present new sharp bounds for Seiffert P in terms of weighted power means of arithmetic mean A and geometric mean G:

$$\left(\tfrac{2}{3}A^{p_1}+\tfrac{1}{3}G^{p_1}\right)^{1/p_1}<P<\left(\tfrac{2}{3}A^{p_2}+\tfrac{1}{3}G^{p_2}\right)^{1/p_2},$$

where $p_1=4/5$ and $p_2=\log_{\pi/2}(3/2)$ are the best possible constants. Moreover, our sharp bounds for P are compared with other known ones, which yields a chain of inequalities involving Seiffert mean P.

1. Introduction and main results

Throughout the paper, we assume that $a,b>0$ with $a\neq b$.

Let $w\in(0,1)$. The r-th weighted power mean of positive numbers $a,b>0$ is defined as

$$M_r(a,b;w):=(wa^r+(1-w)b^r)^{1/r}\text{ if }r\neq 0\text{ and }M_0(a,b;w)=a^wb^{1-w}. \tag{1.1}$$

It is well-known that $M_r(a,b;w)$ is increasing with respect to r on $\mathbb{R}$ (see [1]). In particular, $M_r(a,b):=M_r(a,b;1/2)$ is the standard power mean. As special cases, the arithmetic mean and geometric mean are $A=A(a,b)=M_1(a,b)$ and $G=G(a,b)=M_0(a,b)$, respectively. Let $L=(a-b)/(\log a-\log b)$, $I=e^{-1}\left(b^b/a^a\right)^{1/(b-a)}$ denote the logarithmic mean and identric mean, respectively.

The Seiffert's mean defined by

$$P=P(a,b)=\frac{a-b}{4\arctan\sqrt{a/b}-\pi} \tag{1.2}$$

or

$$P=P(a,b)=\frac{a-b}{2\arcsin\frac{a-b}{a+b}} \tag{1.3}$$

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was introduced in [17], it has attracted many scholars' attention, and the inequalities involving $P(a,b)$ have been the subject of intensive research. In [18], the author proved that

$$L < P < I \tag{1.4}$$

and further showed that [19]:

$$P > \frac{3AG}{2A+G}, \tag{1.5}$$

$$P > \frac{AG}{L}, \tag{1.6}$$

$$\frac{2}{\pi}A < P < A. \tag{1.7}$$

Jagers [9] and Hästo [5] gave bounds for P in terms of power means:

$$M_{1/2} < P < M_{2/3}, \tag{1.8}$$

$$\frac{2\sqrt{2}}{\pi}M_{2/3} < P < M_{2/3}, \tag{1.9}$$

respectively. Later, Hästo obtained a sharp lower bound for P [6]:

$$P > M_{\log_\pi 2}. \tag{1.10}$$

In 2001, Sándor [16] established the following

$$\frac{A+G}{2} < P < \sqrt{\frac{A+G}{2}A}, \tag{1.11}$$

$$A^{2/3}G^{1/3} < P < \frac{2A+G}{3}. \tag{1.12}$$

The more results can be found in [4], [7], [12], [14], [20], [21].

The main purpose of this paper is to strengthen the inequalities (1.12), that is, to determine the best $p \in (0,1)$ such that the inequality

$$P > \left(\tfrac{2}{3}A^p + \tfrac{1}{3}G^p\right)^{1/p} \tag{1.13}$$

or its reverse inequality holds. Our main results are the following

THEOREM 1. *The inequality (1.13) holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leqslant p_1 = 4/5$. Moreover, we have*

$$\alpha_1 \left(\tfrac{2}{3}A^{4/5} + \tfrac{1}{3}G^{4/5}\right)^{5/4} < P < \alpha_2 \left(\tfrac{2}{3}A^{4/5} + \tfrac{1}{3}G^{4/5}\right)^{5/4}, \tag{1.14}$$

where $\alpha_1 = 1$ and $\alpha_2 = 3\sqrt[4]{24}/(2\pi) = 1.0568...$ are the best possible constants.

THEOREM 2. *The inequality* (1.13) *is reversed for all* $a,b>0$ *with* $a\neq b$ *if and only if* $p \geqslant p_2 = \log_{\pi/2}(3/2) = 0.89788\ldots$. *Moreover, we have*

$$\beta_1\left(\frac{2}{3}A^{p_2}+\frac{1}{3}G^{p_2}\right)^{1/p_2} < P < \beta_2\left(\frac{2}{3}A^{p_2}+\frac{1}{3}G^{p_2}\right)^{1/p_2}, \tag{1.15}$$

where $\beta_1 \approx 0.99237$ *and* $\beta_2 = 1$ *are the best possible constants.*

Due to (1.3) and with $x = \arcsin\frac{a-b}{a+b} \in (0,\pi/2)$, we have

$$\frac{P}{A} = \frac{\sin x}{x}, \quad \frac{G}{A} = \cos x.$$

Thus Theorem 1 and 2 can be changed as the following two equivalent theorems.

THEOREM A. *The inequality*

$$\frac{\sin x}{x} > \left(\tfrac{2}{3}+\tfrac{1}{3}(\cos x)^p\right)^{1/p} \tag{1.16}$$

holds for $x \in (0,\pi/2)$ *if and only if* $p \leqslant p_1 = 4/5$. *Moreover, we have*

$$\alpha_1\left(\tfrac{2}{3}+\tfrac{1}{3}(\cos x)^{4/5}\right)^{5/4} < \frac{\sin x}{x} < \alpha_2\left(\tfrac{2}{3}+\tfrac{1}{3}(\cos x)^{4/5}\right)^{5/4}, \tag{1.17}$$

where $\alpha_1 = 1$ *and* $\alpha_2 = 3\sqrt[4]{24}/(2\pi) = 1.0568\ldots$ *are the best possible constants.*

THEOREM B. *The inequality* (1.16) *is reversed for* $x \in (0,\pi/2)$ *if and only if* $p \geqslant p_2 = \log_{\pi/2}(3/2) = 0.89788\ldots$. *Moreover, we have*

$$\beta_1\left(\tfrac{2}{3}+\tfrac{1}{3}(\cos x)^{p_2}\right)^{1/p_2} < \frac{\sin x}{x} < \beta_2\left(\tfrac{2}{3}+\tfrac{1}{3}(\cos x)^{p_2}\right)^{1/p_2}, \tag{1.18}$$

where $\beta_1 \approx 0.99237$ *and* $\beta_2 = 1$ *are the best possible constants.*

REMARK 1. Cusa-Huygens inequality [8] refers to

$$\frac{\sin x}{x} < \tfrac{2}{3}+\tfrac{1}{3}\cos x \tag{1.19}$$

holds for $x \in (0,\pi/2)$. It is obvious that Our Theorem A and B are improvements of (1.19). Other improvements and refinements for Cusa-Huygens inequality can be found in [2], [10], [13], [14], [15].

A hyperbolic counterpart of the inequality (1.16) is due to Zhu [22, Theorem 1.1].

2. Lemmas

LEMMA 1. *Let* $M(a,b)$ *be a homogeneous mean of positive arguments* a *and* b. *Then*

$$M(a,b) = \sqrt{ab}\,M\left(e^t, e^{-t}\right),$$

where $t = \frac{1}{2}\log(a/b)$.

LEMMA 2. *Let the function* $t \mapsto F_p(t)$ *be defined on* $(0,\infty)$ *by*

$$F_p(t) = \begin{cases} \log \frac{2\sinh t}{4\arctan e^t - \pi} - \frac{1}{p}\log\left(\frac{2}{3}\cosh^p t + \frac{1}{3}\right) & \text{if } p \neq 0, \\ \log \frac{2\sinh t}{4\arctan e^t - \pi} - \cosh^{2/3} t & \text{if } p = 0. \end{cases} \tag{2.1}$$

Then we have

$$\lim_{t\to 0^+} \frac{F_p(t)}{t^4} = \frac{1}{45} - \frac{1}{36}p \tag{2.2}$$

$$F_p(\infty) = \lim_{t\to\infty} F_p(t) = \begin{cases} \frac{1}{p}\log\frac{3}{2} - \log\frac{\pi}{2} & \text{if } p > 0, \\ \infty & \text{if } p \leqslant 0. \end{cases} \tag{2.3}$$

Proof. Using power series expansion we have

$$F_p(t) = -\frac{5p-4}{180}t^4 + O\left(t^6\right),$$

which yields (2.2).

To obtain (2.3), we write $F_p(t)$ as

$$F_p(t) = \log 2 - \log\left(4\arctan e^t - \pi\right) - \frac{1}{p}\log\left(\frac{2}{3}\left(\frac{\cosh t}{\sinh t}\right)^p + \frac{1}{3}\left(\frac{1}{\sinh t}\right)^p\right),$$

from which (2.3) easily follows.

The proof ends. □

LEMMA 3. *Let the function* $t \mapsto F_p(t)$ *be defined on* $(0,\infty)$ *by* (2.1)*. Then* F_p *is strictly increasing on* $(0,\infty)$ *if* $p \in (0, 4/5]$.

Proof. Differentiation and arrangement yield

$$F_p'(t) = \frac{2\cosh^p t + \cosh^2 t}{(\cosh t \sinh t)(2\cosh^p t + 1)(4\arctan e^t - \pi)} f_1(t), \tag{2.4}$$

where

$$f_1(t) = 4\arctan e^t - \pi - 2\sinh t + \frac{2\sinh^3 t}{\cosh^2 t + 2\cosh^p t}. \tag{2.5}$$

Differentiation again and factoring lead to

$$f_1'(t) = \frac{4\sinh^2 t}{(\cosh^3 t)\left(1 + 2\cosh^{p-2} t\right)^2} f_2(\cosh t), \tag{2.6}$$

here

$$f_2(x) = (1-p)x^p - 2x^{2p-2} + px^{p-2} + 1, \quad x \in (1,\infty). \tag{2.7}$$

Simple computation reveals that

$$x^{3-p} f_2'(x) = p(1-p)x^2 + 4(1-p)x^p + p(p-2) := f_3(x), \tag{2.8}$$
$$f_3{}'(x) = 2p(1-p)\left(x + 2x^{p-1}\right). \tag{2.9}$$

If $p \in (0, 4/5]$, then

$$f_3{}'(x) = 2p(1-p)\left(x + 2x^{p-1}\right) > 0$$

for all $x > 1$, that is, f_3 is increasing on $(1, \infty)$, it is derived that

$$f_3(x) > f_3(1) = 4 - 5p > 0,$$

which together with (2.8) leads to $f_2'(x) > 0$, that is, f_2 is increasing on $(1, \infty)$. Hence, we have

$$f_2(x) > f_2(1) = 0,$$

which in conjunction with (2.6) implies that $f_1'(t) > 0$ for all $t > 0$, and then, $f_1(t) > f_1(0) = 0$. Thus it is obtained that $F_p'(t) > 0$, that is, the desired result.

The proof is completed. □

From the proof of Lemma 3 it is obtained that

$$f_1(t) = 4\arctan e^t - \pi - 2\sinh t + \frac{2\sinh^3 t}{\cosh^2 t + 2\cosh^p t} > 0,$$

which can be written as

$$\frac{2\sinh t}{4\arctan e^t - \pi} < \frac{\cosh^2 t + 2\cosh^p t}{1 + 2\cosh^p t}, \tag{2.10}$$

where $p \in (0, 4/5]$. It is easy to verify that

$$\frac{d}{dp}\frac{\cosh^2 t + 2\cosh^p t}{1 + 2\cosh^p t} = -\cosh^p t \frac{\log(\cosh t)}{(2\cosh^p t + 1)^2}(\cosh 2t - 1) < 0,$$

that is, the function $p \mapsto \frac{\cosh^2 t + 2\cosh^p t}{1 + 2\cosh^p t}$ is decreasing on $\mathbb{R}$. By Lemma 1 the result can be stated as a corollary of Lemma 3.

COROLLARY 1. *We have*

$$P < \frac{A^2 + 2A^p G^{2-p}}{G^2 + 2A^p G^{2-p}} G, \tag{2.11}$$

where the right hand of (2.11) *decreases as* p *increases on* $(-\infty, 4/5]$. *Particularly, putting* $p = 4/5,\ 0,\ ...,\ \to -\infty$ *we have*

$$P < A^{4/5} G^{-1/5} \frac{A^{6/5} + 2G^{6/5}}{2A^{4/5} + G^{4/5}} < \frac{A^2 + 2G^2}{3G} < \cdots < \frac{A^2}{G}. \tag{2.12}$$

LEMMA 4. *Let $p \in (4/5, 1)$ and the function $t \mapsto F_p(t)$ be defined on $(0, \infty)$ by (2.1). Then there is a unique number $t_3 \in (0, \infty)$ to satisfy $f_1(t_3) = 0$ such that F_p is decreasing on $(0, t_3)$ and increasing on (t_3, ∞).*

Proof. We start with (2.9) to prove this lemma. If $p \in (4/5, 1)$ then

$$f_3{}'(x) = 2p(1-p)\left(x + 2x^{p-1}\right) > 0,$$

and note that

$$f_3(1) = 4 - 5p < 0 \text{ and } f_3(\infty) = \operatorname{sgn}(p(1-p)) > 0,$$

it is seen that there is a unique number $x_1 \in (1, \infty)$ such that $f_3(x) < 0$ for $x \in (1, x_1)$ and $f_3(x) > 0$ for $x \in (x_1, \infty)$. From (2.8) it is deduced that f_2 is decreasing on $(1, x_1)$ and increasing on (x_1, ∞). And then, $f_2(x) < f_2(1) = 0$ for $x \in (1, x_1)$, but $f_2(\infty) = \operatorname{sgn}(1-p) > 0$, it follows that there is a unique number $x_2 \in (x_1, \infty)$ such that $f_2(x) < 0$ for $x \in (1, x_2)$ and $f_2(x) > 0$ for $x \in (x_2, \infty)$. Due to (2.6) this implies that there exits a unique $t_2 \in (0, \infty)$ to satisfy $\cosh t_2 = x_2$ so that the function $t \mapsto f_1(t)$ is decreasing on $(0, t_2)$ and increasing on (t_2, ∞). Hence, we have

$$f_1(t) < f_1(0) = 0 \text{ if } t \in (0, t_2).$$

However,

$$\lim_{t \to \infty} f_1(t) = \frac{\pi}{4} > 0,$$

thus there is a unique number $t_3 \in (t_2, \infty)$ to satisfy $f_1(t_3) = 0$ such that $f_1(t) < 0$ if $t \in (0, t_3)$ and $f_1(t) > 0$ if $t \in (t_3, \infty)$, which from (2.4) reveals that the function $t \mapsto F_p(t)$ is decreasing on $(0, t_3)$ and increasing on (t_3, ∞).

This completes the proof. □

3. Proofs of Main Results

Proof of Theorem 1. By symmetry, we assume that $b > a > 0$. We have

$$P\left(e^t, e^{-t}\right) = \frac{2\sinh t}{4\arctan e^t - \pi}, \quad A\left(e^t, e^{-t}\right) = \cosh t, \quad G\left(e^t, e^{-t}\right) = 1,$$

where $t = \frac{1}{2}\log(b/a) > 0$. From Lemma 1, in order to prove that inequality (1.13) holds if and only if $p \leqslant 4/5$, it is enough to show that inequalities

$$\log \frac{2\sinh t}{4\arctan e^t - \pi} > \frac{1}{p}\log\left(\frac{2}{3}(\cosh t)^p + \frac{1}{3}\right),$$

that is, $F_p(t) > 0$ holds if and only if $p \leqslant 4/5$, where $F_p(t)$ is defined by (2.4).

Necessity. If $F_p(t) > 0$ holds for all $t > 0$, then by Lemma 2 we have

$$\begin{cases} \lim_{t \to 0^+} \frac{F_p(t)}{t^4} = \frac{1}{45} - \frac{1}{36}p \geqslant 0, \\ \lim_{t \to \infty} F_p(t) = \frac{1}{p}\log\frac{3}{2} - \log\frac{\pi}{2} \geqslant 0 \text{ if } p > 0 \end{cases}$$

or

$$\begin{cases} \lim_{t\to 0^+} \frac{F_p(t)}{t^4} = \frac{1}{45} - \frac{1}{36}p \geqslant 0, \\ \lim_{t\to\infty} F_p(t) = \infty \text{ if } p \leqslant 0. \end{cases}$$

Solving the inequalities for p yields $p \leqslant 4/5$.

Sufficiency. Suppose that $p \leqslant 4/5$. Since the function

$$p \mapsto \frac{1}{p} \log\left(\tfrac{2}{3}(\cosh t)^p + \tfrac{1}{3}\right)$$

is clearly increasing, so the function $p \mapsto F_p(t)$ is decreasing, thus it is suffices to show that $F_p(t) > 0$ for all $t > 0$ if $p = p_1 = 4/5$. By Lemma 3, we see that F_{p_1} is strictly increasing on $(0,\infty)$. It follows that

$$0 = F_{p_1}(0) < F_{p_1}(t) < F_{p_1}(\infty) = \frac{5}{4}\log\frac{3}{2} - \log\frac{\pi}{2},$$

which proves the sufficiency and inequalities (1.14). Clearly,

$$\alpha_1 = \exp(0) = 1 \text{ and } \alpha_2 = \exp\left(\frac{5}{4}\log\frac{3}{2} - \log\frac{\pi}{2}\right) = 3\sqrt[4]{24}/(2\pi)$$

are the best possible constants.

Thus the proof of Theorem 1 is finished. □

Proof of Theorem 2. Clearly, the reverse inequality of (1.13) is equivalent to $F_p(t) < 0$ for $t > 0$. Now we show that $F_p(t) < 0$ holds for all $t > 0$ if and only if $p \geqslant p_2 = (\log 3 - \log 2)/(\log\pi - \log 2)$.

Necessity. The condition $p \geqslant p_2$ is necessary. Indeed, if $F_p(t) < 0$ holds for all $t > 0$, then we have

$$\begin{aligned} \lim_{t\to 0^+} \frac{F_p(t)}{t^4} &= \frac{1}{45} - \frac{1}{36}p \leqslant 0, \\ \lim_{t\to\infty} F_p(t) &= \frac{1}{p}\log\frac{3}{2} - \log\frac{\pi}{2} \leqslant 0 \text{ if } p > 0, \end{aligned}$$

which leads to $p \geqslant \log_{\pi/2}(3/2) = p_2$.

Sufficiency. The condition $p \geqslant p_2$ is also sufficient. As mentioned in proof of Theorem 1, the function $p \mapsto F_p(t)$ is decreasing, thus it is suffices to show that $F_p(t) < 0$ for all $t > 0$ if $p = p_2$.

Lemma 4 reveals that for $p \in (4/5, 1)$ there is a unique number $t_3 \in (t_2, \infty)$ to satisfy

$$f_1(t) = \arctan e^t - \frac{2\sinh t + 4\cosh^p t \sinh t + \pi\cosh^2 t + 2\pi\cosh^p t}{8\cosh^p t + 4\cosh^2 t} = 0 \qquad (3.1)$$

such that F_p is decreasing on $(0,t_3)$ and increasing on (t_3,∞). It is acquired that for $p_2 = \log_{\pi/2}(3/2) \in (4/5,1)$

$$F_{p_2}(t_3) < F_{p_2}(t) < F_{p_2}(0) = 0 \text{ if } t \in (0,t_3),$$
$$F_{p_2}(t_3) < F_{p_2}(t) < F_{p_2}(\infty) = 0 \text{ if } t \in (t_3,\infty),$$

that is,

$$F_{p_2}(t_3) < \log \frac{\frac{2\sinh t}{4\arctan e^t - \pi}}{\left(\frac{2}{3}(\cosh t)^{p_2} + \frac{1}{3}\right)^{1/p_2}} < 0,$$

which proves the sufficiency and inequalities (1.15).

Furthermore, for $p = p_2 = \log_{\pi/2}(3/2)$, solving the equation (3.1) for t by mathematical computation software yields $t_3 \approx 2.6630245$, and then

$$\beta_1 = \exp(F_{p_2}(t_3)) \approx 0.99237 \text{ and } \beta_2 = \exp(0) = 1.$$

Clearly, $\beta_1 \approx 0.99237$ and $\beta_2 = 1$ are the best possible constants.

This completes the proof of Theorem 2. □

4. Comparisons of certain bounds for P

It is mentioned in the Introduction that there has many bounds for P, some comparisons of them can refer to [7]. In this section, the bounds in the form of $M_{r_1}(A,G;2/3)$ will be compared with other ones in the form of $M_{r_2}(a,b;1/2)$, where $M_r(a,b;w)$ is defined by (1.1).

LEMMA 5. *The inequalities*

$$\left(\tfrac{2}{3}A^p + \tfrac{1}{3}G^p\right)^{1/p} < \left(\frac{a^{2/3}+b^{2/3}}{2}\right)^{3/2} < \left(\tfrac{2}{3}A^q + \tfrac{1}{3}G^q\right)^{1/q} \tag{4.1}$$

hold if and only if $p \leqslant 10/9$ *and* $q \geqslant \log_2(9/4)$.

Proof. By Lemma 1, in order to prove this lemma, it is enough to prove that for $t > 0$ inequalities

$$\tfrac{1}{p}\log\left(\tfrac{2}{3}(\cosh t)^p + \tfrac{1}{3}\right) < \tfrac{3}{2}\log\cosh\tfrac{2}{3}t < \tfrac{1}{q}\log\left(\tfrac{2}{3}(\cosh t)^q + \tfrac{1}{3}\right) \tag{4.2}$$

hold if and only if $p \leqslant 10/9$ and $q \geqslant \log_2(9/4)$.

Define that

$$G_p(t) := \frac{1}{p}\log\left(\frac{2}{3}(\cosh t)^p + \frac{1}{3}\right) - \frac{3}{2}\log\cosh\frac{2}{3}t. \tag{4.3}$$

Then we easily get

$$\lim_{t\to 0^+}\frac{G_p(t)}{t^4} = \frac{1}{36}p - \frac{5}{162}, \tag{4.4}$$

$$G_p(\infty) = \begin{cases} \frac{1}{2}\log 2 - \frac{1}{p}\log\frac{3}{2} & \text{if } p > 0, \\ -\infty & \text{if } p \leqslant 0. \end{cases} \tag{4.5}$$

On the other hand, differentiation yields

$$G_p'(t) = \frac{\left(2\sinh\frac{t}{3}\cosh t\right)\log\cosh t}{\left(\cosh\frac{2}{3}t\cosh t\right)\left(2\cosh^p t + 1\right)} L\left(\cosh^{p-1} t, \cosh\frac{1}{3}t\right) \times g_1(t), \tag{4.6}$$

where

$$g_1(t) = p - 1 - \frac{\log\cosh\frac{1}{3}t}{\log\cosh t} \tag{4.7}$$

and $L(x,y)$ is the logarithmic mean of positive numbers x and y.

Differentiation again leads to

$$\frac{3\left(\cosh\frac{1}{3}t\cosh t\right)\log^2(\cosh t)}{3\cosh\frac{1}{3}t\sinh t} g_1'(t)$$

$$= \log\left(\cosh\frac{1}{3}t\right) - \frac{\sinh\frac{1}{3}t\cosh t}{3\cosh\frac{1}{3}t\sinh t}\log(\cosh t) := g_2(t), \tag{4.8}$$

$$g_2'(t) = -\frac{2}{9}\frac{\sinh^3\frac{2}{3}t}{\cosh^2\frac{1}{3}t\sinh^2 t}\log(\cosh t) < 0 \tag{4.9}$$

for $t > 0$. It is acquired that $g_2(t) < g_2(0) = 0$, which implies that g_1 is decreasing on $(0,\infty)$.

Now we are in a position to prove the desired results.

(i) We prove the first inequality of (4.2) holds if and only if $p \leqslant 10/9$. In fact, if the first inequality of (4.2) holds, that is, $G_p(t) < 0$ for all $t > 0$, then by (4.4) and (4.5) we have

$$\begin{cases} \lim_{t\to 0^+}\frac{G_p(t)}{t^4} = \frac{1}{36}p - \frac{5}{162} \leqslant 0, \\ G_p(\infty) = \frac{1}{2}\log 2 - \frac{1}{p}\log\frac{3}{2} \leqslant 0 \text{ if } p > 0 \end{cases}$$

or

$$\begin{cases} \lim_{t\to 0^+}\frac{G_p(t)}{t^4} = \frac{1}{36}p - \frac{5}{162} \leqslant 0, \\ G_p(\infty) = -\infty \text{ if } p \leqslant 0. \end{cases}$$

Solving the above inequalities leads to $p \leqslant 10/9$.

Conversely, if $p \leqslant 10/9$, then since g_1 is decreasing on $(0,\infty)$, it is obtained that

$$g_1(t) < g_1\left(0^+\right) = \lim_{t\to 0^+}\left(p - 1 - \frac{\log\cosh\frac{1}{3}t}{\log\cosh t}\right) = p - \frac{10}{9} \leqslant 0,$$

which in combination with (4.6) reveals that $G_p'(t) < 0$. Thus we conclude that $G_p(t) < G_p(0) = 0$, that is, the first inequality of (4.2) holds.

(ii) Next we show that the second inequality of (4.2) holds if and only if $p \geqslant \log_2(9/4)$.

If the second inequality of (4.2) holds, that is, $G_p(t) > 0$ for all $t > 0$, then by (4.4) and (4.5) we have

$$\begin{cases} \lim_{t\to 0^+} \frac{G_p(t)}{t^4} = \frac{1}{36}p - \frac{5}{162} \geqslant 0, \\ G_p(\infty) = \frac{1}{2}\log 2 - \frac{1}{p}\log\frac{3}{2} \geqslant 0 \text{ if } p > 0, \end{cases}$$

which yields $p \geqslant \log_2(9/4)$.

Conversely, if $p \geqslant \log_2(9/4)$, since the function $p \to G_p(t)$ is increasing, then it suffices to show that $G_p(t) > 0$ for all $t > 0$ if $p = \log_2(9/4)$. By the monotonicity of the function g_1 and the fact that

$$g_1(0^+) = \lim_{t\to 0^+}\left(\log_2\frac{9}{4} - 1 - \frac{\log\cosh\frac{1}{3}t}{\log\cosh t}\right) = \log_2\frac{9}{4} - \frac{10}{9} > 0,$$

$$g_1(\infty) = \lim_{t\to\infty}\left(\log_2\frac{9}{4} - 1 - \frac{\log\cosh\frac{1}{3}t}{\log\cosh t}\right) = \log_2\frac{9}{4} - \frac{4}{3} < 0,$$

it is seen that there is a unique number $t_0 \in (0,\infty)$ such that $g_1(t) > 0$ if $t \in (0,t_0)$ and $g_1(t) < 0$ if $t \in (t_0,\infty)$, which together with (4.6) indicates that the function $t \to G_p(t)$ is increasing on $(0,t_0)$ and decreasing on (t_0,∞). Therefore, we conclude that

$$G_p(t) > G_p(0) = 0 \text{ for } t \in (0,t_0),$$

$$G_p(t) > G_p(\infty) = 0 \text{ for } t \in (t_0,\infty),$$

which is the desired result.

Thus the proof ends. □

LEMMA 6. *Let $r_0 = (\log 2)/\log\pi$. Then inequalities*

$$\left(\tfrac{2}{3}A^p + \tfrac{1}{3}G^p\right)^{1/p} > \left(\frac{a^{r_0} + b^{r_0}}{2}\right)^{1/r_0} \tag{4.10}$$

hold if and only if $p \geqslant \log_{\pi/2}(3/2)$, and the two sides of (4.10) are not comparable for all $a,b > 0$ with $a \neq b$ if $p < \log_{\pi/2}(3/2)$.

Proof. From (1.10) and Theorem 2 it follows that (4.10) holds if $p \geqslant \log_{\pi/2}(3/2)$, that is, the condition $p \geqslant \log_{\pi/2}(3/2)$ is sufficient to (4.10) holds for all $a,b > 0$ with $a \neq b$.

We now show that the condition $p \geqslant \log_{\pi/2}(3/2)$ is necessary. Indeed, by symmetry of a and b, we assume that $b > a$ and let $x = a/b \in (0,1)$. Then inequality (4.10) is equivalent with

$$U_p(x) := \frac{1}{p}\log\left(\tfrac{2}{3}\left(\frac{x+1}{2}\right)^p + \tfrac{1}{3}\left(\sqrt{x}\right)^p\right) - \frac{1}{r_0}\log\left(\frac{x^{r_0}+1}{2}\right) > 0,$$

where $x \in (0,1)$.

If (4.10) holds for all $a,b > 0$ with $a \neq b$, then

$$U_p\left(0^+\right) = \begin{cases} \frac{1}{p}\log\frac{2}{3} - \log 2 + \frac{1}{r_0}\log 2 \text{ if } p > 0 \\ -\infty \text{ if } p \leqslant 0 \end{cases}$$

has to be nonnegative, which leads to $p \geqslant \log_{\pi/2}(3/2)$. This completes the proof of sufficiency.

Next we show that the two sides of (4.10) are not comparable for all $a,b > 0$ with $a \neq b$ if $p < \log_{\pi/2}(3/2)$. Clearly, $U_p\left(0^+\right) < 0$ and

$$\lim_{x\to 1^-}\frac{U_p(x)}{(x-1)^2} = \frac{1}{12} - \frac{1}{8}r_0 = \frac{2\log\pi - 3\log 2}{24\log\pi} > 0.$$

From this it is seen that there exits $x_1, x_2 \in (0,1)$ such that $U_p(x) < 0$ for $x \in (0,x_1)$ and $U_p(x) > 0$ for $x \in (x_2,1)$, that is, the sign of $U_p(x)$ is not a constant. Thus the proof is completed. □

LEMMA 7. *The inequality*

$$\left(\tfrac{2}{3}A^p + \tfrac{1}{3}G^p\right)^{1/p} > \left(\frac{a^{1/2}+b^{1/2}}{2}\right)^2 \tag{4.11}$$

holds if and only if $p \geqslant \log_2 3 - 1 = 0.58496...$, *and the two sides of* (4.11) *are not comparable for all* $a,b > 0$ *with* $a \neq b$ *if* $p < \log_2 3 - 1$.

Proof. Since the right hand of (4.11) can be written as $(A+G)/2$, then the inequality (4.11) is equivalent with

$$V_p(x) = \frac{1}{p}\log\left(\frac{2}{3}x^p + \frac{1}{3}\right) - \log\left(\frac{1}{2}x + \frac{1}{2}\right) > 0,$$

where $x = A/G > 1$.

If $V_p(x) > 0$ for all $x > 1$ then $p > 0$. If not, then

$$\lim_{x\to\infty} V_p(x) = -\infty \text{ if } p \leqslant 0, \tag{4.12}$$

which yields a contradiction. Thus we get

$$V_p(\infty) = \lim_{x\to\infty} V_p(x) = \frac{1}{p}\log\frac{2}{3} - \log\frac{1}{2} \geqslant 0 \text{ if } p > 0, \tag{4.13}$$

which yields $p \geqslant \log_2 3 - 1$.

Conversely, if $p \geqslant \log_2 3 - 1$, then since the function $p \mapsto V_p(x)$ is increasing, we need to prove $V_p(x) > 0$ for all $x > 1$.

Differentiation leads to

$$V_p'(x) = \frac{-x^p}{x(x+1)(2x^p+1)}\left(x^{1-p} - 2\right),$$

which indicates that there is a unique number $x_0 = 2^{1/(1-p)}$ such that $V_p'(x) > 0$ if $x \in (1, x_0)$ and $V_p'(x) < 0$ if $x \in (x_0, \infty)$. Thus we conclude that

$$V_p(x) > V_p(1) = 0 \text{ if } x \in (1, x_0) \text{ and } V_p(x) > V_p(\infty) \geqslant 0 \text{ if } x \in (x_0, \infty),$$

that is, the desired result.

We now illustrate that the two sides of (4.11) are not comparable for all $a, b > 0$ with $a \neq b$ if $p < \log_2 3 - 1$. In fact, if $p < \log_2 3 - 1$, then via (4.12) and (4.13) it is easily seen that $V_p(\infty) < 0$. On the other hand, it is easy to derive

$$\lim_{x \to 1^+} \frac{V_p(x)}{x-1} = \frac{1}{6} > 0.$$

Consequently, there exits $x_1, x_2 \in (1, \infty)$ such that $V_p(x) > 0$ for $x \in (1, x_1)$ and $V_p(x) < 0$ for $x \in (x_2, \infty)$, that is, $\operatorname{sgn}(V_p(x))$ is not a constant.

This completes the proof. □

Using Theorem 1, 2 and Lemma 5, 6, 7, the following theorem is immediate.

THEOREM 3. *Let* $q \geqslant \log_2(9/4)$, $\log_{\pi/2}(3/2) \leqslant r \leqslant 10/9$, $\log_2 3 - 1 \leqslant s \leqslant 4/5$. *Then we have*

$$\begin{aligned}\left(\tfrac{2}{3}A^q + \tfrac{1}{3}G^q\right)^{1/q} &> M_{2/3} > \left(\tfrac{2}{3}A^r + \tfrac{1}{3}G^r\right)^{1/r} \\ &> P > \left(\tfrac{2}{3}A^s + \tfrac{1}{3}G^s\right)^{1/s} > M_{1/2}.\end{aligned} \tag{4.14}$$

REMARK 2. In [11] Kouba proved that inequalities

$$\left(\tfrac{2}{3}A^p + \tfrac{1}{3}G^p\right)^{1/p} < I < \left(\tfrac{2}{3}A^q + \tfrac{1}{3}G^q\right)^{1/q} \tag{4.15}$$

hold if and only if $p \leqslant 6/5$ and $q \geqslant (\log 3 - \log 2)/(1 - \log 2)$.

Relation (4.14) in combination with (4.15) leads to

$$\begin{aligned}\left(\tfrac{2}{3}A^p + \tfrac{1}{3}G^p\right)^{1/p} &> I > \left(\tfrac{2}{3}A^q + \tfrac{1}{3}G^q\right)^{1/q} > M_{2/3} \\ &> \left(\tfrac{2}{3}A^r + \tfrac{1}{3}G^r\right)^{1/r} > P > \left(\tfrac{2}{3}A^s + \tfrac{1}{3}G^s\right)^{1/s} > M_{1/2},\end{aligned} \tag{4.16}$$

where $p \geqslant (\log 3 - \log 2)/(1 - \log 2)$, $\log_2(9/4) \leqslant q \leqslant 6/5$, $\log_{\pi/2}(3/2) \leqslant r \leqslant 10/9$, $\log_2 3 - 1 \leqslant s \leqslant 4/5$.

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Zhen-Hang Yang
System Division Zhejiang Province
Electric Power Test and Research Institute
Hangzhou, Zhejiang
China, 31001
e-mail: yzhkm@163.com