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GEOMETRIC CONSTANTS AND CHARACTERIZATIONS OF INNER PRODUCT SPACES

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Abstract. Let X be a real normed space, let Ψ_2 denote the set of all convex functions on $[0,1]$ such that $\max\{1-t,t\} \leqslant \psi(t) \leqslant 1$, and let Φ_2 denote the set of all concave function on $[0,1]$ such that $\psi(0)=\psi(1)=1$. For each $\psi \in \Phi_2 \cup \Psi_2$, it is shown that $\| \|x\|^{-1}x + \|y\|^{-1}y\| \leqslant C_\psi \|x-y\| \|(x,y)\|_\psi^{-1}$ for all nonzero vectors x, $y \in X$, where $C_\psi = 4\max \psi(t)$. The case of $\psi = \psi_p$ $(p>0)$, defined as $\psi_p(t) = ((1-t)^p + t^p)^{1/p}$, is due to Al-Rashed, and is due to Dunkl and Williams when $p=1$. In particular, it is shown that for certain $\psi \in \Phi_2$, the inequality holds for $C_\psi = 2\psi(1/2)$ if and only if X is an inner product space; this generalizes the works of Al-Rashed and Kirk-Smiley.

1. Introduction and Preliminaries

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is called *absolute* if $\|(x,y)\| = \|(|x|,|y|)\|$ for all $(x,y) \in \mathbb{R}^2$, and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. Let AN_2 denote the family of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ_2 denote the family of all continuous convex functions ψ on $[0,1]$ such that $\max\{1-t,t\} \leqslant \psi(t) \leqslant 1$ for $t \in [0,1]$. Then as in Bonsall and Duncan [3], AN_2 and Ψ_2 are in a one-to-one correspondence under the equation $\psi(t) = \|(1-t,t)\|$ for $t \in [0,1]$.

The notion of ψ*-direct sum* of Banach spaces was introduced in Takahashi–Kato–Saito [14]. More precisely, for $\psi \in \Psi_2$ and Banach spaces X, Y, ψ*-direct sum* $X \oplus_\psi Y$ is defined to be their direct sum equipped with the norm

$$\|(x,y)\|_\psi = \|(\|x\|,\|y\|)\|_\psi$$

where $\|\cdot\|_\psi$ term in the right-hand side is the absolute normalized norm on $\mathbb{R}^2$.

Throughout this paper, unless otherwise stated, let X be a real normed linear space of dimension not less than two. For each $x \in X$, let $\operatorname{sgn} x = \|x\|^{-1}x$. In order to make

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a detailed analysis of the triangle inequality in uniformly convex spaces, Clarkson [4] introduced the concept of angular distance $\alpha[x,y]$, which is defined by

$$\alpha[x,y] = \|\operatorname{sgn} x - \operatorname{sgn} y\|$$

for each $x,\ y \in X \setminus \{0\}$. In Al-Rashed [1], for each $p > 0$, the function f_p is defined by

$$f_p(x,y) = \frac{(\|x\|^p + \|y\|^p)^{1/p}}{\|x-y\|}\alpha[x,y]$$

for $x,\ y \in X \setminus \{0\}$ with $x \neq y$, and proved that X is an inner product space if and only if $f_p(x,y) \leqslant 2^{1/p}$ for any $x,\ y \in X \setminus \{0\}$ with $x \neq y$. The case of $p = 1$ is due to Kirk and Smiley [11]. The aim of this paper is to give a generalization of this result.

Notice that $f_p(x,y) \leqslant 2^{1/p}$ for any $x,\ y \in X \setminus \{0\}$ with $x \neq y$ implies $0 < p \leqslant 1$. So it is worth considering concave version of the set Ψ_2. Let Φ_2 denote the family of all continuous concave functions ψ on $[0,1]$ such that $\psi(0) = \psi(1) = 1$. For each $\psi \in \Phi_2$, we define the function $\|\cdot\|_\psi$ on $\mathbb{R}^2$ by

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\frac{|y|}{|x|+|y|}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Let $\psi_p(t) = ((1-t)^p + t^p)^{1/p}$ for $0 < p < \infty$ and $\psi_\infty(t) = \max\{1-t, t\}$. Then $\psi_p \in \Phi_2$ if $0 < p \leqslant 1$ and $\psi_p \in \Psi_2$ if $1 \leqslant p \leqslant \infty$. Obviously, $\Phi_2 \cap \Psi_2 = \{\psi_1\}$.

For each $\psi \in \Phi_2$ and Banach spaces X, Y, we define the function $\|\cdot\|_\psi$ on $X \times Y$ by

$$\|(x,y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi.$$

where $\|\cdot\|_\psi$ term in the right-hand side is the function on $\mathbb{R}^2$ defined in above.

For each $\psi \in \Phi_2 \cup \Psi_2$, we define the function f_ψ by

$$f_\psi(x,y) = \frac{\|(x,y)\|_\psi}{\|x-y\|}\alpha[x,y]$$

for $x,\ y \in X \setminus \{0\}$ with $x \neq y$, and define the constant $C_\psi(X)$ by

$$C_\psi(X) = \sup\{f_\psi(x,y) : x,\ y \in X \setminus \{0\},\ x \neq y\}.$$

Notice that $f_{\psi_p} = f_p$ and $C_{\psi_1}(X) = DW(X)$, where $DW(X)$ is the Dunkl–Williams constant (cf. [10]).

2. Basic properties of the constant $C_\psi(X)$

Some basic properties of f_ψ are collected in the following lemma.

LEMMA 1. *For $\psi \in \Phi_2 \cup \Psi_2$ and for all nonzero vectors $x,\ y \in X$ with $x \neq y$, we have*
(a) $f_\psi \geqslant 0$.
(b) $f_\psi(x,y) = 2\psi(1/2)$, *whenever* $\|x\| = \|y\|$.
(c) $f_\psi(x,y) \leqslant 2(\|(x,y)\|_\psi / \|x-y\|)$.
(d) $f_\psi(x,y) = f_\psi(\gamma x, \gamma y)$, *for any* $\gamma \neq 0$.
(e) *If $y = \gamma x$ with $\gamma \neq 1$, then*

$$f_\psi(x,y) = \begin{cases} 0 & \text{if } \gamma > 0, \\ 2\frac{\|(1,\gamma)\|_\psi}{1-\gamma} & \text{if } \gamma < 0. \end{cases}$$

PROPOSITION 1. *Let $\psi \in \Phi_2 \cup \Psi_2$. Then,* $2\max\psi(t) \leqslant C_\psi(X)$.

Proof. By part (e) of Lemma 1, we have

$$f_\psi\left(x, \frac{-t}{1-t}x\right) = 2\psi(t)$$

for each $t \in (0,1)$. Also, we have $2(\|(1,\gamma)\|_\psi/(1-\gamma)) \to 2$ as $\gamma \to -0$. Thus we obtain $2\max\psi(t) \leqslant C_\psi(X)$. □

The following result is due to Dunkl and Williams [6].

THEOREM 1. $2 \leqslant DW(X) = C_{\psi_1}(X) \leqslant 4$.

The next result shows the basic property of the constant $C_\psi(X)$.

THEOREM 2. *Let $\phi,\ \psi \in \Phi_2 \cup \Psi_2$. Put*

$$m = \min \frac{\psi(t)}{\phi(t)} \text{ and } M = \max \frac{\psi(t)}{\phi(t)}.$$

Then

$$mC_\phi(X) \leqslant C_\psi(X) \leqslant MC_\phi(X).$$

Proof. This follows from the fact that

$$m\|\cdot\|_\phi \leqslant \|\cdot\|_\psi \leqslant M\|\cdot\|_\phi. \quad \square$$

By letting $\phi = \psi_1$ in the preceding theorem, we have

COROLLARY 1. *Let $\psi \in \Phi_2 \cup \Psi_2$. Then*

$$DW(X)\min\psi(t) \leqslant C_\psi(X) \leqslant DW(X)\max\psi(t).$$

From Proposition 1 and the fact that $DW(X) \leqslant 4$, we have

COROLLARY 2. *Let* $\psi \in \Phi_2 \cup \Psi_2$. *Then*

$$2\max\psi(t) \leqslant C_\psi(X) \leqslant 4\max\psi(t).$$

From the fact that X is an inner product space if and only if $DW(X) = 2$, Corollaries 1 and 2 imply

THEOREM 3. *Let* X *be an inner product space and let* $\psi \in \Phi_2 \cup \Psi_2$. *Then* $C_\psi(X) = 2\max\psi(t)$.

As in Thoerem 6, if ψ is concave and satisfying the certain condition, then the fact that $C_\psi(X) = 2\max\psi(t)$ characterizes inner product spaces. So, the case of $\psi \in \Phi_2$ is essential in this direction.

3. Characterizations of inner product spaces

We recall some definitions and facts that will be needed in the sequel. For $x, y \in X$, x is said to be *BJ-orthogonal* to y, denoted by $x \perp_B y$, if $\|x + \gamma y\| \geqslant \|x\|$ for all real number γ. The BJ-orthogonality is *homogeneous*, that is, $x \perp_B y$ implies $\alpha x \perp_B \beta y$ for any real numbers α, β. However, it is not *symmetric* in general, that is, $x \perp_B y$ does not necessarily imply $y \perp_B x$. It is known that for a normed linear space X with $\dim X \geqslant 3$, BJ-orthogonality is symmetric if and only if X is an inner product space (cf. [5, 8]). For more details in this direction, the reader is referred to Birkhoff [2] and James [7, 8, 9].

We need the following lemma.

LEMMA 2. *Let* $\psi \in \Phi_2$. *Then the function* $t \mapsto \psi(t)/(1-t)$ *is strictly increasing on* $[0,1)$.

Proof. Let s, t be real numbers such that $0 \leqslant s < t < 1$. Then by concavity of ψ, we have

$$\begin{aligned}
\psi(t) &= \psi\left(\frac{1-t}{1-s}s + \frac{t-s}{1-s}1\right) \\
&\geqslant \frac{1-t}{1-s}\psi(s) + \frac{t-s}{1-s}\psi(1) \\
&> \frac{1-t}{1-s}\psi(s). \quad \square
\end{aligned}$$

The next result shows the relationship between the constant $C_\psi(X)$ and the BJ-orthogonality.

THEOREM 4. *Let* $\psi \in \Phi_2$ *such that* $\max\psi(t) = \psi(1/2)$. *If* $C_\psi(X) = 2\psi(1/2)$, *then the* BJ*-orthogonality is symmetric.*

Proof. Let $x,\ y \in X \setminus \{0\}$ such that $x \perp_B y$. Then

$$\|\alpha x + \beta y\| \geqslant \|\alpha x\|$$

for any real numbers $\alpha,\ \beta$. Let γ be a nonzero real number. Since

$$\begin{aligned}\alpha[\gamma x + y, y] &= \left\| \frac{\gamma x}{\|\gamma x + y\|} + \left(\frac{1}{\|\gamma x + y\|} - \frac{1}{\|y\|} \right) y \right\| \\ &\geqslant \frac{\|\gamma x\|}{\|\gamma x + y\|},\end{aligned}$$

we have

$$\begin{aligned}2\psi(1/2) \geqslant f_\psi(\gamma x + y, y) &= \frac{\|(\gamma x + y, y)\|_\psi}{\|\gamma x\|}\alpha[\gamma x + y, y] \\ &\geqslant \frac{\|(\gamma x + y, y)\|_\psi}{\|\gamma x + y\|} \\ &= \frac{\|\gamma x + y\| + \|y\|}{\|\gamma x + y\|}\psi\left(\frac{\|y\|}{\|\gamma x + y\| + \|y\|} \right).\end{aligned}$$

Putting $t = \|y\|/(\|\gamma x + y\| + \|y\|)$, then

$$\frac{\psi(1/2)}{1/2} \geqslant \frac{\psi(t)}{1-t}.$$

By Lemma 2, we have $t \leqslant 1/2$. Thus

$$\|y + \gamma x\| \geqslant \|y\|$$

holds for all $\gamma \in \mathbb{R}$, and so $y \perp_B x$. This completes the proof. □

COROLLARY 3. ([1]) *Let* $0 < p \leqslant 1$. *If* $C_{\psi_p}(X) = 2^{1/p}$, *then the BJ-orthogonality is symmetric.*

As was mentioned in the beginning of this section, the symmetry of the *BJ*-orthogonality characterizes inner product spaces among all normed spaces with dimension not less than 3. Thus we have

COROLLARY 4. *Suppose that* $\dim X \geqslant 3$. *Let* $\psi \in \Phi_2$ *such that* $\max \psi(t) = \psi(1/2)$. *Then* X *is an inner product space if and only if* $C_\psi(X) = 2\psi(1/2)$.

As will be seen in Theorem 6, it turns out that the restriction on the dimension on X is redundant if we add a certain assumption on ψ.

The following result is due to Lorch [12, p. 525] which is useful for our purpose.

THEOREM 5. *The following are equivalent*:

(i) X *is an inner product space.*

(ii) *For any* $x,\ y \in X$ *with* $\|x\| = \|y\|$ *and for any* $\alpha \in \mathbb{R} \setminus \{0\}$, *we have*

$$\|\alpha x + \alpha^{-1} y\| \geqslant \|x + y\|.$$

The next lemma will be needed.

LEMMA 3. *Let* $\psi \in \Phi_2$. *Then the following are equivalent*:
(i) *For all* $t \in [0,1]$, *we have*

$$\psi(t) \geqslant 2\psi(1/2)\sqrt{t(1-t)}.$$

(ii) *For all nonzero real number* α, *we have*

$$\|(\alpha,\alpha^{-1})\|_\psi \geqslant \|(1,1)\|_\psi.$$

Proof. Suppose that (i) holds. For each $\alpha > 0$,

$$(\alpha,\alpha^{-1}) = \frac{\alpha^2+1}{\alpha}\left(\frac{\alpha^2}{\alpha^2+1},\frac{1}{\alpha^2+1}\right).$$

Put $t = 1/(\alpha^2+1)$. Then we have

$$\|(\alpha,\alpha^{-1})\|_\psi = \frac{\psi(t)}{\sqrt{t(1-t)}} \geqslant 2\psi(1/2) = \|(1,1)\|_\psi.$$

Conversely, assume that (ii) holds. For each $t \in (0,1)$, we have

$$\frac{\psi(t)}{\sqrt{t(1-t)}} = \left\|\left(\frac{\sqrt{1-t}}{\sqrt{t}},\frac{\sqrt{t}}{\sqrt{1-t}}\right)\right\|_\psi \geqslant \|(1,1)\|_\psi = 2\psi(1/2).$$

This completes the proof. □

There are many elements of Φ_2 satisfying the condition (i) in Lemma 3. First example is the following.

EXAMPLE 1. Let $0 < p \leqslant 1$ and let α be a nonzero real number. Since $(|\alpha|^p-1)^2 \geqslant 0$, we have

$$\|(\alpha,\alpha^{-1})\|_p = (|\alpha|^p+|\alpha|^{-p})^{1/p} \geqslant 2^{1/p} = \|(1,1)\|_p.$$

Hence, by Lemma 3,

$$\psi_p(t) \geqslant 2\psi_p(1/2)\sqrt{t(1-t)}\ (t \in [0,1]).$$

The next example contains elements of Φ_2 that do not satisfy the condition (i) in Lemma 3.

EXAMPLE 2. For each $1 \leqslant \gamma \leqslant 3/2$, let $\psi_\gamma(t) = \min\{1+t,\gamma,2-t\}$. Then, $\psi_\gamma(t) = \psi_\gamma(1-t)$ and

$$\frac{\psi_\gamma(t)^2}{t(1-t)} = \begin{cases} \frac{(1+t)^2}{t(1-t)} & \text{if } t \in (0,\gamma-1], \\ \frac{\gamma^2}{t(1-t)} & \text{if } t \in [\gamma-1,1/2]. \end{cases}$$

Also, we have

$$\frac{d}{dt}\frac{\psi_\gamma(t)^2}{t(1-t)} = \begin{cases} \frac{(1+t)(3t-1)}{t^2(1-t)^2} & \text{if } t \in (0, \gamma-1), \\ \frac{\gamma^2(2t-1)}{t^2(1-t)^2} & \text{if } t \in (\gamma-1, 1/2]. \end{cases}$$

This implies that the function

$$t \mapsto \frac{\psi_\gamma(t)^2}{t(1-t)}$$

is decreasing on $(0,1/3] \cup [\gamma-1, 1/2]$ and increasing on $[1/3, \gamma-1]$. Thus, if $\gamma \leqslant 4/3$, we have

$$\frac{\psi_\gamma(t)^2}{t(1-t)} \geqslant \frac{\psi_\gamma(1/2)^2}{(1/2)(1-(1/2))} = 4\psi_\gamma(1/2)^2.$$

On the other hand, if $\gamma > 4/3$, then

$$\tfrac{\psi_\gamma(1/3)^2}{(1/3)(1-(1/3))} = 8 \text{ and } \tfrac{\psi_\gamma(1/2)^2}{(1/2)(1-(1/2))} = 4\psi_\gamma(1/2)^2 = 4\gamma^2.$$

Therefore, we have

$$\psi_\gamma(t) \geqslant 2\psi_\gamma(1/2)\sqrt{t(1-t)} \ (t \in [0,1])$$

if $1 \leqslant \gamma \leqslant \sqrt{2}$, and

$$\psi_\gamma(1/3) < 2\psi_\gamma(1/2)\sqrt{(1/3)(1-(1/3))}$$

if $\sqrt{2} < \gamma \leqslant 3/2$.

Our aim in this note is the following

THEOREM 6. *Let* $\psi \in \Phi_2$ *such that* $\max \psi(t) = \psi(1/2)$ *and*

$$\psi(t) \geqslant 2\psi(1/2)\sqrt{t(1-t)} \ (t \in [0,1]).$$

Then X *is an inner product space if and only if* $C_\psi(X) = 2\psi(1/2)$.

Proof. Let α be a nonzero real number and let $x,\ y \in X \setminus \{0\}$ with $\|x\| = \|y\|$. If $\|\alpha x + \alpha^{-1} y\| = 0$, then $\alpha^2 = 1$, in which case $\|\alpha x + \alpha^{-1} y\| = \|x+y\|$. So, we assume that $\|\alpha x + \alpha^{-1} y\| > 0$. Then by Lemma 3, we have

$$\begin{aligned} 2\psi(1/2) &\geqslant f_\psi(\alpha x, -\alpha^{-1} y) \\ &= \frac{\|(\alpha, \alpha^{-1})\|_\psi}{\|\alpha x + \alpha^{-1} y\|}\|x+y\| \\ &\geqslant \frac{\|(1,1)\|_\psi}{\|\alpha x + \alpha^{-1} y\|}\|x+y\| \\ &= \frac{2\psi(1/2)}{\|\alpha x + \alpha^{-1} y\|}\|x+y\|. \end{aligned}$$

This implies $\|\alpha x + \alpha^{-1} y\| \geqslant \|x+y\|$. Thus, by Theorem 5, X is an inner product space. This completes the proof. $\square$

COROLLARY 5. ([1]) *Let* $0 < p \leqslant 1$. *Then* X *is an inner product space if and only if* $C_{\psi_p}(X) = 2^{1/p}$.

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