Mathematical
Inequalities
& Applications
Volume 17, Number 2 (2014), 627–639 doi:10.7153/mia-17-46

MORREY SPACES ARE CLOSELY EMBEDDED BETWEEN VANISHING STUMMEL SPACES

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(*Communicated by L. Maligranda*)

Abstract. We prove a new property of Morrey function spaces by showing that the generalized local Morrey spaces are embedded between weighted Lebesgue spaces with weights differing only by a logarithmic factor. This leads to the statement that the generalized global Morrey spaces are embedded between two generalized Stummel classes whose characteristics similarly differ by a logarithmic factor. We give examples proving that these embeddings are strict. For the generalized Stummel spaces we also give an equivalent norm.

1. Introduction

The classical Morrey spaces $\mathscr{L}^{p,\lambda}(\Omega)$, over an open set $\Omega \subseteq \mathbb{R}^n$, defined by the norm

$$\|f\|_{p,\lambda} := \sup_{x\in\Omega, r>0} \left(\frac{1}{r^\lambda} \int\limits_{\widetilde{B}(x,r)} |f(y)|^p\, dy \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty, \quad 0 \leqslant \lambda \leqslant n, \tag{1.1}$$

where $\widetilde{B}(x,r) = \Omega \cap B(x,r)$, are well known, in particular, because of their usage in the study of regularity properties of solutions to PDE, see for instance the books [7], [11], [19] and references therein. There are also known various generalizations of the classical Morrey spaces $\mathscr{L}^{p,\lambda}$, we refer for instance to the surveying paper [16]. One of the direct generalizations is obtained by replacing r^λ in (1.1) by a function $\varphi(r)$, mainly satisfying some monotonicity type conditions. We also denote it as $\mathscr{L}^{p,\varphi}(\Omega)$ without danger of confusion. Such spaces appeared in [20], and were widely studied by various authors, see for instance [2], [3], [8], [9], [15] and references therein. The spaces $\mathscr{L}^{p,\varphi}_{\mathrm{loc};\{x_0\}}(\Omega)$, defined by the norm

$$\|f\|_{p,\varphi;\mathrm{loc}} := \sup_{r>0} \left(\frac{1}{\varphi(r)} \int\limits_{\widetilde{B}(x_0,r)} |f(y)|^p\, dy \right)^{\frac{1}{p}}, \tag{1.2}$$

Mathematics subject classification (2010): Primary 46E15; Secondary 42B35.
Keywords and phrases: function space, local Morrey space, global Morrey space, generalized Morrey space, weighted space, Stummel class.

Paper MIA-17-46

where $x_0 \in \Omega$, are known as generalized *local Morrey spaces*. Spaces of type (1.1) are often called *global Morrey spaces*, in comparison with the local ones.

Strange enough, in spite of enormous literature on Morrey type spaces, it seems that it became unobserved that the local "Morrey type behaviour" of a function f determined by the finiteness of the norm (1.2), is very close to the p-integrability of f with the weight

$$\rho_{x_0}(x) = \frac{1}{\varphi(|x-x_0|)}. \tag{1.3}$$

Namely, the local space $\mathscr{L}^{p,\varphi}_{\mathrm{loc};\{x_0\}}(\Omega)$ contains such a weighted Lebesgue spaces, but is embedded into a close Lebesgue space with and the same weight multiplied by a power of logarithm. More precisely, with the notation

$$L^p(\Omega,\rho) := \left\{ f : \int\limits_{\Omega} \rho(x)|f(x)|^p\,dx < \infty \right\},$$

in this note we prove the continuous embeddings

$$L^p\left(\Omega,\rho_{x_0}\right) \hookrightarrow \mathscr{L}^{p,\varphi}_{\{x_0\}}(\Omega) \hookrightarrow \bigcap_{\varepsilon>0} L^p\left(\Omega, \frac{\rho_{x_0}(x)}{\ln^{1+\varepsilon}\frac{A}{|x-x_0|}}\right), \tag{1.4}$$

where $\rho_{x_0}(x)$ is the weight function (1.3), Ω is a bounded open set and $A >$ diam Ω (a modification for unbounded sets is obvious). The space in the upper embedding is assumed tto be equipeed with the natural topology. The lower embedding in (1.4) is known, the novelty is in the upper embedding.

By means of these localized embeddings, we prove that the following embeddings

$$V\mathfrak{S}^{p,\varphi}(\Omega) \hookrightarrow \mathscr{L}^{p,\varphi}(\Omega) \hookrightarrow \bigcap_{\varepsilon>0} V\mathfrak{S}^{p,\varphi\ln^{1+\varepsilon}}(\Omega)$$

hold between the global Morrey spaces $\mathscr{L}^{p,\varphi}(\Omega)$ and the so called *Stummel classes* $V\mathfrak{S}^{p,\varphi}(\Omega)$, with the "logarithmic gap" between the lower and upper embeddings (we prefer to call these Stummel classes *vanishing Stummel spaces*).

We also provide examples of functions showing that the above embeddings are strict under the corresponding assumptions on the function φ.

There are also known generalizations of Morrey spaces where $\sup_{r>0}$ is replaced by the $\|\cdot\|_{L^\theta}$-norm (mainly for $\Omega = \mathbb{R}^n$):

$$\|f\|_{\mathscr{L}^{p,\theta,\varphi}(\Omega)} := \sup_{x\in\Omega}\left(\int_0^\ell \left(\frac{1}{\varphi(r)}\int_{B(x,r)}|f(y)|^p dy\right)^{\frac{\theta}{p}} \frac{dr}{r}\right)^{\frac{1}{\theta}}, \quad \ell = \mathrm{diam}\,\Omega. \tag{1.5}$$

Such spaces first appeared (in the case $\varphi(r) = r^\lambda$) in [1], p. 44, but their wide study was made in [8], [9], [4], see also references therein.

The embeddings 1.4 are proved in Section 2. In Section 2 we study the generalized Stummel spaces and construct nontrivial examples of functions in Stummel spaces not belonging to vanishing Stummel spaces (Subsection 3.2) and for generalized Stummel spaces we provide an equivalent norm (Subsection 3.3). Embeddings corresponding to the generalized global Morrey spaces are proved in Section 4.

2. Embeddings of local Morrey spaces

The function $\varphi(r)$ defining Morrey spaces is assumed to be, as usual, a bounded non-negative continuous function on $[0,\ell]$, $0<\ell\leqslant\infty$ (in fact it suffice to assume that it is continuous only in a neigbourhood of the origin), strictly positive for $r>0$ and non-decreasing.

Note that Morrey spaces are often considered with almost increasing function φ. Since an almost increasing function is equivalent to a non-decreasing function and Morrey space remains the same under replacement of φ by an equivalent function, we assume that φ is non-decreasing.

We also suppose that φ is absolutely continuous and its derivative satisfies the condition

$$P := \sup_{0<t<\ell} \frac{t\varphi'(t)}{\varphi(t)} < \infty. \tag{2.1}$$

For some goals we will also impose the Zygmund condition

$$\int_0^r \frac{\varphi(t)}{t}dt \leqslant C\varphi(r), \quad 0<r\leqslant\ell. \tag{2.2}$$

on the function φ.

THEOREM 2.1. *Let Ω be a bounded open set, $1\leqslant p<\infty$ and φ be non-decreasing and satisfy the condition* (2.1). *Then the embeddings* (1.4) *hold:*

$$c_\varepsilon\|f\|_{L^p\left(\Omega,\frac{\rho_{x_0}(x)}{\ln^{1+\varepsilon}\frac{A}{|x-x_0|}}\right)} \leqslant \|f\|^p_{\mathscr{L}^{p,\varphi}_{loc;\{x_0\}}(\Omega)} \leqslant \|f\|_{L^p(\Omega,\rho_{x_0})} \tag{2.3}$$

where $c_\varepsilon = \left(2+\frac{1+P\ln\frac{A}{\ell}}{\varepsilon}\right)^{-1}\ln^{-1-\varepsilon}\frac{A}{\ell}$. *The upper embedding is strict, i.e.*

$$L^p\left(\Omega,\frac{\rho_{x_0}(x)}{\ln^{1+\varepsilon}\frac{A}{|x-x_0|}}\right) \neq \mathscr{L}^{p,\varphi}_{\{x_0\}}(\Omega).$$

The lower embedding in (1.4) *is strict, if we additionally suppose that φ satisfies the condition* (2.2) *(in particular, if $\varphi(r)=r^\lambda$, $\lambda>0$).*

Proof. We take $x_0=0$ for simplicity, supposing that $0\in\Omega$.

1°. The lower embedding is trivial: since φ is non-decreasing, we have

$$\left(\int_\Omega \frac{|f(y)|^p}{\varphi(|y|)}\,dy\right)^{\frac{1}{p}} \geqslant \left(\int_{\widetilde{B}(0,r)} \frac{|f(y)|^p}{\varphi(|y|)}\,dy\right)^{\frac{1}{p}} \geqslant \left(\frac{1}{\varphi(r)}\int_{\widetilde{B}(0,r)} |f(y)|^p\,dy\right)^{\frac{1}{p}} \qquad (2.4)$$

so that $\|f\|_{L^p\left(\Omega,\frac{1}{\varphi}\right)} \geqslant \|f\|_{\mathscr{L}^{p,\varphi}_{\mathrm{loc};\{0\}}(\Omega)}$.

2°. The upper embedding. With the notation

$$\psi_\varepsilon(|x|) := \frac{1}{\varphi(|x|)\ln^{1+\varepsilon}\frac{A}{|x|}}$$

we have

$$\int_{\widetilde{B}(0,r)} |f(y)|^p\psi_\varepsilon(|y|)dy = C_\varepsilon\int_{\widetilde{B}(0,r)} |f(y)|^p\,dy - \int_{\widetilde{B}(0,r)} |f(y)|^p\left(\int_{|y|}^{\ell}\frac{d}{dt}\psi_\varepsilon(t)dt\right)dy, \qquad (2.5)$$

where $C_\varepsilon = \psi_\varepsilon(\ell)$. Therefore,

$$\begin{aligned}
&\int_{\widetilde{B}(0,r)} |f(y)|^p\psi_\varepsilon(|y|)dy\\
&\leqslant C_\varepsilon\int_{\widetilde{B}(0,r)} |f(y)|^p\,dy + \int_0^\ell |\psi_\varepsilon'(t)|\left(\int_{\{y\in\Omega:|y|<\min\{r,t\}\}} |f(y)|^p dy\right)dt\\
&\leqslant C_\varepsilon\|f\|^p_{L^p(\widetilde{B}(0,r))} + \int_0^\ell |\psi_\varepsilon'(t)|\cdot\|f\|^p_{L^p(\widetilde{B}(0,t))}dt\\
&\leqslant \left[C(\varepsilon)\varphi(\ell) + \int_0^\ell |\psi_\varepsilon'(t)|\,\varphi(t)\,dt\right]\|f\|^p_{\mathscr{L}^{p,\varphi}_{\mathrm{loc};\{0\}}(\Omega)} \qquad (2.6)
\end{aligned}$$

where the last integral converges for every $\varepsilon > 0$, since

$$\psi_\varepsilon'(t) = \frac{\frac{1+\varepsilon}{\ln\frac{A}{t}} - \frac{t\varphi'(t)}{\varphi(t)}}{t\varphi(t)\ln^{1+\varepsilon}\frac{A}{t}},$$

and then

$$|\psi_\varepsilon'(t)|\varphi(t) \leqslant \frac{D_\varepsilon}{t\left(\ln\frac{A}{t}\right)^{1+\varepsilon}}, \qquad D_\varepsilon = \frac{1+\varepsilon}{\ln\frac{A}{\ell}} + P. \qquad (2.7)$$

3°. The strictness of the lower embedding. The function $f_0(x) = \frac{\varphi^{\frac{1}{p}}(|x|)}{|x|^{\frac{n}{p}}}$ is the corresponding example:

$$f_0 \in \mathscr{L}^{p,\varphi}_{\{0\}}(\Omega), \quad \text{but} \quad f_0 \notin L^p\left(\Omega, \frac{1}{\varphi(|x|)}\right).$$

Indeed, passing to polar coordinates, we have

$$\frac{1}{\varphi(r)}\int\limits_{B(0,r)} |f_0(y)|^p\,dy \leqslant |\mathbb{S}^{n-1}|\,\frac{1}{\varphi(r)}\int\limits_0^r \frac{\varphi(t)}{t}\,dt, \quad 0 < r < \ell.$$

where the right-hand side is bounded, by the assumption on φ.

4°. The strictness of the upper embedding. The corresponding counterexample for

$$g_0 \in \bigcap_{\varepsilon>0} L^p\left(\Omega, \frac{\ln^{1+\varepsilon}\frac{A}{|x-x_0|}}{\varphi(|x-x_0|)}\right), \quad \text{but} \quad g_0 \notin \mathscr{L}^{p,\varphi}_{\{x_0\}}(\Omega).$$

is $g_0(x) = \frac{\varphi^{\frac{1}{p}}(|x|)}{|x|^{\frac{n}{p}}} \ln\left(\ln\frac{B}{|x|}\right)$, $B > \ell e^e$. Indeed,

$$\|g_0\|^p_{L^p(\Omega,\psi_\varepsilon)} = \int_\Omega \frac{\ln^p\left(\ln\frac{B}{|x|}\right)}{|x|^n\left(\ln\frac{A}{|x|}\right)^{1+\varepsilon}} dx \leqslant |\mathbb{S}^{n-1}| \int_0^\ell \frac{\ln^p\left(\ln\frac{B}{t}\right)}{t\left(\ln\frac{A}{t}\right)^{1+\varepsilon}} dt < \infty$$

for every $\varepsilon > 0$. However, for $r \in (0,\delta)$, where $\delta = dist(0,\partial\Omega)$, we obtain

$$\frac{1}{\varphi(r)} \int_{B(0,r)} g_0^p(|x|)\,dx = \frac{1}{\varphi(r)} \int_{B(0,r)} \frac{\varphi(|x|)\ln^p\left(\ln\frac{B}{|x|}\right)dx}{|x|^n}$$

$$= \frac{|\mathbb{S}^{n-1}|}{\varphi(r)} \int_0^r \frac{\varphi(t)\ln^p\left(\ln\frac{B}{t}\right)dt}{t} \geqslant \frac{|\mathbb{S}^{n-1}|}{\varphi(r)} \int_{\frac{r}{2}}^r \frac{\varphi(t)\ln^p\left(\ln\frac{B}{t}\right)dt}{t}.$$

Taking into account that $\varphi(t)$ is non-decreasing and satisfies the doubling condition (the latter follows from the fact that (2.1) implies that $t^{-P}\varphi(t)$ is non-increasing), we get

$$\frac{1}{\varphi(r)} \int_{B(0,r)} g_0^p(|x|)\,dx \geqslant C\ln^p\left(\ln\frac{B}{r}\right) \int_{\frac{r}{2}}^r \frac{dt}{t} = C\ln 2\ln^p\left(\ln\frac{B}{r}\right) \to \infty \quad \text{as} \quad r \to 0,$$

which completes the proof of the theorem. □

REMARK 2.1. Note that local inequalities of the type (2.3) for the so called complementary Morrey spaces were proved in [10] in the case of a power function φ and in [14] in the general case.

3. Stummel spaces

Since the constants of embeddings in (2.3) do not depend on the local point $x_0 \in \Omega$, the inequalities in (2.3) allow to immediately pass to the global Morrey spaces. Taking supremum with respect to $x_0 \in \Omega$ we observe that the lower and upper spaces in the embeddings in (2.3) turn to be spaces related with the so called Stummel classes. Before to proceed to the corresponding formulation, in the next subsection we recall the notion of Stummel spaces and Stummel classes.

3.1. Stummel type classes

Recall that related in a sense to the notion of Morrey-type regularity of functions, the Stummel type class consists of locally p-integrable functions such that

$$\lim_{r\to 0}\eta_{p,\lambda}(f,r)=0, \tag{3.1}$$

where

$$\eta_{p,\lambda}(f,r):=\sup_{x\in\Omega}\int\limits_{\widetilde{B}(x,r)}\frac{|f(y)|^p\,dy}{|x-y|^\lambda},\qquad 1\leqslant p<\infty,\quad 0<\lambda<n.$$

Such type of conditions on a function f appeared in [18] in the case $p=2$; in the case $p=1$ this class was studied in [17] and [5]. In the case $p=1$ and $\lambda=n-2$ it is also called Stummel-Kato class.

By $\mathfrak{S}^{p,\lambda}(\Omega)$ we denote the space defined by the norm

$$\|f\|_{\mathfrak{S}^{p,\lambda}}:=\sup_{x\in\Omega}\left(\int\limits_{\Omega}\frac{|f(y)|^p\,dy}{|x-y|^\lambda}\right)^{\frac{1}{p}}. \tag{3.2}$$

We will call it Stummel space. Such a space was used in particular in [12] and [13] in applications to PDE.

Note that a more general hybrid $M^{p,\lambda}_\beta(X,\mu)$ of Morrey and Stummel type spaces was introduced in [6] in a general setting of a quasimetric measure space (X,ρ,μ), with the norm

$$\|f\|_{M^{p,\lambda}_\beta}:=\sup_{\substack{x\in X\\ r>0}}\left(\frac{1}{r^\lambda}\int_{\rho(x,y)<r}|f(y)|^p\rho^\beta(x,y)d\mu(y)\right)^{\frac{1}{p}}.$$

As regards the Stummel class defined by the condition (3.1), we find it natural to call it *vanishing Stummel space* following the tradition known for vanishing Morrey spaces and spaces of vanishing mean oscillation (VMO). We will denote it by $V\mathfrak{S}^{p,\lambda}(\Omega)$. With respect to the norm (3.2), the space $V\mathfrak{S}^{p,\lambda}(\Omega)$ is a closed subspace of $\mathfrak{S}^{p,\lambda}(\Omega)$. A generalization of such Stummel spaces, similar to that of Morrey spaces, may be defined in a natural way via the norm

$$\|f\|_{\mathfrak{S}^{p,\varphi}}:=\sup_{x\in\Omega}\left(\int\limits_{\Omega}\frac{|f(y)|^p\,dy}{\varphi(|x-y|)}\right)^{\frac{1}{p}}. \tag{3.3}$$

Obviously $\|f\|_{\mathscr{L}^{p,\varphi}}\leqslant\|f\|_{\mathfrak{S}^{p,\varphi}}$ for non-decreasing functions φ, so that in this case

$$\mathfrak{S}^{p,\varphi}(\Omega)\hookrightarrow\mathscr{L}^{p,\varphi}(\Omega).$$

We denote the corresponding Stummel classes (the vanishing generalized Stummel spaces) by $V\mathfrak{S}^{p,\varphi}$. Such generalized Stummel classes in the case $p=1$ were considered in [17] and [5].

3.2. $V\mathfrak{S}^{p,\lambda}$ is a proper subspace of $\mathfrak{S}^{p,\lambda}$

By standard arguments it is easily shown that $V\mathfrak{S}^{p,\varphi}$ is a closed space with respect to the norm (3.3), i.e. $\left\|\int_\Omega |f(y)|^p \varphi^{-1}(|y-x|)dy\right\|_{L^\infty}$. With this norm, the space $\mathfrak{S}^{p,\varphi}$ itself is expected to be non-separable. We do not dwell on proving this in the general case, but in Lemma 3.2 provide a family of examples of functions in $\mathfrak{S}^{p,\lambda}$ which are not in $V\mathfrak{S}^{p,\lambda}$, from which the non-separability of $\mathfrak{S}^{p,\lambda}$ follows. We pay a special attention to such examples to better understand the nature of Stummel classes (in spite of a number of papers devoted to Stummel type spaces, such examples were never given, up to author's knowledge). Functions with singularities, for instance, of the type $|x|^{-\gamma}$, $\gamma < \frac{n-\lambda}{p}$ or $|x|^{-\frac{n-\lambda}{p}} \ln^{-a} \frac{1}{|x|}$, $a > \frac{1}{p}$, belong to both the spaces $\mathfrak{S}^{p,\lambda}$ and $V\mathfrak{S}^{p,\lambda}$ under the same conditions and in general it is obvious that examples with a singularity at a fixed point only, or at a finite number of points, are helpless.

For simplicity, we consider the one-dimensional case and $\Omega = (0,1)$; the constructions below can be similarly adapted for the multidimensional case with intervals I_k of decreasing size replaced by balls of decreasing radius.

Let $\Omega = (0,1)$ and let $\{x_k\}_{k=1}^\infty$ be any sequence of numbers $x_k \in (0,1)$ monotonously tending to zero. Let $I_k := (x_k(1-h_k),\ x_k(1+h_k))$, where $0 < h_k < 1$, be symmetrical intervals of the points x_k. We choose h_k so that the intervals I_k adjoin each other:

$$x_k(1+h_k) = x_{k-1}(1-h_{k-1}) \quad \Longleftrightarrow \quad h_k = \frac{x_{k-1}-x_k}{x_k+x_{k-1}}. \tag{3.4}$$

We construct a family of counterexamples of the form

$$\Phi(x) = \sum_{k=1}^\infty a_k \frac{\chi_{I_k}(x)}{|x-x_k|^{1-\lambda-\alpha_k}}, \qquad a_k > 0, \quad \alpha_k > 0 \tag{3.5}$$

with three "discrete parameters" x_k, a_k, α_k. Clearly, the counterexample may be expected when $\alpha_k \to 0$ as $k \to \infty$.

LEMMA 3.1. *Let the function Φ be given by (3.5) with a decreasing sequence of numbers $x_k \in (0,1)$, $\lim_{k\to\infty} x_k = 0$, subject to the relation (3.4). The conditions*

$$\sum_{k=1}^\infty a_k < \infty \tag{3.6}$$

and

$$\sup_{m\in\mathbb{N}} \frac{a_m}{\alpha_m}(h_m x_m)^{\alpha_m} < \infty \tag{3.7}$$

are sufficient for the function $f := \Phi^{\frac{1}{p}}$ to belong to the Stummel space $\mathfrak{S}^{p,\lambda}(0,1)$, $1 \leqslant p < \infty$, $0 < \lambda < 1$. The condition (3.7) is also necessary.

Proof. We have

$$\int_0^1 \frac{f^p(y)\,dy}{|x-y|^\lambda} = \sum_{k=1}^\infty a_k \int_{I_k} \frac{dy}{|x-y|^\lambda |y-x_k|^{1-\lambda-\alpha_k}}. \tag{3.8}$$

The change $y = x_k + t|x - x_k|$ yields

$$\int_0^1 \frac{f^p(y)\,dy}{|x-y|^\lambda} = \sum_{k=1}^{\infty} a_k |x-x_k|^{\alpha_k} \int_{-\rho_k(x)}^{\rho_k(x)} \frac{dt}{|t|^{1-\lambda-\alpha_k}|t - sign(x-x_k)|^\lambda},$$

where $\rho_k(x) := \frac{h_k x_k}{|x-x_k|}$. Note that for $x \in I_m$ we have $\rho_k(x) \leqslant 1$ if $k \neq m$, but $\rho_m(x) \geqslant 1$. With $x \in I_m$ we have

$$\begin{aligned}\int_0^1 \frac{f^p(y)\,dy}{|x-y|^\lambda} \ &= \sum_{k=1}^{m-1} a_k (x_k - x)^{\alpha_k} \int_{-\rho_k(x)}^{\rho_k(x)} \frac{dt}{|t|^{1-\lambda-\alpha_k}(t+1)^\lambda} \\ &\quad + a_m |x - x_m|^{\alpha_m} \int_{-\rho_m(x)}^{\rho_m(x)} \frac{dt}{|t|^{1-\lambda-\alpha_m}|t - sign(x-x_m)|^\lambda} \\ &\quad + \sum_{k=m+1}^{\infty} a_k (x-x_k)^{\alpha_k} \int_{-\rho_k(x)}^{\rho_k(x)} \frac{dt}{|t|^{1-\lambda-\alpha_k}(1-t)^\lambda} \\ &=: \mathbb{A}_m + \mathbb{B}_m + \mathbb{C}_m.\end{aligned}$$

Here

$$\mathbb{A}_m = \sum_{k=1}^{m-1} a_k (x_k - x)^{\alpha_k} \int_0^{\rho_k(x)} \frac{1}{t^{1-\lambda-\alpha_k}} \left[\frac{1}{(1+t)^\lambda} + \frac{1}{(1-t)^\lambda} \right] dt$$

so that

$$\mathbb{A}_m \leqslant c_\lambda \sum_{k=1}^{m-1} a_k (x_k - x_m(1-h_m))^{\alpha_k} \leqslant c_\lambda \sum_{k=1}^{m-1} a_k,$$

where $c_\lambda = \int_0^1 \frac{1}{t^{1-\lambda}} \left[\frac{1}{(1+t)^\lambda} + \frac{1}{(1-t)^\lambda} \right] dt$. Similarly

$$\mathbb{C}_m \leqslant c_\lambda \sum_{k=m+1}^{\infty} a_k ((1+h_m)x_m - x_k)^{\alpha_k} \leqslant c_\lambda \sum_{k=m+1}^{\infty} a_k.$$

For $\mathbb{B}_m$ we obtain

$$\mathbb{B}_m = a_m |x - x_m|^{\alpha_m} \int_0^{\rho_m(x)} \frac{1}{t^{1-\lambda-\alpha_k}} \left[\frac{1}{(t+1)^\lambda} + \frac{1}{|t-1|^\lambda} \right] dt.$$

It is easily checked that

$$c_1 \frac{\rho_m^{\alpha_m}(x)}{\alpha_m} \leqslant \int_0^{\rho_m(x)} \frac{1}{t^{1-\lambda-\alpha_m}} \left[\frac{1}{(t+1)^\lambda} + \frac{1}{|t-1|^\lambda} \right] dt \leqslant c_2 \frac{\rho_m^{\alpha_m}(x)}{\alpha_m}$$

where the constants c_1 and c_2 do not depend on m and x. Consequently,

$$c_1 \frac{a_m}{\alpha_m}(h_m x_m)^{\alpha_m} \leqslant \mathbb{B}_m \leqslant c_2 \frac{a_m}{\alpha_m}(h_m x_m)^{\alpha_m}. \tag{3.9}$$

Hence

$$\sup_{x\in(0,1)} \int_0^1 \frac{f^p(y)\,dy}{|x-y|^\lambda} = \sup_{m\in\mathbb{N}} \sup_{x\in I_m} \int_0^1 \frac{f(y)\,dy}{|x-y|^\lambda} \leqslant c_\lambda \sum_{k=1}^\infty a_k + c_2 \sup_{m\in\mathbb{N}} \frac{a_m}{\alpha_m}(h_m x_m)^{\alpha_m},$$

which proves the lemma. The necessity of the condition (3.7) is seen from (3.9). □

LEMMA 3.2. *Let the function Φ be given by (3.5) with x_k the same as in Lemma 3.1, and coefficients a_k satisfying the condition (3.6). Let the exponents α_k be defined by the condition*

$$\frac{a_m}{\alpha_m}(h_m x_m)^{\alpha_m} = C, \tag{3.10}$$

where $C > 0$ does not depend on m. Then $f = \Phi^{\frac{1}{p}} \in \mathfrak{S}^{p,\lambda}(0,1)$, but $f \notin V\mathfrak{S}^{p,\lambda}(0,1)$.

Proof. First note that, given a_m and $h_m x_m$, the condition (3.10) always determines (unique) value of α_m, since the equation of form $ab^t = t$ with $a > 0$ and $0 < b \leqslant 1$ has a positive solution t (it may be written via the Lambert function $W : t = \frac{1}{\ln\frac{1}{b}} W\left(a \ln \frac{1}{b}\right)$).

By Lemma 3.1, we have only to show that $f \notin V\mathfrak{S}^{p,\lambda}(0,1)$. For

$$g(x,r) := \int_{y\in[0,1]:|y-x|<r} \frac{|f^p(y)|\,dy}{|x-y|^\lambda},$$

as in (3.8), we have

$$g(x,r) = \sum_{k=1}^\infty a_k \int_{y\in I_k:|y-x|<r} \frac{dy}{|x-y|^\lambda |y-x_k|^{1-\lambda-\alpha_k}} \geqslant a_m \int_{y\in I_m:|y-x|<r} \frac{dy}{|x-y|^\lambda |y-x_m|^{1-\lambda-\alpha_m}},$$

where we can choose $m = m(r)$ sufficiently large so that $\{y \in I_m : |y-x| < r\} = I_m$, when $x \in I_m$. Since $|y-x| \leqslant |y|+|x| \leqslant 2x_m(1+h_m) \leqslant 4x_m$ in this case, such a value m may be chosen by the condition $x_m < \frac{r}{4}$. Then for $x \in I_m$

$$g(x,r) \geqslant a_m \int_{y\in I_m} \frac{dy}{|x-y|^\lambda |y-x_m|^{1-\lambda-\alpha_m}} = \mathbb{B}_m \geqslant c_1 \frac{a_m}{\alpha_m}(h_m x_m)^{\alpha_m} = const$$

by (3.9) and the assumption (3.10). Hence

$$\lim_{r\to 0} \sup_{x\in(0,1)} \int_{y\in[0,1]:|y-x|<r} \frac{|f^p(y)|\,dy}{|x-y|^\lambda} > 0,$$

and consequently, $f \notin V\mathfrak{S}^{p,\lambda}(0,1)$. □

3.3. Another norm for the generalized Stummel space

LEMMA 3.3. *Let $\ell = diam\,\Omega \leqslant \infty$ and φ be a non-decreasing absolutely continuous function on $[0,\ell]$ such that $\inf_{\delta<r<\ell}\varphi(r) > 0$ for every $\delta \in (0,\ell)$. Let also a function $f(x)$ defined on Ω be extended as zero beyond Ω to $\mathbb{R}^n$. Then for all $r \in (0,\ell]$ the following identity holds*

$$\int\limits_{B(x,r)} \frac{|f(y)|^p\,dy}{\varphi(|y-x|)} + \frac{1}{\varphi(r)}\int\limits_{B(x,r)} |f(y)|^p\,dy = \int\limits_0^r \frac{\varphi'(t)}{\varphi^2(t)}dt \int\limits_{B(x,t)} |f(y)|^p\,dy. \tag{3.11}$$

Proof. It is obvious that $\frac{1}{\varphi(t)} = \int\limits_t^r \frac{\varphi'(s)}{\varphi^2(s)}ds - \frac{1}{\varphi(r)}$ for all $0<t<r$. Therefore,

$$\int\limits_{B(x,r)} \frac{|f(y)|^p\,dy}{\varphi(|y-x|)} = \int\limits_{B(x,r)} |f(y)|^p \left[\int\limits_{|x-y|}^r \frac{\varphi'(s)}{\varphi^2(s)}ds - \frac{1}{\varphi(r)}\right] dy$$

from which (3.11) follows. □

Relation (3.11) (with $r=\ell$) in the case $\varphi(t)=t^\lambda$ was proved in [12] by another way.

The identity (3.11) shows that the norm in $\mathfrak{S}^{p,\varphi}(\Omega)$ may be equivalently replaced by the norm generated by the right-hand side of (3.11). More precisely, the following statement is valid.

COROLLARY 3.4. *Under the assumptions of Lemma 3.3, the Stummel space $\mathfrak{S}^{p,\varphi}(\Omega)$ coincides with the generalized Morrey space $\mathscr{L}^{p,p,\phi}(\Omega)$:*

$$\mathfrak{S}^{p,\varphi}(\Omega) = \mathscr{L}^{p,p,\phi}(\Omega), \quad where \quad \phi(r) = \frac{r\varphi'(r)}{\varphi^2(r)},$$

up to the equivalence of norms: $\|f\|_{\mathfrak{S}^{p,\varphi}(\Omega)} \leqslant \|f\|_{\mathscr{L}^{p,p,\phi}(\Omega)} \leqslant 2^{\frac{1}{p}}\|f\|_{\mathfrak{S}^{p,\varphi}(\Omega)}$.

Proof. From (3.11) with $r=\ell$ we have

$$\|f\|_{\mathscr{L}^{p,p,\phi}(\Omega)} = \sup_{x\in\Omega}\left(\int\limits_\Omega \frac{|f(y)|^p\,dy}{\varphi(|y-x|)} + \frac{1}{\varphi(\ell)}\int\limits_{B(x,\ell)} |f(y)|^p\,dy\right)^{\frac{1}{p}} \geqslant \|f\|_{\mathfrak{S}^{p,\varphi}(\Omega)}.$$

On the other hand, since the function φ is non-decreasing, we have

$$\|f\|_{\mathscr{L}^{p,p,\phi}(\Omega)} \leqslant \sup_{x\in\Omega}\left(\int\limits_\Omega \frac{|f(y)|^p\,dy}{\varphi(|y-x|)} + \int\limits_\Omega \frac{|f(y)|^p}{\varphi(|x-y|)}dy\right)^{\frac{1}{p}} \leqslant 2^{\frac{1}{p}}\|f\|_{\mathfrak{S}^{p,\varphi}(\Omega)}. \quad \square$$

4. Embeddings for global Morrey spaces

With the above notation for the Stummel spaces, from the embeddings (2.3) we arrive at the next statement.

THEOREM 4.1. *Let Ω, p and φ be the same as in Theorem 2.1. Then:*

$$\mathfrak{S}^{p,\varphi}(\Omega) \hookrightarrow \mathscr{L}^{p,\varphi}(\Omega) \hookrightarrow \bigcap_{\varepsilon>0} \mathfrak{S}^{p,\varphi \ln^{1+\varepsilon}}(\Omega), \tag{4.1}$$

where $\varphi \ln^{1+\varepsilon}$ stands for $\varphi(r)\ln^{1+\varepsilon}\frac{A}{r}$; the embeddings are strict under the same assumptions on φ as in Theorem 2.1.

Proof. The embeddings (4.1) are obtained from (2.3) by taking sup with respect to $x_0 \in \Omega$, which is possible since the embedding constants do not depend on x_0. The strictness of the embeddings is justified by the same examples as in Theorem 2.1. □

Apart from the above immediate consequence of the local embeddings of Theorem 2.1, we are interested also in a similar comparison of the Morrey spaces $\mathscr{L}^{p,\varphi}(\Omega)$ with the vanishing Stummel spaces $V\mathfrak{S}^{p,\varphi}$, since the latter spaces are of wider application to PDE, see [17]. Comparison of such a kind was made in the case $p=1$, i.e. between $\mathscr{L}^{1,\mu}(\Omega)$ and $V\mathfrak{S}^{1,\lambda}(\Omega)$ with $\mu > \lambda$, in [17], which was extended in [5] to the case of function $\varphi(r)$ instead of r^{λ}. The estimates obtained above, allow us to obtain a finer result for an arbitrary p and only with the "logarithmical scale of the gap" between the spaces. More precisely, the following theorem is valid.

THEOREM 4.2. *Let Ω, p and φ be the same as in Theorem 2.1 and additionally let $\varphi(0)=0$. Then*

$$V\mathfrak{S}^{p,\varphi}(\Omega) \hookrightarrow \mathfrak{S}^{p,\varphi}(\Omega) \hookrightarrow \mathscr{L}^{p,\varphi}(\Omega) \hookrightarrow \bigcap_{\varepsilon>0} V\mathfrak{S}^{p,\varphi \ln^{1+\varepsilon}}(\Omega). \tag{4.2}$$

Proof. The embeddings $V\mathfrak{S}^{p,\varphi}(\Omega) \hookrightarrow \mathfrak{S}^{p,\varphi}(\Omega) \hookrightarrow \mathscr{L}^{p,\varphi}(\Omega)$ are obvious in view of Theorem 4.1. The proof of the remaining embedding is prepared by estimates in the proof of Theorem 2.1. Let

$$S_{p,\varphi}(f;x,r) := \int_{\widetilde{B}(x,r)} \frac{|f(y)|^p\,dy}{\varphi(|x-y|)}.$$

We have to show that

$$f \in \mathscr{L}^{p,\varphi}(\Omega) \quad \Longrightarrow \quad \lim_{r\to 0}\sup_{x\in\Omega} S_{p,\varphi \ln^{1+\varepsilon}}(f;x,r) = 0 \tag{4.3}$$

for every $\varepsilon > 0$. Following the arguments in part 2° of the proof of Theorem 2.1 (see the estimates in (2.5) and (2.6)), we have

$$S_{p,\varphi\ln^{1+\varepsilon}}(f;x,r) = \int\limits_{\widetilde{B}(x,r)} \frac{|f(y)|^p\,dy}{\varphi(|x-y|)\ln^{1+\varepsilon}\frac{A}{|x-y|}}$$
$$\leqslant C_\varepsilon \int\limits_{\widetilde{B}(x,r)} |f(y)|^p dy + \int_0^\ell |\psi_\varepsilon'(t)| \int\limits_{\widetilde{B}(x,t_r)} |f(y)|^p dy\,dt$$

where $\psi_\varepsilon(t)$ is the same as in (2.5) and $t_r = \min\{t,r\}$. Hence

$$S_{p,\varphi\ln^{1+\varepsilon}}(f;x,r) \leqslant \left[C_\varepsilon\varphi(r) + \int_0^\ell |\psi_\varepsilon'(t)|\varphi(t_r)\,dt\right]\|f\|_{\mathscr{L}^{p,\varphi}(\Omega)}.$$

Therefore, to prove (4.3), it remains to show that $\lim\limits_{r\to 0}\int_0^\ell |\psi_\varepsilon'(t)|\varphi(t_r)\,dt = 0$. Since $\varphi(t_r) \leqslant \varphi(t)$ and $\varphi(0) = 0$, this follows from the Lebesgue dominated convergence theorem in view of (2.7). □

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(Received July 27, 2012)

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