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LÉVY–KHINTCHINE REPRESENTATION OF THE GEOMETRIC MEAN OF MANY POSITIVE NUMBERS AND APPLICATIONS

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Abstract. In the paper, the authors establish, by Cauchy integral formula in the theory of complex functions, Lévy-Khintchine representation for the geometric mean of many positive numbers, find that the geometric mean of many positive numbers is a complete Bernstein function, and supply a new proof of the well known arithmetic-geometric mean inequality.

1. Introduction

We recall some definitions and notions.

Throughout this paper, the notation $\mathbb{N}$ stands for the set of all positive integers.

DEFINITION 1.1. ([26, Chapter IV]) An infinitely differentiable function f on an interval I is said to be completely monotonic on I if it satisfies

$$(-1)^{n-1} f^{(n-1)}(t) \geqslant 0$$

for $x \in I$ and $n \in \mathbb{N}$.

We denote the class of all completely monotonic functions on an interval I by the notation $\mathscr{C}[I]$. The class $\mathscr{C}[(0,\infty)]$ is characterized by the famous Hausdorff-Bernstein-Widder theorem below.

PROPOSITION 1.1 ([26, Theorem 12b]) *A necessary and sufficient condition that* $f(x)$ *should be completely monotonic for* $0 < x < \infty$ *is that*

$$f(x) = \int_0^\infty e^{-xt} \, \mathrm{d}\alpha(t), \tag{1.1}$$

where $\alpha(t)$ *is non-decreasing and the integral converges for* $0 < x < \infty$.

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DEFINITION 1.2. ([1, 13, 15]) An infinitely differentiable function f on an interval I is said to be logarithmically completely monotonic on I if its logarithm $\ln f$ satisfies

$$(-1)^k[\ln f(t)]^{(k)} \geqslant 0$$

for $k \in \mathbb{N}$ on I.

We denote the set of all logarithmically completely monotonic functions on an interval I by $\mathscr{L}[I]$. When $I=(0,\infty)$, Definition 1.2 becomes [24, Definiton 5.8] and [25, Definition 5.10]. See also [25, p. 67].

DEFINITION 1.3. ([24, Definition 2.1]) If a function $f:(0,\infty)\to[0,\infty)$ can be written in the form

$$f(x)=\frac{a}{x}+b+\int_0^\infty \frac{1}{s+x}\,\mathrm{d}\mu(s), \tag{1.2}$$

then it is called a Stieltjes function or a Stieltjes transform, where $a,b \geqslant 0$ are constants and μ is a measure on $(0,\infty)$ such that $\int_0^\infty \frac{1}{1+s}\,\mathrm{d}\mu(s)<\infty$.

We denote the family of all Stieltjes functions by $\mathscr{S}$.

There exist inclusions

$$\mathscr{L}[I]\subset\mathscr{C}[I] \quad \text{and} \quad \mathscr{S}\subset\mathscr{L}[(0,\infty)],$$

which are called Qi-Berg's inclusions in the literature. For more detailed information on these inclusions, please refer to [3, Theorem 1.1], [7, Theorem 4], [12, Section 1], [13, Theorem 1], [15, Theorem 4], [16, Remark 8], [17, Section 1], [18, Remark 4.7], [24, Theorem 5.9], and plenty of references therein.

DEFINITION 1.4. An infinitely differentiable function $f: I\to[0,\infty)$ is called a Bernstein function on an interval I if $f'(t)$ is completely monotonic on I.

When $I=(0,\infty)$, Definition 1.4 becomes [24, Definition 3.1]. We denote the group of all Bernstein functions on an interval I by $\mathscr{B}[I]$. The class $\mathscr{B}[(0,\infty)]$ can be characterized by

PROPOSITION 1.2 ([24, Theorem 3.2]) *A function $f:(0,\infty)\to[0,\infty)$ is a Bernstein function if and only if it admits the representation*

$$f(x)=a+bx+\int_0^\infty \left(1-e^{-xt}\right)\mathrm{d}\mu(t), \tag{1.3}$$

where $a,b\geqslant 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_0^\infty \min\{1,t\}\,\mathrm{d}\mu(t)<\infty$. In particular, the triplet (a,b,μ) determines f uniquely and vice versa.

The formula (1.3) is called Lévy-Khintchine representation of f. The representing measure μ and the characteristic triplet (a,b,μ) from (1.3) are often respectively called Lévy measure and Lévy triplet of the Bernstein function f.

In [5, pp. 161–162, Theorem 3] and [24, Proposition 5.17], it was discovered that the reciprocal of any Bernstein function is logarithmically completely monotonic.

DEFINITION 1.5. ([1]) If for some non-negative integer k the derivative $f^{(k)}(t)$ is completely monotonic on an interval I, but $f^{(k-1)}(t)$ is not completely monotonic on I, then $f(t)$ is called a completely monotonic function of k-th order on I.

It is clear that a completely monotonic function of first order on I is a Bernstein function on I if and only if it is non-negative on I.

DEFINITION 1.6. ([24, Definition 6.1]) If Lévy measure μ from (1.3) has a completely monotonic density $m(t)$ with respect to Lebesgue measure, that is, the integral representation

$$f(x) = a + bx + \int_0^\infty (1 - e^{-xt}) m(t)\,\mathrm{d}t \tag{1.4}$$

holds for $a, b \geqslant 0$, where $m(t)$ is a completely monotonic function on $(0,\infty)$ and satisfies $\int_0^\infty \min\{1,t\} m(t)\,\mathrm{d}t < \infty$, then f is said to be a complete Bernstein function on $(0,\infty)$.

We denote the collection of all complete Bernstein functions on $(0,\infty)$ by $\mathscr{CB}$.

DEFINITION 1.7. ([24, Definition 8.1]) Under conditions of Definition 1.6, if the function $tm(t)$ is completely monotonic on $(0,\infty)$, then f is said to be a Thorin-Bernstein function on $(0,\infty)$.

We use $\mathscr{TB}$ to denote the class of all Thorin-Bernstein functions on $(0,\infty)$. It is clear that $\mathscr{TB} \subset \mathscr{CB}$.

We now begin to introduce the motivation of this paper.

For $\lambda \in (0,1)$ and $x, y > 0$, let

$$G_\lambda(x,y) = x^\lambda y^{1-\lambda},$$

which is called the weighted geometric mean of two positive numbers x and y with unit weight $(\lambda, 1-\lambda)$. For $z \in \mathbb{C} \setminus (-\infty, -\min\{x,y\})$ and $x, y \in \mathbb{R}$, let

$$G_{x,y;\lambda}(z) = G_\lambda(x+z, y+z) = (z+x)^\lambda (z+y)^{1-\lambda}. \tag{1.5}$$

In [14, Corollary 1] it was proved that $G'_{x,y;\lambda}(t) \in \mathscr{L}[(-\min\{x,y\},\infty)]$ and that $G_{x,y;\lambda}(t)$ is a completely monotonic function of first order in $t \in (-\min\{x,y\},\infty)$. In other words, $G_{x,y;\lambda}(t) \in \mathscr{B}[(-\min\{x,y\},\infty)]$.

In [23], among other things, the fact that $G_{x,y;1/2}(t) \in \mathscr{B}[(-\min\{x,y\},\infty)]$ was recovered by several approaches, Lévy-Khintchine representation of the principal branch of $G_{x,y;1/2}(z)$ for $x > y > 0$ and $z \in \mathbb{C} \setminus (-\infty, -y]$ was established, and the conclusion $G_{x,y;1/2}(t) \in \mathscr{TB}$ for $x > y > 0$ was verified. See also [27, Chapter 2].

In [21], among other things, Lévy-Khintchine representation of the principal branch of the weighted geometric mean $G_{x,y;\lambda}(z)$, defined by (1.5), for $\lambda \in (0,1)$ and $x > y > 0$, was established and the result $G_{x,y;\lambda}(t) \in \mathscr{TB}$ for $\lambda \in (0,1)$ and $x, y > 0$ was concluded.

Let $n \in \mathbb{N}$ and $\mathbf{a} = (a_1, a_2, \dots, a_n)$ be a positive sequence, that is, $a_k > 0$ for $1 \leqslant k \leqslant n$. It is well known that the arithmetic and geometric means $A_n(\mathbf{a})$ and $G_n(\mathbf{a})$ of the positive sequence $\mathbf{a}$ are defined respectively as

$$A_n(\mathbf{a}) = \frac{1}{n}\sum_{k=1}^{n} a_k \quad \text{and} \quad G_n(\mathbf{a}) = \left(\prod_{k=1}^{n} a_k\right)^{1/n}.$$

It is general knowledge that

$$G_n(\mathbf{a}) \leqslant A_n(\mathbf{a}), \tag{1.6}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. This is called in the literature the arithmetic-geometric mean inequality.

For $z \in \mathbb{C} \setminus (-\infty, -\min\{a_k, 1 \leqslant k \leqslant n\}]$ and $n \geqslant 2$, let $\mathbf{e} = (\overbrace{1, 1, \dots, 1}^{n})$ and

$$G_n(\mathbf{a} + z\mathbf{e}) = \left[\prod_{k=1}^{n} (a_k + z)\right]^{1/n}.$$

The first aim of this paper is to establish, by using Cauchy integral formula in the theory of complex functions, Lévy-Khintchine representation of the geometric mean $G_n(\mathbf{a} + z\mathbf{e})$ and to deduce that $G_n(\mathbf{a} + t\mathbf{e}) \in \mathscr{C}\mathscr{B}$ for $t \in (-\min\{a_k, 1 \leqslant k \leqslant n\}, \infty)$.

THEOREM 1.1 *Let σ be a permutation of the sequence $\{1, 2, \dots, n\}$ such that the sequence $\sigma(\mathbf{a}) = \left(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}\right)$ is a rearrangement of $\mathbf{a}$ in an ascending order $a_{\sigma(1)} \leqslant a_{\sigma(2)} \leqslant \cdots \leqslant a_{\sigma(n)}$. Then the principal branch of the geometric mean $G_n(\mathbf{a} + z\mathbf{e})$ has the integral representation*

$$G_n(\mathbf{a} + z\mathbf{e}) = A_n(\mathbf{a}) + z - \frac{1}{\pi}\sum_{\ell=1}^{n-1} \sin\frac{\ell\pi}{n} \int_{a_{\sigma(\ell)}}^{a_{\sigma(\ell+1)}} \left|\prod_{k=1}^{n} (a_k - t)\right|^{1/n} \frac{\mathrm{d}t}{t+z} \tag{1.7}$$

for $z \in \mathbb{C} \setminus (-\infty, -\min\{a_k, 1 \leqslant k \leqslant n\}]$. Equivalently, the principal branch of the geometric mean $G_n(\mathbf{a} + z\mathbf{e})$ has Lévy-Khintchine representation

$$G_n(\mathbf{a} + z\mathbf{e}) = G_n(\mathbf{a}) + z + \int_0^{\infty} \left(1 - e^{-zu}\right) Q(u)\,\mathrm{d}u, \tag{1.8}$$

where

$$Q(u) = \frac{1}{\pi}\sum_{\ell=1}^{n-1} \sin\frac{\ell\pi}{n} \int_{a_{\sigma(\ell)}}^{a_{\sigma(\ell+1)}} \left|\prod_{k=1}^{n} (a_k - t)\right|^{1/n} e^{-tu}\,\mathrm{d}t.$$

Consequently, $G_n(\mathbf{a} + t\mathbf{e}) \in \mathscr{C}\mathscr{B}$ for $t \in (-\min\{a_k, 1 \leqslant k \leqslant n\}, \infty)$.

The second aim of this paper is to, with the help of the integral representation (1.7), supply a new proof of the arithmetic-geometric mean inequality (1.6).

2. Lemmas

In order to prove our main results, we need the following lemmas.

LEMMA 2.1 (CAUCHY INTEGRAL FORMULA [6, p. 113]) *Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D and extendable smoothly to the boundary of D, then*

$$f(z)=\frac{1}{2\pi i}\oint_{\partial D}\frac{f(w)}{w-z}\,\mathrm{d}w,\quad z\in D. \tag{2.1}$$

LEMMA 2.2 *For $z\in\mathbb{C}\setminus(-\infty,-\min\{a_k,1\leqslant k\leqslant n\}]$, the principal branch of the complex function*

$$f_{\mathbf{a},n}(z)=G_n(\mathbf{a}+z\mathbf{e})-z \tag{2.2}$$

fulfils

$$\lim_{z\to\infty}f_{\mathbf{a},n}(z)=A_n(\mathbf{a}). \tag{2.3}$$

Proof. By L'Hôspital's rule in the theory of complex functions, we have

$$\begin{aligned}\lim_{z\to\infty}f_{\mathbf{a},n}(z)&=\lim_{z\to\infty}\Big\{z\Big[G_n\Big(\mathbf{e}+\frac{\mathbf{a}}{z}\Big)-1\Big]\Big\}\\&=\lim_{z\to 0}\frac{G_n(1+z\mathbf{a})-1}{z}=\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left[\prod_{k=1}^{n}(1+a_kz)\right]^{1/n}=A_n(\mathbf{a}).\end{aligned}$$

Lemma 2.2 is thus proved.

LEMMA 2.3 *Let σ be a permutation of the sequence $\{1,2,\dots,n\}$ such that the sequence $\sigma(\mathbf{a})=\big(a_{\sigma(1)},a_{\sigma(2)},\dots,a_{\sigma(n)}\big)$ is a rearrangement of $\mathbf{a}$ in an ascending order $a_{\sigma(1)}\leqslant a_{\sigma(2)}\leqslant\cdots\leqslant a_{\sigma(n)}$. For $z\in\mathbb{C}\setminus(-\infty,0]$, let*

$$h_n(z)=G_n\big(\sigma(\mathbf{a})-a_{\sigma(1)}\mathbf{e}+z\mathbf{e}\big)-z. \tag{2.4}$$

Then the principal branch of $h_n(z)$ satisfies

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\Im h_n(-t+i\varepsilon)\\&=\begin{cases}\left[\displaystyle\prod_{k=1}^{n}\big|a_{\sigma(k)}-a_{\sigma(1)}-t\big|\right]^{1/n}\sin\dfrac{\ell\pi}{n}, & t\in\big(a_{\sigma(\ell)}-a_{\sigma(1)},a_{\sigma(\ell+1)}-a_{\sigma(1)}\big]\\ 0, & t\geqslant a_{\sigma(n)}-a_{\sigma(1)}\end{cases}\end{aligned} \tag{2.5}$$

for $1\leqslant\ell\leqslant n-1$.

Proof. For $t\in(0,\infty)\setminus\big\{a_{\sigma(\ell+1)}-a_{\sigma(1)},1\leqslant\ell\leqslant n-1\big\}$ and $\varepsilon>0$, we have

$$
\begin{aligned}
h_n(-t+i\varepsilon) &= G_n\big(\sigma(\mathbf{a})-a_{\sigma(1)}\mathbf{e}-t\mathbf{e}+i\varepsilon\mathbf{e}\big)+t-i\varepsilon\\
&= \exp\left[\frac{1}{n}\sum_{k=1}^{n}\ln\big(a_{\sigma(k)}-a_{\sigma(1)}-t+i\varepsilon\big)\right]+t-i\varepsilon\\
&= \exp\left\{\frac{1}{n}\sum_{k=1}^{n}\big[\ln\big|a_k-a_{\sigma(1)}-t+i\varepsilon\big|+i\arg\big(a_{\sigma(k)}-a_{\sigma(1)}-t+i\varepsilon\big)\big]\right\}+t-i\varepsilon\\
&\to \begin{cases}\exp\left(\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\ln\big|a_{\sigma(k)}-a_{\sigma(1)}-t\big|+\dfrac{\ell\pi}{n}i\right)+t, & t\in\big(a_{\sigma(\ell)}-a_{\sigma(1)},a_{\sigma(\ell+1)}-a_{\sigma(1)}\big)\\
\exp\left(\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\ln\big|a_{\sigma(k)}-a_{\sigma(1)}-t\big|+\pi i\right)+t, & t>a_{\sigma(n)}-a_{\sigma(1)}\end{cases}\\
&= \begin{cases}\left(\displaystyle\prod_{k=1}^{n}\big|a_{\sigma(k)}-a_{\sigma(1)}-t\big|\right)^{1/n}\exp\left(\dfrac{\ell\pi}{n}i\right)+t, & t\in\big(a_{\sigma(\ell)}-a_{\sigma(1)},a_{\sigma(\ell+1)}-a_{\sigma(1)}\big)\\
\left(\displaystyle\prod_{k=1}^{n}\big|a_{\sigma(k)}-a_{\sigma(1)}-t\big|\right)^{1/n}\exp(\pi i)+t, & t>a_{\sigma(n)}-a_{\sigma(1)}\end{cases}
\end{aligned}
$$

as $\varepsilon\to0^+$. As a result, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to0^+}\Im h_n(-t+i\varepsilon)\\
&\quad=\begin{cases}\left(\displaystyle\prod_{k=1}^{n}\big|a_{\sigma(k)}-a_{\sigma(1)}-t\big|\right)^{1/n}\sin\dfrac{\ell\pi}{n}, & t\in\big(a_{\sigma(\ell)}-a_{\sigma(1)},a_{\sigma(\ell+1)}-a_{\sigma(1)}\big);\\
0, & t>a_{\sigma(n)}-a_{\sigma(1)}.\end{cases}
\end{aligned}
$$

For $t=a_{\sigma(\ell+1)}-a_{\sigma(1)}$ for $1\leqslant\ell\leqslant n-1$, we have

$$
\begin{aligned}
h_n(-t+i\varepsilon) &= \exp\left[\frac{1}{n}\sum_{k\ne\ell+1}^{n}\ln\big(a_{\sigma(k)}-a_{\sigma(1)}-t+i\varepsilon\big)+\frac{1}{n}\ln(i\varepsilon)\right]+t-i\varepsilon\\
&= \exp\left[\frac{1}{n}\sum_{k\ne\ell+1}^{n}\ln\big(a_{\sigma(k)}-a_{\sigma(1)}-t+i\varepsilon\big)\right]\exp\left[\frac{1}{n}\left(\ln|\varepsilon|+\frac{\pi}{2}i\right)\right]+t-i\varepsilon\\
&\to \exp\left[\frac{1}{n}\sum_{k\ne\ell+1}^{n}\ln\big(a_{\sigma(k)}-a_{\sigma(1)}-t\big)\right]\lim_{\varepsilon\to0^+}\exp\left[\frac{1}{n}\left(\ln|\varepsilon|+\frac{\pi}{2}i\right)\right]+t\\
&= t
\end{aligned}
$$

as $\varepsilon\to0^+$. Hence, when $t=a_{\sigma(\ell+1)}-a_{\sigma(1)}$ for $1\leqslant\ell\leqslant n-1$, we have

$$
\lim_{\varepsilon\to0^+}\Im h_n(-t+i\varepsilon)=0.
$$

The proof of Lemma 2.3 is completed.

3. Proof of Theorem 1.1

We now turn our attention to the proof of Theorem 1.1.

By standard arguments, it is not difficult to see that

$$\lim_{z\to 0^+}[zh_n(z)] = 0 \quad \text{and} \quad h_n(\overline{z}) = \overline{h_n(z)}, \tag{3.1}$$

where $h_n(z)$ is defined by (2.4).

For any fixed point $z_0 = x_0 + iy_0 \in \mathbb{C}\setminus(-\infty,0]$, choose ε and r such that

$$\begin{cases} 0<\varepsilon<|y_0|\leqslant|z_0|<r, & y_0\neq 0,\\ 0<\varepsilon<x_0=|z_0|<r, & y_0=0, \end{cases}$$

and consider the positively oriented contour $C(\varepsilon,r)$ in $\mathbb{C}\setminus(-\infty,0]$ consisting of the half circle $z=\varepsilon e^{i\theta}$ for $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and the half lines $z=x\pm i\varepsilon$ for $x\leqslant 0$ until they cut the circle $|z|=r$, which close the contour at the points $-r(\varepsilon)\pm i\varepsilon$, where $0<r(\varepsilon)\to r$ as $\varepsilon\to 0$. See Figure 1 below.

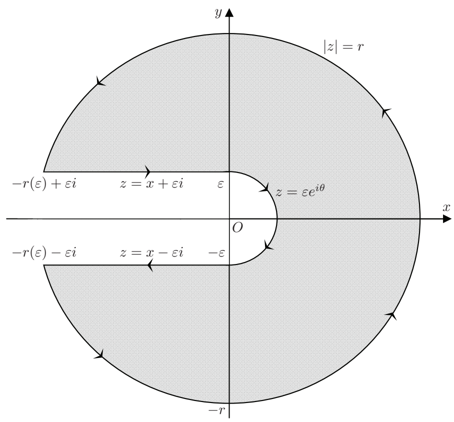

Figure 1. *The contour* $C(\varepsilon,r)$

By Cauchy integral formula, that is, Lemma 2.1, we have

$$\begin{aligned} h_n(z_0) &= \frac{1}{2\pi i}\oint_{C(\varepsilon,r)}\frac{h_n(w)}{w-z_0}\,\mathrm{d}w\\ &= \frac{1}{2\pi i}\left[\int_{\pi/2}^{-\pi/2}\frac{i\varepsilon e^{i\theta}h\big(\varepsilon e^{i\theta}\big)}{\varepsilon e^{i\theta}-z_0}\,\mathrm{d}\theta + \int_{\arg[-r(\varepsilon)-i\varepsilon]}^{\arg[-r(\varepsilon)+i\varepsilon]}\frac{ire^{i\theta}h\big(re^{i\theta}\big)}{re^{i\theta}-z_0}\,\mathrm{d}\theta\right.\\ &\quad\left.+\int_{-r(\varepsilon)}^{0}\frac{h_n(x+i\varepsilon)}{x+i\varepsilon-z_0}\,\mathrm{d}x+\int_0^{-r(\varepsilon)}\frac{h_n(x-i\varepsilon)}{x-i\varepsilon-z_0}\,\mathrm{d}x\right]. \end{aligned} \tag{3.2}$$

By the limit in (3.1), it follows that

$$\lim_{\varepsilon\to 0^+}\int_{\pi/2}^{-\pi/2}\frac{i\varepsilon e^{i\theta}h_n\big(\varepsilon e^{i\theta}\big)}{\varepsilon e^{i\theta}-z_0}\,\mathrm{d}\theta = 0. \tag{3.3}$$

By virtue of the limit (2.3) in Lemma 2.2, we deduce that

$$\begin{aligned}\lim_{\substack{\varepsilon\to0^+\\ r\to\infty}}\int_{\arg[-r(\varepsilon)-i\varepsilon]}^{\arg[-r(\varepsilon)+i\varepsilon]}\frac{ire^{i\theta}h_n\bigl(re^{i\theta}\bigr)}{re^{i\theta}-z_0}\,\mathrm{d}\,\theta&=\lim_{r\to\infty}\int_{-\pi}^{\pi}\frac{ire^{i\theta}h_n\bigl(re^{i\theta}\bigr)}{re^{i\theta}-z_0}\,\mathrm{d}\,\theta\\&=2A_n\bigl(\sigma(\mathbf{a})-a_{\sigma(1)}\mathbf{e}\bigr)\pi i.\end{aligned}\tag{3.4}$$

Utilizing the second formula in (3.1) and the limit (2.5) in Lemma 2.3 results in

$$\begin{aligned}&\int_{-r(\varepsilon)}^{0}\frac{h_n(x+i\varepsilon)}{x+i\varepsilon-z_0}\,\mathrm{d}\,x+\int_{0}^{-r(\varepsilon)}\frac{h_n(x-i\varepsilon)}{x-i\varepsilon-z_0}\,\mathrm{d}\,x\\
&=\int_{-r(\varepsilon)}^{0}\left[\frac{h_n(x+i\varepsilon)}{x+i\varepsilon-z_0}-\frac{h_n(x-i\varepsilon)}{x-i\varepsilon-z_0}\right]\mathrm{d}\,x\\
&=\int_{-r(\varepsilon)}^{0}\frac{(x-i\varepsilon-z_0)h_n(x+i\varepsilon)-(x+i\varepsilon-z_0)h_n(x-i\varepsilon)}{(x+i\varepsilon-z_0)(x-i\varepsilon-z_0)}\,\mathrm{d}\,x\\
&=\int_{-r(\varepsilon)}^{0}\frac{(x-z_0)[h_n(x+i\varepsilon)-h_n(x-i\varepsilon)]-i\varepsilon[h_n(x-i\varepsilon)+h_n(x+i\varepsilon)]}{(x+i\varepsilon-z_0)(x-i\varepsilon-z_0)}\,\mathrm{d}\,x\\
&=2i\int_{-r(\varepsilon)}^{0}\frac{(x-z_0)\Im h_n(x+i\varepsilon)-\varepsilon\Re h_n(x+i\varepsilon)}{(x+i\varepsilon-z_0)(x-i\varepsilon-z_0)}\,\mathrm{d}\,x\\
&\to2i\int_{-r}^{0}\frac{\lim_{\varepsilon\to0^+}\Im h_n(x+i\varepsilon)}{x-z_0}\,\mathrm{d}\,x\\
&=-2i\int_{0}^{r}\frac{\lim_{\varepsilon\to0^+}\Im h_n(-t+i\varepsilon)}{t+z_0}\,\mathrm{d}\,t\\
&\to-2i\int_{0}^{\infty}\frac{\lim_{\varepsilon\to0^+}\Im h_n(-t+i\varepsilon)}{t+z_0}\,\mathrm{d}\,t\\
&=-2i\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{\sigma(\ell)}-a_{\sigma(1)}}^{a_{\sigma(\ell+1)}-a_{\sigma(1)}}\left[\prod_{k=1}^{n}\bigl|a_{\sigma(k)}-a_{\sigma(1)}-t\bigr|\right]^{1/n}\frac{\mathrm{d}\,t}{t+z_0}\end{aligned}\tag{3.5}$$

as $\varepsilon\to0^+$ and $r\to\infty$. Substituting equations (3.3), (3.4), and (3.5) into (3.2) and simplifying generate

$$\begin{aligned}h_n(z_0)=A_n\bigl(\sigma(\mathbf{a})-a_{\sigma(1)}\mathbf{e}\bigr)\\-\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{\sigma(\ell)}-a_{\sigma(1)}}^{a_{\sigma(\ell+1)}-a_{\sigma(1)}}\left[\prod_{k=1}^{n}\bigl|a_{\sigma(k)}-a_{\sigma(1)}-t\bigr|\right]^{1/n}\frac{\mathrm{d}\,t}{t+z_0}.\end{aligned}\tag{3.6}$$

From (2.2) and (2.4), it is easy to obtain that

$$f_{\mathbf{a},n}(z_0)=h_n\bigl(z+a_{\sigma(1)}\bigr)+a_{\sigma(1)}.$$

Combining this with (3.6) and changing the variables of integrals, it is immediate to deduce that

$$f_{\mathbf{a},n}(z_0)=A_n\bigl(\sigma(\mathbf{a})-a_{\sigma(1)}\mathbf{e}\bigr)+a_{\sigma(1)}$$

$$
-\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{\sigma(\ell)}-a_{\sigma(1)}}^{a_{\sigma(\ell+1)}-a_{\sigma(1)}}\left[\prod_{k=1}^{n}\left|a_{\sigma(k)}-a_{\sigma(1)}-t\right|\right]^{1/n}\frac{\mathrm{d}t}{t+z_0+a_{\sigma(1)}}
$$

$$
=A_n(\sigma(\mathbf{a}))-\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{\sigma(\ell)}}^{a_{\sigma(\ell+1)}}\left[\prod_{k=1}^{n}\left|a_{\sigma(k)}-t\right|\right]^{1/n}\frac{\mathrm{d}t}{t+z_0},
$$

from which and the facts that

$$
A_n(\sigma(\mathbf{a}))=A_n(\mathbf{a}) \quad \text{and} \quad \prod_{k=1}^{n}|a_{\sigma(k)}-t|=\prod_{k=1}^{n}|a_k-t|,
$$

the integral representation (1.7) follows.

Since $\frac{1}{t+z}=\int_0^\infty e^{-(t+z)u}\,\mathrm{d}u$ for $\Re(t+z)>0$, the integral representation (1.7) may be rewritten as

$$
G_n(\mathbf{a}+z\mathbf{e})=A_n(\mathbf{a})-\int_0^\infty Q(u)\,\mathrm{d}u+z+\int_0^\infty\left(1-e^{-zu}\right)Q(u)\,\mathrm{d}u. \tag{3.7}
$$

Letting $z\to0$ on both sides of (3.7) gives $\int_0^\infty Q(u)\,\mathrm{d}u=A_n(\mathbf{a})-G_n(\mathbf{a})$. Hence, the integral representation (1.8) follows.

It is clear that the function $Q(u)$ is completely monotonic on $(0,\infty)$, which means that the geometric mean $G_n(\mathbf{a}+x\mathbf{e})$ is a complete Bernstein function on $(0,\infty)$. The proof of Theorem 1.1 is complete.

4. A new proof of the arithmetic-geometric mean inequality

There has been a large number, presumably over one hundred, of proofs of the arithmetic-geometric mean inequality (1.6) in the mathematical literature. The most complete information, so far, can be found in the monographs [2, 4, 8, 9, 10] and a lot of references therein.

As an application of the integral representation (1.7) in Theorem 1.1, a new proof for the arithmetic-geometric mean inequality (1.6) may be formulated as follows.

Taking $z=0$ in the integral representation (1.7) yields

$$
G_n(\mathbf{a})=A_n(\mathbf{a})-\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{\sigma(\ell)}}^{a_{\sigma(\ell+1)}}\left[\prod_{k=1}^{n}|a_k-t|\right]^{1/n}\frac{\mathrm{d}t}{t}\leqslant A_n(\mathbf{a}), \tag{4.1}
$$

from which the inequality (1.6) follows.

From (4.1), it is also immediate that the equality in (1.6) is valid if and only if $a_{\sigma(1)}=a_{\sigma(2)}=\cdots=a_{\sigma(n)}$, that is, $a_1=a_2=\cdots=a_n$. The proof of the arithmetic-geometric mean inequality (1.6) is complete.

REMARK 4.1. The integral representation (1.7) in Theorem 1.1 has been applied in [11] to find an integral representation of Stirling numbers of the first kind. It has also been generalized in [20].

REMARK 4.2. This article combines and upgrades the preprints [19, 22].

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