A HALANAY–TYPE INEQUALITY ON TIME
SCALES IN HIGHER DIMENSIONAL SPACES

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Abstract. In this paper, we investigate a certain class of Halanay-type inequalities on time scales in higher dimensional spaces. By means of the obtained inequality, we derive some new global stability conditions for linear delay dynamic systems on time scales. An example is given to illustrate the results.

1. Introduction

The analysis of the stability of dynamic systems using differential and difference inequalities has attracted a great deal of attention in the literature. One of the main motivations arises from some results due to Halanay, [4]. In the investigation of the stability of the delay differential equation

$$x'(t) = -px(t) + qx(t - \tau), \quad \tau > 0,$$

Halanay proved the following result (see [4]).

LEMMA 1.1. *If*

$$f'(t) \leqslant -\alpha f(t) + \beta \sup_{s \in [t-\tau,t]} f(s), \quad \textit{for} \quad t \geqslant t_0$$

where $\alpha > \beta > 0$, *then there exist* $\gamma > 0$ *and* $K > 0$ *such that*

$$f(t) \leqslant K e^{-\gamma(t-t_0)}, \quad \textit{for } t \geqslant t_0.$$

In 2000, Mohamad and Gopalsamy obtained a generalized discrete Halanay inequality (see [7]). A very detailed discussion of the application of Halanay's lemma and its generalizations may be found in [3].

In this paper, we derive an analog of the Halanay-type inequality on time scales in higher dimensional spaces. By means of this inequality, we obtain new global stability conditions for linear delay dynamic systems on time scales.

For completeness, we recall some basic results for dynamic equations and the calculus on time scales, (see [8] and [9] for elementary results for the time scale calculus). Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where $\sup \varnothing := \inf \mathbb{T}$, where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is left-dense. Given a time scale interval $[c,d]_{\mathbb{T}} := \{t \in \mathbb{T} : c \leqslant t \leqslant d\}$ in $\mathbb{T}$ the notation $[c,d]^{\kappa}_{\mathbb{T}}$ denotes the interval $[c,d]_{\mathbb{T}}$ in case $\rho(d) = d$ and denotes the interval $[c,d)_{\mathbb{T}}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We also recall that the notation C_{rd} denotes the set of all functions which are continuous at all right-dense points and have finite left-sided limits at left-dense points. We say that $x : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^{\Delta}(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s},$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. Hence our results contain the discrete and continuous cases as special cases and generalizes these results to time scales with bounded graininess.

2. Main Theorems

LEMMA 2.1. *Let*

$$P(t) := (P_1(t), P_2(t), \cdots, P_r(t)), \quad Q(t) := (Q_1(t), Q_2(t), \cdots, Q_r(t))$$

be r dimensional vector functions defined on $[t_0, \infty)_{\mathbb{T}}$, where $P(t), Q(t) \in C_{rd}$. Let $\tau > 0$, $t_0 \in \mathbb{T}$. Suppose the following dynamic inequalities hold:

$$P_i^{\Delta}(t) \leqslant \sum_{j=1}^{r} [a_{ij} P_j(t) + b_{ij} P_j(t, \tau)], \quad (i = 1, 2, \cdots, r), \quad t > t_0 \qquad (2.1)$$

$$Q_i^\Delta(t) > \sum_{j=1}^{r} [a_{ij}Q_j(t) + b_{ij}Q_j(t,\tau)], \quad (i=1,2,\cdots,r), \quad t > t_0 \qquad (2.2)$$

and

$$P(t) < Q(t), \quad t \in [t_0 - \tau, t_0]_\mathbb{T} \qquad (2.3)$$

where $a_{ii} < 0$, $1 + \mu(t)a_{ii} > 0$, $(i=1,2,\cdots,r)$, $t \geqslant t_0$,

$$a_{ij} \geqslant 0, \quad (i \neq j, \ i,j = 1,2,\cdots,r), \quad b_{ij} \geqslant 0, \quad (i,j=1,2,\cdots,r),$$

$$P_j(t,\tau) = \sup_{-\tau \leqslant \theta \leqslant 0} P_j(\theta + t), \quad Q_j(t,\tau) = \sup_{-\tau \leqslant \theta \leqslant 0} Q_j(\theta + t), \quad t \geqslant t_0.$$

Then

$$P(t) < Q(t), \quad t_0 < t < \infty.$$

Proof. Let $\xi = \sup\{t : P(t) < Q(t)\} > t_0$. We will show $\xi = \infty$. Suppose that $\xi < \infty$. Clearly we have $P(\xi) \leqslant Q(\xi)$. Then we have two cases:

Case (1). There exist $i_0 \in \{1,2,\cdots,r\}$ such that

$$P_{i_0}(\xi) = Q_{i_0}(\xi), \quad P(t) \leqslant Q(t), \quad t_0 - \tau \leqslant t \leqslant \xi.$$

So

$$\sup_{\xi - \tau \leqslant \theta \leqslant \xi} P(\theta) \leqslant \sup_{\xi - \tau \leqslant \theta \leqslant \xi} Q(\theta).$$

$$\begin{aligned}
P_{i_0}^\Delta(\xi) &\leqslant \sum_{j=1}^{r} [a_{i_0 j}P_j(\xi) + b_{i_0 j}P_j(\xi,\tau)] \qquad (2.4) \\
&\leqslant \sum_{j=1}^{r} [a_{i_0 j}Q_j(\xi) + b_{i_0 j}Q_j(\xi,\tau)] \\
&< Q_{i_0}^\Delta(\xi).
\end{aligned}$$

When $t_0 < t < \xi$, $P(t) \leqslant Q(t)$ and $P_{i_0}(\xi) = Q_{i_0}(\xi)$, so $P_{i_0}^\Delta(\xi) \geqslant Q_{i_0}^\Delta(\xi)$, which contradicts (2.4).

Case (2). $P(\xi) < Q(\xi)$. In this case, ξ must be right-scattered, for otherwise if ξ is right-dense, then we have $P(t) < Q(t)$, for $t \in [t_0 - \tau, \xi]_\mathbb{T}$. Therefore, there exists δ sufficiently small so that $P(t) < Q(t)$, for $t \in [t_0 - \tau, \xi + \delta]_\mathbb{T}$. This contradicts the definition of ξ, using the fact that $P(t), Q(t)$ are rd-continuous. Hence, since ξ is right-scattered, we have $P(t) < Q(t)$, for $t \in [t_0 - \tau, \xi]_\mathbb{T}$ and there exists $i_0 \in \{1,2,\cdots,r\}$ such that

$$P_{i_0}(\sigma(\xi)) \geqslant Q_{i_0}(\sigma(\xi)). \qquad (2.5)$$

Noticing $1 + \mu(t)a_{ii} > 0$ and

$$P_{i_0}^\Delta(\xi) = \frac{P_{i_0}(\sigma(\xi)) - P_{i_0}(\xi)}{\mu(\xi)}, \qquad (2.6)$$

$$Q_{i_0}^{\Delta}(\xi) = \frac{Q_{i_0}(\sigma(\xi)) - Q_{i_0}(\xi)}{\mu(\xi)}, \tag{2.7}$$

from (2.1), (2.6), (2.7), (2.2), and $1 + \mu(t)a_{i_0 i_0} > 0$, we get that

$$P_{i_0}(\sigma(\xi)) \leqslant P_{i_0}(\xi) + \mu(\xi)\sum_{j=1}^{r}[a_{i_0 j}P_j(\xi) + b_{i_0 j}P_j(\xi, \tau)]$$

$$= [1 + \mu(\xi)a_{i_0 i_0}]P_{i_0}(\xi) + \mu(\xi)b_{i_0 i_0}P_{i_0}(\xi, \tau)]$$

$$+ \mu(\xi)\sum_{j=1, j \neq i_0}^{r}[a_{i_0 j}P_j(\xi) + b_{i_0 j}P_j(\xi, \tau)]$$

$$\leqslant [1 + \mu(\xi)a_{i_0 i_0}]Q_{i_0}(\xi) + \mu(\xi)b_{i_0 i_0}Q_{i_0}(\xi, \tau)$$

$$+ \mu(\xi)\sum_{j=1, j \neq i_0}^{r}[a_{i_0 j}Q_j(\xi) + b_{i_0 j}Q_j(\xi, \tau)]$$

$$= Q_{i_0}(\xi) + \mu(\xi)\sum_{j=1}^{r}[a_{i_0 j}Q_j(\xi) + b_{i_0 j}Q_j(\xi, \tau)]$$

$$< Q_{i_0}(\sigma(\xi)),$$

which contradicts (2.5).

This completes the proof. □

The following lemmas may be found in [8] and [10], respectively.

LEMMA 2.2. *If $p \in \mathfrak{R}$, then the semigroup property*

$$e_p(t, r)e_p(r, s) = e_p(t, s), \quad \text{for all} \quad r, s, t \in \mathbb{T}$$

is satisfied.

LEMMA 2.3. *For a nonnegative φ with $-\varphi \in \mathfrak{R}^+$, we have the inequalities*

$$1 - \int_s^t \varphi(u)\Delta u \leqslant e_{-\varphi}(t, s) \leqslant exp\left\{-\int_s^t \varphi(u)\Delta u\right\}, \quad \text{for all} \quad t \geqslant s.$$

If φ is rd-continuous and nonnegative, then

$$1 + \int_s^t \varphi(u)\Delta u \leqslant e_{\varphi}(t, s) \leqslant exp\left\{\int_s^t \varphi(u)\Delta u\right\}, \quad \text{for all} \quad t \geqslant s.$$

We now present the main result of the paper.

THEOREM 2.4. *Let $P(t) = (P_1(t), P_2(t), \cdots, P_r(t))$ be an r dimensional C_{rd} (non-negative) vector function defined on $[t_0, +\infty)_{\mathbb{T}}$, ($\tau > 0, t_0 \in \mathbb{T}$), with $t - \tau \in \mathbb{T}$ for all $t \in [t_0, +\infty)_T$. If*

$$P_i^{\Delta}(t) \leqslant \sum_{j=1}^{r}[a_{ij}P_j(t) + b_{ij}P_j(t, \tau)], \quad (i = 1, 2, \cdots, r), \quad t \geqslant t_0, \tag{2.8}$$

where

$$a_{ii} < 0, \quad 1 + \mu(t)a_{ii} > 0, \quad (i = 1, 2, \cdots, r), \quad t \geqslant t_0,$$

$$a_{ij} \geqslant 0, \quad (i \neq j, i, j = 1, 2, \cdots, r), \quad b_{ij} \geqslant 0, \quad (i, j = 1, 2, \cdots, r),$$

$$Re\lambda(a_{ij} + b_{ij})_{r \times r} < 0$$

and

$$P_j(t, \tau) = \sup_{-\tau \leqslant \theta \leqslant 0} P_j(\theta + t), \quad j = 1, 2, \cdots, r, \quad t \geqslant t_0.$$

Then there exist $K > 1$, $\alpha_i > 0$ and $\alpha > 0$ such that

$$P_i(t) \leqslant K\alpha_i \sum_{j=1}^{r} P_j(t_0, \tau)e_{-\alpha}(t, t_0), \quad (i = 1, 2, \cdots, r), \quad t \geqslant t_0. \tag{2.9}$$

Proof. From $Re\lambda(a_{ij} + b_{ij})_{r \times r} < 0$ and [5], it follows that the matrix $(a_{ij} + b_{ij})_{r \times r}$ is quasi negative diagonally dominant. So there exist $\alpha_i > 0$, $(i = 1, 2, \cdots, r)$ such that

$$\sum_{j=1}^{r} (a_{ij} + b_{ij})\alpha_j < 0, \quad (i = 1, 2, \cdots, r).$$

From the continuity, there exists a small $\alpha > 0$ such that

$$\alpha\alpha_i + \sum_{j=1}^{r} [a_{ij}\alpha_j + b_{ij}\alpha_j(1 - \alpha\tau)^{-1}] < 0. \tag{2.10}$$

Using Lemma 2.3, there exists a large K such that for $t \in [t_0 - \tau, t_0]_{\mathbb{T}}$, we have

$$K\alpha_i e_{-\alpha}(t, t_0) \geqslant K\alpha_i[1 - \alpha(t - t_0)] \geqslant K\alpha_i > 1. \tag{2.11}$$

In the first place, we will prove that for any $\varepsilon > 0$,

$$P_i(t) < K\alpha_i \left[\sum_{j=1}^{r} P_j(t_0, \tau) + \varepsilon \right] e_{-\alpha}(t, t_0), \quad (i = 1, 2, \cdots, r), \quad t \geqslant t_0. \tag{2.12}$$

Since $e_{-\alpha}^{\Delta}(t, t_0) = -\alpha e_{-\alpha}(t, t_0) < 0$, for $t > t_0$, using Lemma 2.2, we have

$$\sup_{t-\tau \leqslant \theta \leqslant t} e_{-\alpha}(\theta, t_0) = e_{-\alpha}(t - \tau, t_0) \tag{2.13}$$

$$= e_{-\alpha}(t - \tau, t)e_{-\alpha}(t, t_0).$$

From Lemma 2.3, we have $e_{-\alpha}(t - \tau, t) = e_{-\alpha}^{-1}(t, t - \tau) \leqslant (1 - \alpha\tau)^{-1}$. So from (2.13), we get that

$$\sup_{t-\tau \leqslant \theta \leqslant t} e_{-\alpha}(\theta, t_0) \leqslant \frac{e_{-\alpha}(t, t_0)}{1 - \alpha\tau}, \quad t \geqslant t_0. \tag{2.14}$$

Let

$$Q_i(t) = K\alpha_i \left[\sum_{j=1}^{r} P_j(t_0, \tau) + \varepsilon \right] e_{-\alpha}(t, t_0), \quad (i = 1, 2, \cdots, r), \quad t \in [t_0 - \tau, +\infty)_{\mathbb{T}}. \tag{2.15}$$

Using (2.10), we have that for $t \geqslant t_0$

$$Q_i^{\Delta}(t) = K\alpha_i \left[\sum_{j=1}^{r} P_j(t_0, \tau) + \varepsilon \right] (-\alpha) e_{-\alpha}(t, t_0)$$

$$> K \sum_{j=1}^{r} [a_{ij}\alpha_j + b_{ij}\alpha_j(1 - \alpha\tau)^{-1}] \left[\sum_{j=1}^{r} P_j(t_0, \tau) + \varepsilon \right] e_{-\alpha}(t, t_0).$$

From (2.14) and (2.15), we get that

$$Q_i^{\Delta}(t) > \sum_{j=1}^{r} [a_{ij}Q_j(t) + b_{ij} \sup_{t-\tau \leqslant \theta \leqslant t} Q_j(\theta)], \quad t \geqslant t_0. \tag{2.16}$$

From (2.11) and (2.15), it is easy to see that

$$P_i(t) < Q_i(t), \quad \text{for} \quad t \in [t_0 - \tau, t_0]_{\mathbb{T}}. \tag{2.17}$$

From (2.8), (2.16), (2.17) and Lemma 2.1, we get that (2.12) holds.
Letting $\varepsilon \to 0^+$, we obtain that (2.9) holds. $\quad\square$

3. Examples

EXAMPLE 3.1. Consider the delay dynamic system with $n > 1$

$$x_i^{\Delta}(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t - \tau), \quad (i = 1, 2, \cdots, n), \quad t \in [t_0, +\infty)_{\mathbb{T}}, \tag{3.1}$$

where

$$a_{ii} < 0, \quad 1 + \mu(t)a_{ii} > 0, \quad (i = 1, 2, \cdots, n),$$

$$a_{ij} \geqslant 0, \quad (i \neq j, \ i, j = 1, 2, \cdots, n), \quad b_{ij} \geqslant 0, \quad (i, j = 1, 2, \cdots),$$

$$\operatorname{Re}\lambda(a_{ij} + b_{ij})_{n \times n} < 0.$$

We equivalently rewrite (3.1) as

$$(1 + \mu(t)a_{ii})x_i^{\Delta}(t) = a_{ii}x_i^{\sigma}(t) + \sum_{j=1, j \neq i}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t - \tau), \tag{3.2}$$

$(i = 1, 2, \cdots, n)$. So

$$x_i(t) = x_i^0 e_{a_{ii}}(t, t_0) + \int_{t_0}^{t} e_{a_{ii}}(t, \sigma(s)) \left[\sum_{j=1, j \neq i}^{n} a_{ij}x_j(s) + \sum_{j=1}^{n} b_{ij}x_j(s - \tau) \right] \Delta s,$$

$(i = 1, 2, \cdots, n)$.

Let the functions $y_i(t)$, $(i = 1, 2, \cdots, n)$ be defined as follows:

$$y_i(t) = |x_i(t)|, \quad \text{for} \quad t \in [t_0 - \tau, t_0]_{\mathbb{T}}$$

and

$$y_i(t) = |x_i^0| e_{a_{ii}}(t, t_0) + \int_{t_0}^t e_{a_{ii}}(t, \sigma(s)) \left[\sum_{j=1, j \neq i}^n a_{ij} |x_j(s)| + \sum_{j=1}^n b_{ij} \sup_{s-\tau \leqslant \theta \leqslant s} |x_j(\theta)| \right] \Delta s,$$

for $t > t_0$. Then we have $|x_i(t)| \leqslant y_i(t)$, $(i = 1, 2, \cdots, n)$ for all $t \in [t_0 - \tau, \infty)_{\mathbb{T}}$.

By [8, Theorem 1.117], we get that

$$y_i^{\Delta}(t)$$
$$= a_{ii} \left(|x_i^0| e_{a_{ii}}(t, t_0) + \int_{t_0}^t e_{a_{ii}}(t, \sigma(s)) \left[\sum_{j=1, j \neq i}^n a_{ij} |x_j(s)| + \sum_{j=1}^n b_{ij} \sup_{s-\tau \leqslant \theta \leqslant s} |x_j(\theta)| \right] \Delta s \right)$$
$$+ \sum_{j=1, j \neq i}^n a_{ij} |x_j(t)| + \sum_{j=1}^n b_{ij} \sup_{t-\tau \leqslant \theta \leqslant t} |x_j(\theta)|$$
$$\leqslant a_{ii} y_i(t) + \sum_{j=1, j \neq i}^n a_{ij} y_j(t) + \sum_{j=1}^n b_{ij} \sup_{t-\tau \leqslant \theta \leqslant t} y_j(\theta)$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. Therefore, it follows from Theorem 2.4 that there exists a constant $M > 1$ and $\alpha > 0$ such that

$$|x_i(t)| \leqslant y_i(t) \leqslant M e_{-\alpha}(t, t_0), \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

In the following, we let $\mathbb{T} = \mathbb{Z}$ and choose some explicit values for a_{ij}, b_{ij}, $(i, j = 1, 2, \cdots, n)$. Let

$$(a_{ij})_{n \times n} = \frac{1}{2n^2} \begin{pmatrix} 1 - 2n^2 & 1 & 1 & \cdots & 1 \\ 1 & 1 - 2n^2 & 1 & \cdots & 1 \\ 1 & 1 & 1 - 2n^2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 - 2n^2 \end{pmatrix}$$

$$(b_{ij})_{n \times n} = \frac{1}{2n^2} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

We have

$$(a_{ij} + b_{ij})_{n \times n} = \frac{1}{n^2} \begin{pmatrix} 1 - n^2 & 1 & 1 & \cdots & 1 \\ 1 & 1 - n^2 & 1 & \cdots & 1 \\ 1 & 1 & 1 - n^2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 - n^2 \end{pmatrix}.$$

The $k-th$ order leading principal minor of $(a_{ij}+b_{ij})_{n\times n}$ is

$$\Delta_k = \begin{vmatrix} n^{-2}-1 & n^{-2} & n^{-2} & \cdots & n^{-2} \\ n^{-2} & n^{-2}-1 & n^{-2} & \cdots & n^{-2} \\ n^{-2} & n^{-2} & n^{-2}-1 & \cdots & n^{-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n^{-2} & n^{-2} & n^{-2} & \cdots & n^{-2}-1 \end{vmatrix}_{k\times k}$$

$(k=1,2,\cdots,n)$. In the following, using induction we will prove that

$$\Delta_k = (-1)^k(1-\frac{k}{n^2}).\qquad(3.3)$$

When $k=1$, it is easy to see that (3.3) holds.

Suppos that $k=m-1$, (3.3) holds. Then when $k=m$,

$$\Delta_m = n^{-2m}\begin{vmatrix} 1-n^2 & 1 & 1 & \cdots & 1 \\ 1 & 1-n^2 & 1 & \cdots & 1 \\ 1 & 1 & 1-n^2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1-n^2 \end{vmatrix}_{m\times m}$$

$$= n^{-2m}\begin{vmatrix} 1-n^2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1-n^2 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1-n^2 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-n^2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix}_{m\times m}$$

$$+n^{-2m}\begin{vmatrix} 1-n^2 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1-n^2 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1-n^2 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-n^2 & 0 \\ 1 & 1 & 1 & \cdots & 1 & -n^2 \end{vmatrix}_{m\times m}$$

$$= n^{-2m}\begin{vmatrix} -n^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -n^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -n^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n^2 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix}_{m\times m} - \Delta_{m-1}$$

So

$$\Delta_m = (-1)^{m-1}n^{-2} - \Delta_{m-1}$$
$$= (-1)^m\left[1-\frac{m}{n^2}\right].$$

So (3.3) holds. Therefore $Re\lambda (a_{ij} + b_{ij})_{n\times n} < 0$.

When $\mathbb{T} = \mathbb{Z}$ and α is constant, then we have $e_{-\alpha}(t, t_0) = (1 - \alpha)^{t-t_0}$. Therefore we can find $M > 1$ and a sufficiently small $0 < \alpha < 1$ such that

$$|x_i(n)| \leqslant M(1 - \alpha)^{n-n_0}, \quad n \geqslant n_0, \quad (i = 1, 2, \cdots, n).$$

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