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A WEIGHTED NORM INEQUALITY FOR MULTILINEAR FOURIER MULTIPLIER OPERATOR

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Abstract. In this paper, a weighted norm inequality with multiple-weight is established for the multilinear Fourier multiplier operator.

1. Introduction

The study of the multilinear Fourier multiplier was originated by Coifman and Meyer in their celebrated work [2], [3]. Let $\sigma \in L^\infty(\mathbb{R}^{mn})$. Define the multilinear Fourier multiplier operator T_σ by

$$T_\sigma(f_1,\dots,f_m)(x) = \int_{\mathbb{R}^{mn}} \exp(2\pi i x(\xi_1+\ldots+\xi_m))\sigma(\xi_1,\dots,\xi_2)\widehat{f_1}(\xi_1)\ldots\widehat{f_m}(\xi_m)d\vec{\xi} \tag{1.1}$$

for $f_1,\ldots,f_m \in \mathscr{S}(\mathbb{R}^n)$, where and in the following, $d\vec{\xi} = d\xi_1 \ldots d\xi_m$. Coifman and Meyer [3] proved that if $\sigma \in C^s(\mathbb{R}^{nm}\backslash\{0\})$ satisfying that

$$|\partial_{\xi_1}^{\alpha_1}\ldots\partial_{\xi_m}^{\alpha_m}\sigma(\xi_1,\ldots,\xi_m)| \leqslant C_{\alpha_1,\ldots\alpha_m}(|\xi_1|+\ldots+|\xi_m|)^{-(|\alpha_1|+\ldots+|\alpha_m|)} \tag{1.2}$$

for all $|\alpha_1|+\ldots+|\alpha_m| \leqslant s$ with $s \geqslant 2mn+1$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times\ldots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1,\ldots,p_m,p < \infty$ with $1/p = \sum_{1\leqslant k\leqslant m} 1/p_k$. For the case of $s \geqslant nm+1$, Grafakos and Torres [7] improved the multiplier theorem of Coifman and Meyer to the indices $1/m \leqslant p \leqslant 1$, using the multilinear Calderón-Zygmund operator theory. A very important progress in this topics was given by Tomita. Let $\Phi \in \mathscr{S}(\mathbb{R}^{nm})$ such that $\operatorname{supp}\Phi \subset \{(\xi_1,\ldots,\xi_m) : 1/2 \leqslant |\xi_1|+\ldots+|\xi_m| \leqslant 2\}$ and for all $(\xi_1,\ldots,\xi_m) \in \mathbb{R}^{mn}\backslash\{0\}$.

$$\sum_{l\in\mathbb{Z}} \Phi(2^{-l}\xi_1,\ldots,2^{-l}\xi_m) = 1.$$

Set

$$\sigma_l(\xi_1,\ldots,\xi_m) = \Phi(\xi_1,\ldots,\xi_m)\sigma(2^l\xi_1,\ldots,2^l\xi_m), \tag{1.3}$$

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Paper MIA-17-65

and

$$\|\sigma_l\|_{W^s(\mathbb{R}^{mn})} = \Big(\int_{\mathbb{R}^{mn}} (1+|\xi_1|^2+\ldots+|\xi_m|^2)^s |\mathscr{F}\sigma_l(\xi_1,\ldots,\xi_m)|^2 d\vec{\xi}\Big)^{1/2},$$

with $\mathscr{F}\sigma_l$ the Fourier transform of σ_l. Tomita [10] proved that if

$$\sup_{l\in\mathbb{Z}} \|\sigma_l\|_{W^s(\mathbb{R}^{mn})} < \infty, \tag{1.4}$$

for some $s \in (mn/2, mn]$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times\ldots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $p_1,\ldots,p_m,p\in(1,\infty)$ and $1/p=\sum_{1\leqslant k\leqslant m} 1/p_k$. Grafakos and Si [6] considered the mapping properties from $L^{p_1}(\mathbb{R}^n)\times\ldots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for T_σ when $p\leqslant 1$. Particularly, the argument used in [6] shows that if σ satisfies (1.4) for some $s>n$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times\ldots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $p_1,\ldots,p_m\in(mn/s,\infty)$ and $1/p=\sum_{1\leqslant k\leqslant m}1/p_k$, see the proof of Theorem 1.1 in [6].

Now we consider the weighted estimate for the operator T_σ. As it is well known, when σ satisfies (1.2) for some $s\geqslant mn+1$, then T_σ is a standard multilinear Calderón-Zygmund operator, and then by the weighted estimates with multiple weights for multilinear Calderón-Zygmund operators, which was established by Lerner [8], we know that for any $p_1,\ldots,p_m\in[1,\infty)$ and $p\in(0,\infty)$ with $1/p=\sum_{1\leqslant k\leqslant m}1/p_k$, and weights $w_1,\ldots,w_m$ such that $\vec{w}=(w_1,\ldots,w_m)\in A_{\vec{p}}(\mathbb{R}^{mn})$ (for the definition of $A_{\vec{p}}(\mathbb{R}^{mn})$, see Definition 1.1 below),

$$\|T_\sigma(f_1,\ldots,f_m)\|_{L^{p,\infty}(\mathbb{R}^n,\nu_{\vec{w}})} \lesssim \prod_{k=1}^m \|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)},$$

where and in the following, for index $p_1,\ldots,p_m$, we set $\vec{p}=(p_1,\ldots,p_m)$ and $p\in(0,\infty)$ such that $1/p=1/p_1+\ldots+1/p_m$. By a suitable kernel estimate and the theory of multilinear singular integral operator, Anh and Duong [1] established the weighted estimates with multiple weights for T_σ when σ satisfies (1.2) for $m=2$ and $s\in(n,2n]$. Our purpose in this paper is to give a weighted norm inequality with multiple weights for T_σ, which is a generalization of the result in [1]. To state our main result, we first introduce a class of weights.

DEFINITION 1.1. Let $m\geqslant 1$ be an integer, $w_1,\ldots,w_m$ be weights, $p_1,\ldots,p_m,p\in(0,\infty)$ with $1/p=\sum_{k=1}^m 1/p_k$, $r_k\in(0,p_k]$ $(1\leqslant k\leqslant m)$. Set $\vec{w}=(w_1,\ldots,w_m)$, $\vec{p}=(p_1,\ldots,p_m)$ and $\nu_{\vec{w}}=\prod_{k=1}^m w_k^{p/p_k}$. We say that $\vec{w}\in A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$ if

$$\sup_{Q\subset\mathbb{R}^n}\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}(x)\,dx\Big)^{1/p}\prod_{k=1}^m\Big(\frac{1}{|Q|}\int_Q w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(x)\,dx\Big)^{1/r_k-1/p_k}<\infty,$$

when $p_k=r_k$, $\Big(\frac{1}{|Q|}\int_Q w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(x)\,dx\Big)^{1/r_k-1/p_k}$ is understood as $(\inf_{x\in Q}w_k)^{-1/p_k}$

REMARK 1.1. For the case of $r_1 = \ldots = r_m = 1$, $A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$ is denoted by $A_{\vec{p}}(\mathbb{R}^{mn})$, which was introduced in [8], while when $r_1 = \ldots = r_m = r > 1$, $A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$ was given in [1].

Our main result in this paper is the following weighted norm inequality for T_σ.

THEOREM 1.1. *Let σ be a multiplier which satisfies (1.4) for some $s \in (mn/2, mn]$, $t_1, \ldots, t_m \in [1,2)$ such that $1/t_1 + \ldots + 1/t_m = s/n$. If $p_k \in (t_k, \infty)$ for $k = 1, \ldots, m$ and the weights $w_1, \ldots, w_m$ satisfy $\vec{w} \in A_{\vec{p}/\vec{t}}(\mathbb{R}^{mn})$, then*

$$\|T_\sigma(f_1, \ldots, f_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \lesssim \prod_{k=1}^{m} \|f_k\|_{L^{p_k}(\mathbb{R}^n, w_k)}. \tag{1.5}$$

REMARK 1.2. For the case of $t_1 = \ldots = t_m = mn/s$, (1.5) was proved by Anh and Duong [1].

We make some conventions. In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \leqslant CB$. For $p \in [1, \infty)$, $A_p(\mathbb{R}^n)$ denotes the Muckenhoupt class. For any set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. For $\vec{p} = (p_1, \ldots, p_m)$ and $\vec{t} = (t_1, \ldots, t_m)$, we say that $\vec{t} < \vec{p}$ if $t_k < p_k$ for $1 \leqslant k \leqslant 2$. For $\vec{t} = (t_1, \ldots, t_m)$ and a $\delta > 0$, set $\delta\vec{t} = (\delta t_1, \ldots, \delta t_m)$.

2. A multi(sub)linear maximal operator

In this section, we will introduce a multi(sub)linear maximal operator which will be useful in the proof of Theorem 1.1. Let $r_1, \ldots, r_m \in (0, \infty)$, and set $\vec{r} = (r_1, \ldots, r_m)$. Define the maximal operator $\mathscr{M}_{\vec{r}}$ by

$$\mathscr{M}_{\vec{r}}(f_1, \ldots, f_m)(x) = \sup_{B \ni x} \prod_{k=1}^{m} \left(\frac{1}{|B|} \int_B |f_k(x)|^{r_k} \, dx \right)^{1/r_k},$$

where the sup is taken over all balls containing x. When $r_1 = \ldots = r_m = 1$, $\mathscr{M}_{\vec{r}}$ is the maximal operator which controls the multilinear Calderón-Zygmund operators and introduced by Lerner et. in [8]. Our result in this section can be stated as follows.

THEOREM 2.1. *Let $m \geqslant 2$ be an integer, $w_1, \ldots, w_m$ be weights, $p_1, \ldots, p_m, p \in (0, \infty)$ with $1/p = \sum_{k=1}^{m} 1/p_k$, $r_k \in (0, p_k]$ $(1 \leqslant k \leqslant m)$. Then the following three conditions are equivalent*

(i) *$\mathscr{M}_{\vec{r}}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times \ldots \times L^{p_m}(\mathbb{R}^n, w_m)$ to $L^{p,\infty}(\mathbb{R}^n, \nu_{\vec{w}})$;*

(ii) *$\vec{w} \in A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$;*

(iii) *$\nu_{\vec{w}} \in A_{p/r}(\mathbb{R}^n)$, and for any k with $1 \leqslant k \leqslant m$, $w_k^{-1/(p_k/r_k - 1)} \in A_{\frac{p_k r_k}{r(p_k - r_k)}}(\mathbb{R}^n)$ if $r_k \neq p_k$ or $w_k^{r/p_k} \in A_1(\mathbb{R}^n)$ if $r_k = p_k$, here $1/r = \sum_{1 \leqslant k \leqslant m} 1/r_k$.*

Moreover, if $r_k \in (0, p_k)$ $(1 \leqslant k \leqslant m)$ *and* $\vec{w} \in A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$*, then* $\mathscr{M}_{\vec{r}}$ *is bounded from* $L^{p_1}(\mathbb{R}^n, w_1) \times ... \times L^{p_m}(\mathbb{R}^n, w_m)$ *to* $L^p(\mathbb{R}^n, v_{\vec{w}})$*.*

To prove Theorem 2.1, we will employ a preliminary lemma.

LEMMA 2.1. *Let* $u, v_1, ..., v_m$ *be weights,* $\vec{r} = (r_1, ..., r_m)$ *with* $r_k \in (0, \infty)$ *and* $\vec{p} = (p_1, ..., p_m)$ *with* $p_k \in [r_k, \infty)$*. Let* $1/p = \sum_{k=1}^m 1/p_k$ *and* $1/r = \sum_{k=1}^m 1/r_k$*. The following conditions are equivalent*

(a) $\mathscr{M}_{\vec{r}}$ *is bounded from* $L^{p_1}(\mathbb{R}^n, v_1) \times ... \times L^{p_m}(\mathbb{R}^n, v_m)$ *to* $L^{p,\infty}(\mathbb{R}^n, u)$*;*

(b)

$$\sup_{Q \subset \mathbb{R}^n} \Big(\frac{1}{|Q|}\int_Q u(x)dx\Big)^{1/p} \prod_{k=1}^m \Big(\frac{1}{|Q|}\int_Q v_k^{-\frac{1}{p_k/r_k-1}}(x)dx\Big)^{1/r_k-1/p_k} < \infty.$$

Proof. The proof is fairly standard. We first prove (b) implies (a). For each fixed $\lambda > 0$, set

$$\Omega_\lambda = \{x \in \mathbb{R}^n : \mathscr{M}_{\vec{r}}(f_1, ..., f_m)(x) > \lambda\}.$$

For each $x \in \Omega_\lambda$, we can choose a cube Q_x containing x such that

$$\prod_{k=1}^m \Big(\frac{1}{|Q_x|}\int_{Q_x} |f_k(y)|^{r_k}\, dy\Big)^{1/r_k} > \lambda.$$

Thus,

$$|Q_x|^{1/r} \leqslant \lambda^{-1} \prod_{k=1}^m \Big(\int_{Q_x} |f_k(y)|^{p_k} v_k(y)\, dy\Big)^{1/p_k} \Big(\int_{Q_x} v_k^{-\frac{1}{p_k/r_k-1}}\, dy\Big)^{1/r_k-1/p_k}.$$

and so

$$|Q_x| \leqslant \lambda^{-p} \prod_{k=1}^m \Big(\int_{Q_x} |f_k(y)|^{p_k} v_k(y)\, dy\Big)^{p/p_k} \Big(\frac{1}{|Q_x|}\int_{Q_x} v_k^{-\frac{1}{p_k/r_k-1}}(y)\, dy\Big)^{p/r_k-p/p_k}.$$

Let $K \subset \mathbb{R}^n$ be a compact set. From the cubes $\{Q_x\}_{x\in\Omega_\lambda}$ we can choose some cube $\{Q_i\}_{i=1}^N$ with disjoint interiors, such that $K \subset \cup_{i=1}^N 2Q_i$. Therefore,

$$u(K) \lesssim \sum_{i=1}^N \frac{u(2Q_i)}{|2Q_i|}|Q_i| \lesssim \lambda^{-p} \prod_{k=1}^m \Big(\int_{\mathbb{R}^n} |f_k(y)|^{p_k} w_k(y)dy\Big)^{p/p_k}.$$

We then get that

$$u(\{x \in \mathbb{R}^n : \mathscr{M}_{\vec{r}}(f_1, ..., f_m)(x) > \lambda\}) \lesssim \lambda^{-p} \prod_{k=1}^m \Big(\int_{\mathbb{R}^n} |f_k(y)|^{p_k} v_k(y)\, dy\Big)^{p/p_k}.$$

Now we prove the converse. Observe that (a) implies that

$$\Big(\frac{1}{|Q|}\int_Q u(x)\, dx\Big)^{1/p} \prod_{k=1}^m \Big(\frac{1}{|Q|}\int_Q |f_k(x)|^{r_k} dx\Big)^{1/r_k} \lesssim \prod_{k=1}^m \Big(\frac{1}{|Q|}\int_Q |f_k(x)|^{p_k} v_k dx\Big)^{1/p_k},$$

for suitable functions $f_1, \dots, f_m$. Setting $\varepsilon > 0$ and $f_k(x) = (v_k(x)+\varepsilon)^{-\frac{1}{p_k-r_k}}$ with $1 \leqslant k \leqslant m$ then leads to that

$$\Big(\frac{1}{|Q|}\int_Q u(x)\,dx\Big)^{1/p}\prod_{k=1}^{m}\Big(\frac{1}{|Q|}\int_Q (v_k(x)+\varepsilon)^{-\frac{1}{p_k/r_k-1}}dx\Big)^{1/r_k-1/p_k} \lesssim 1.$$

Taking $\varepsilon \to \infty$ then leads to that the condition (b). □

Proof of Theorem 2.1. We will invoke the idea used in the proof of [8]. By Lemma 2.1, it suffices to prove that (ii)$\Leftrightarrow$ (iii), and $\mathcal{M}_{\vec{r}}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1)\times \dots \times L^{p_m}(\mathbb{R}^n, w_m)$ to $L^p(\mathbb{R}^n, v_{\vec{w}})$ when $\min_{1\leqslant k\leqslant m} p_k/r_k > 1$ and $\vec{w} \in A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$.

(ii)$\Rightarrow$(iii). We assume that for some l with $0 \leqslant l < m$, $p_k = r_k$ for $1 \leqslant k \leqslant l$, and $p_k > r_k$ for $k = l+1, \dots, m$. Our first goal is to prove that for $l+1 \leqslant k \leqslant m$, $w_k^{-1/(p_k/r_k-1)} \in A_{\frac{p_k r_k}{r(p_k-r_k)}}(\mathbb{R}^n)$, that is, for any cube Q,

$$\Big(\frac{1}{|Q|}\int_Q w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q w_k^{\frac{p}{p_kq_k}}(x)\,dx\Big)^{\frac{p_kq_k}{p}\frac{1}{\frac{p_k}{r_k}-1}} \lesssim 1, \tag{2.2}$$

where

$$q_k = \frac{p}{r_k}\Big(\frac{r_k}{r} - 1 + \frac{r_k}{p_k}\Big).$$

To do this, set

$$q_j = \frac{p_j}{\frac{p_j}{r_j}-1}\frac{q_k}{p}, \quad l+1 \leqslant j \leqslant m, \quad j \neq k.$$

Note that

$$\frac{p}{p_jq_k}q_j = \frac{1}{\frac{p_j}{r_j}-1}$$

and

$$\frac{1}{q_k} + \sum_{l+1\leqslant j\leqslant m,\, j\neq k}\frac{1}{q_j} = \frac{1}{q_k} + \frac{p}{q_k}\Big(\frac{1}{r} - \frac{1}{p} - \frac{1}{r_k} + \frac{1}{p_k}\Big) = 1$$

It follows from the Hölder inequality that

$$\begin{aligned}\frac{1}{|Q|}\int_Q w_k^{\frac{p}{p_kq_k}}(x)\,dx &= \frac{1}{|Q|}\int_Q \prod_{j=l+1}^{m} w_j^{\frac{p}{p_jq_k}}(x)\prod_{l+1\leqslant j\leqslant m,\, j\neq k} w_j^{-\frac{p}{p_jq_k}}(x)\,dx\\ &\lesssim \Big(\frac{1}{|Q|}\int_Q \prod_{j=l+1}^{m} w_j^{p/p_j}\Big)^{1/q_k}\prod_{l+1\leqslant j\leqslant m,\, j\neq k}\Big(\frac{1}{|Q|}\int_Q w_j^{-\frac{pq_j}{p_jq_k}}\Big)^{1/q_j}.\end{aligned} \tag{2.3}$$

The inequality (2.2) follows from (2.3) and the fact that $\vec{w} \in A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$.

To prove that $v_{\vec{w}} \in A_{p/r}(\mathbb{R}^n)$, let $s_k = \frac{p_kr_k}{p_k-r_k}\frac{p-r}{pr}$. It is easy to verify that $s_k \in (1,\infty)$ and $\sum_{k=1}^{m} 1/s_k = 1$. Now write

$$\Big(\prod_{k=l+1}^{m} w_k^{p/p_k}\Big)^{-\frac{1}{p/r-1}} = \prod_{k=l+1}^{m} w_k^{-\frac{1}{\frac{p_k}{r_k}-1}\frac{1}{s_k}}.$$

It then follows from the Hölder inequality that for cube Q,

$$\int_Q \Big(\prod_{k=l+1}^{m} w_k^{p/p_k}(x)\Big)^{-\frac{1}{p/r-1}} dx \leqslant \prod_{k=l+1}^{m} \Big(\int_Q w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(x)\,dx\Big)^{\frac{1}{s_k}},$$

and so

$$\begin{aligned}&\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}^{-\frac{1}{p/r-1}}\,dx\Big)^{p/r-1}\\ &\leqslant \Big\{\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}(x)dx\Big)^{1/p}\prod_{j=1}^{l}\{\inf_Q w_j\}^{-\frac{1}{p_j}}\prod_{k=l+1}^{m}\Big(\frac{1}{|Q|}\int_Q w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(x)dx\Big)^{\frac{1}{r_k}-\frac{1}{p_k}}\Big\}^p\\ &\lesssim 1.\end{aligned}$$

We now prove that $w_k^{r/p_k} \in A_1(\mathbb{R}^n)$ for $k = 1, ..., l$. Again by the Hölder inequality, we have that

$$\begin{aligned}\int_Q w_k^{r/p_k}(x)\,dx &\leqslant \Big(\int_Q \Big(w_k^{r/p_k}\prod_{j=l+1}^{m} w_j^{r/p_j}\Big)^{p/r}\Big)^{r/p}\Big(\int_Q \Big(\prod_{j=l+1}^{m} w_j^{-r/p_j}\Big)^{(p/r)'}\Big)^{1-r/p}\\ &\leqslant \Big(\int_Q w_k^{p/p_k}\prod_{j=l+1}^{m} w_j^{p/p_j}\Big)^{r/p}\prod_{j=l+1}^{m}\Big(\int_Q w_j^{-\frac{1}{\frac{p_j}{r_j}-1}}\Big)^{(1-r/p)/s_j}\\ &\leqslant \Big(\int_Q \nu_{\vec{w}}(x)dx\Big)^{r/p}\prod_{i=1}^{l}(\inf_Q w_i)^{-r/p_i}\prod_{j=l+1}^{m}\Big(\int_Q w_j^{-\frac{1}{\frac{p_j}{r_j}-1}}\Big)^{(1-r/p)/s_j}\inf_Q w_k^{r/p_k}\\ &\lesssim \inf_{y\in Q} w_k^{r/p_k}(y).\end{aligned}$$

(iii)$\Rightarrow$(ii). Note that (iii) implies that

$$\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}^{-\frac{1}{p/r-1}}(x)\,dx\Big)^{p/r-1} \lesssim 1$$

and for each fixed k with $1 \leqslant k \leqslant m$,

$$\Big(\frac{1}{|Q|}\int_Q w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w_k^{\frac{1}{p_k(1/r-1/r_k)+1}}(x)dx\Big)^{\frac{p_k r_k}{r(p_k-r_k)}-1} \lesssim 1.$$

If we can prove that for any cube Q,

$$\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}^{-\frac{1}{p/r-1}}(x)dx\Big)^{1/r-1/p}\prod_{k=1}^{m}\Big(\frac{1}{|Q|}\int_Q w_k^{\frac{1}{p_k(1/r-1/r_k)+1}}(x)dx\Big)^{1/r-1/r_k+1/p_k} \geqslant 1, \tag{2.4}$$

$\vec{w} \in A_{\vec{p}/\vec{r}}(\mathbb{R}^n)$ then follows from (iii) directly. On the other hand, (2.4) is an easy consequence of the Hölder inequality. In fact, if we set

$$\alpha = \frac{1}{p(m-1)/r+1},\ \alpha_k = \frac{(m-1)/r+1/p}{1/r-1/r_k+1/p_k},$$

it is easy to verify that $\sum_{k=1}^{m}\frac{1}{\alpha_k}=1$, and then by the Hölder inequality,

$$\int_Q \nu_{\vec{w}}^{\alpha}(x)\,dx \leqslant \prod_{k=1}^{m}\Big(\int_Q w_k^{\frac{\alpha p\alpha_k}{p_k}}(x)\,dx\Big)^{1/\alpha_k} = \prod_{k=1}^{m}\Big(\int_Q w_k^{\frac{1}{p_k(1/r-1/r_k)+1}}(x)\,dx\Big)^{1/\alpha_k}. \quad (2.5)$$

Let $\tau=\alpha(p/r-1)+1$. We also have by the Hölder inequality that

$$\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}^{\alpha}(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}^{-\frac{1}{p/r-1}}(x)\,dx\Big)^{\alpha(p/r-1)} \geqslant \frac{1}{|Q|}\int_Q \nu_{\vec{w}}^{\alpha/\tau}(x)\nu_{\vec{w}}^{-\alpha/\tau}(x)\,dx \geqslant 1. \quad (2.6)$$

Combining the inequalities (2.5) and (2.6) then gives (2.4).

It remains to prove that $\mathscr{M}_{\vec{r}}$ is bounded from $L^{p_1}(\mathbb{R}^n,w_1)\times\ldots\times L^{p_m}(\mathbb{R}^n,w_m)$ to $L^{p}(\mathbb{R}^n,\nu_{\vec{w}})$ when $\min_{1\leqslant k\leqslant m}p_k/r_k\in(1,\infty]$ and $\vec{w}=(w_1,\ldots,w_m)\in A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$. Again we follows the idea used in the proof of [8, Theorem 3.7]. It suffices to prove that for some $q\in(0,1)$,

$$\mathscr{M}_{\vec{r}}(f_1,\ldots,f_m)(x)\lesssim \prod_{k=1}^{m}\Big\{M_{\nu_{\vec{w}}}^{c}\Big((|f_j|^{p_j}w_j/\nu_{\vec{w}})^{q}\Big)(x)\Big\}^{1/(qp_k)}, \quad (2.7)$$

where $M_{\nu_{\vec{w}}}^{c}$ is the centred maximal operator defined by

$$M_{\nu_{\vec{w}}}^{c}h(x)=\sup_{I:x\,\text{the center of}\,I}\frac{1}{\nu_{\vec{w}}(I)}\int_I |h(y)|\nu_{\vec{w}}(y)\,dy.$$

As we have proved, for each fixed k, $w_k^{-1/(p_k/r_k-1)}\in A_{\frac{p_k r_k}{r(p_k-r_k)}}(\mathbb{R}^n)$, and so there exists a positive constant $\sigma_k>1$ such that for any cube Q,

$$\Big(\frac{1}{|Q|}\int_Q w_k^{-\frac{\sigma_k}{\frac{p_k}{r_k}-1}}(x)\,dx\Big)^{1/\sigma_k}\lesssim \frac{1}{|Q|}\int_Q w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(x)\,dx. \quad (2.8)$$

Let

$$\sigma=\min_{1\leqslant k\leqslant m}\sigma_k,\; q=\frac{pr_k}{pr_k+r(p_k-r_k)(1-1/\sigma)}.$$

An application of the Hölder inequality gives us that

$$\Big(\int_Q |f_k(x)|^{r_k}\,dx\Big)^{1/r_k}\lesssim \Big(\int_Q |f_k|^{qp_k}w_k^{q}\nu_{\vec{w}}^{1-q}\Big)^{\frac{1}{qp_k}}\Big(\int_Q\Big(w_k^{q}\nu_{\vec{w}}^{1-q}\Big)^{-\frac{1}{\frac{p_kq}{r_k}-1}}\Big)^{\frac{1}{r_k}-\frac{1}{qp_k}}. \quad (2.9)$$

Let $\gamma_k=\frac{r(p_kq-r_k)}{r_k(p-r)(1-q)}$. Note that $\gamma_k>1$. Again by the Hölder inequality,

$$\int_Q\Big(w_k^{q}(x)\nu_{\vec{w}}^{1-q}(x)\Big)^{-\frac{1}{\frac{p_kq}{r_k}-1}}dx\leqslant\Big(\int_Q w_k^{-\frac{q\gamma_k'}{\frac{p_kq}{r_k}-1}}(x)\,dx\Big)^{\frac{1}{\gamma_k'}}\Big(\int_Q \nu_{\vec{w}}^{-\frac{1}{p/r-1}}(x)\,dx\Big)^{\frac{1}{\gamma_k}}. \quad (2.10)$$

On the other hand, we can verify that

$$\frac{q\gamma_k'}{p_kq-r_k}=\frac{qr}{r(p_kq-r_k)-r_k(p-r)(1-q)},\ \frac{q(p_k-r_k)}{(p_kq-r_k)-r_k(p/r-1)(1-q)}\leqslant\sigma_k.$$

Thus by the inequality (2.8),

$$\begin{aligned}\int_Q w_k^{-\frac{q\gamma_k'}{p_kq/r_k-1}}(x)\,dx&=\int_Q w_k^{-\frac{1}{p_k/r_k-1}\frac{q(p_k-r_k)}{(p_kq-r_k)-r_k(p/r-1)(1-q)}}(x)\,dx\\&\lesssim|Q|^{1-\frac{q\gamma_k'(p_k-r_k)}{p_kq-r_k}}\Big(\int_Q w_k^{-\frac{1}{p_k/r_k-1}}(x)\,dx\Big)^{\frac{q\gamma_k'(p_k-r_k)}{p_kq-r_k}}.\end{aligned}\tag{2.11}$$

The inequalities (2.9)–(2.11), along with the fact that $\vec{w}\in A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$ then leads to our desired conclusion (2.7). □

By Theorem 2.1, we deduce that

COROLLARY 2.1. *If $\vec{w}=(w_1,\dots,w_m)\in A_{\vec{p}/\vec{t}}(\mathbb{R}^{mn})$ for some $\vec{p}$ and $\vec{t}$ with $\vec{p}<\vec{t}$, then there exists a constant $\delta\in(0,1)$ such that $\vec{w}\in A_{\delta\vec{p}/\vec{t}}(\mathbb{R}^{mn})$.*

3. Proof of Theorem 1.1

We begin with a weighted estimate for the multilinear singular integral operators.

Let $m\geqslant 1$ be a positive integer, $K(x;y_1,\dots,y_m)$ be a locally integrable function defined away from the diagonal $x=y_1=y_2=\dots=y_m$ in $(\mathbb{R}^n)^{m+1}$. An operator T defined on m-fold products of $\mathscr{S}(\mathbb{R}^n)$ (Schwartz space) and taking values in the space of tempered distributions, is said to be an m-linear singular integral operator with kernel K if T is m-linear, and satisfies that

$$T(f_1,\dots,f_m)(x)=\int_{(\mathbb{R}^n)^m}K(x;y_1,\dots,y_m)f(y_1)\dots f(y_m)d\vec{y},\tag{3.1}$$

for bounded functions $f_1,\dots,f_m$ with compact supports, and a. e. $x\in\mathbb{R}^n\setminus\cap_{j=1}^m\operatorname{supp}f_j$, where and in the following, $d\vec{y}=dy_1\dots dy_m$. For the mapping properties of this operator, see [7], [5], [8] and [1].

THEOREM 3.1. *Let $m\geqslant 2$ be an integer, T be an m-linear singular integral operator with kernel K in the sense of (3.1). For $x,x',y_1,\dots,y_m\in\mathbb{R}^n$, set*

$$V(x,x';y_1,\dots,y_m)=|K(x;y_1,\dots,y_m)-K(x';y_1,\dots,y_m)|.$$

Suppose that for some fixed $r_1,\dots,r_m\in(1,\infty)$,

(i) *T is bounded from $L^{r_1}(\mathbb{R}^n)\times\dots\times L^{r_m}(\mathbb{R}^n)$ to $L^{r,\infty}(\mathbb{R}^n)$, with $1/r=\sum_{1\leqslant k\leqslant m}1/r_k$;*

(ii) *there exists a constant* $\rho > 0$*, such that for any ball* B *with radial* R*, and* $x, x' \in B$*, and nonnegative integers* $j_1, ..., j_m$ *with* $j^* = \max_{1 \leqslant k \leqslant m} j_k > 2$*,*

$$\Big(\int_{S_{j_1}(B)} \cdots \Big(\int_{S_{j_m}(B)} |V(x,x';y_1,...,y_m)|^{r_m'} dy_m\Big)^{\frac{r_{m-1}'}{r_m'}} \cdots\Big)^{\frac{r_2'}{r_1'}} dy_1\Big)^{\frac{1}{r_1'}} \lesssim \frac{R^\rho}{|2^{j^*}B|^{1/r_1+...+1/r_m+\rho/n}}.$$

Then for any $p_1, ..., p_m$ *such that* $p_k \in (r_k, \infty)$*,* $1 \leqslant k \leqslant m$*, and weights* $w_1, ..., w_m$ *such that* $\vec{w} \in A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$*,* T *is bounded from* $L^{p_1}(\mathbb{R}^n, w_1) \times ... \times L^{p_m}(\mathbb{R}^n, w_m)$ *to* $L^p(\mathbb{R}^n, \nu_{\vec{w}})$*.*

Proof. For $\delta \in (0, \min\{1, r\})$, let M_δ be the sharp maximal operator defined by

$$M_\delta^\sharp f(x) = \sup_{B \ni x} \inf_{c \in \mathbb{C}} \Big(\frac{1}{|B|}\int_B |f(y) - f_B|^\delta dy\Big)^{1/\delta},$$

where the sup is taken over all balls containing x, and f_B denotes the mean value of f on the ball B. We first claim that under the assumptions of Lemma 2.2,

$$M_\delta^\sharp(T(f_1, ..., f_m))(x) \lesssim \mathscr{M}_{\vec{r}}(f_1, ..., f_m)(x). \tag{3.2}$$

To prove this, we will employ the ideas used in [1]. For any $x \in \mathbb{R}^n$, any ball B containing x and suitable functions $f_1, ..., f_m$, we decomposition $f_1, ..., f_m$ as

$$f_k^1(y) = f_k(y)\chi_{4B}(y),\ f_k^2(y) = f_k(y)\chi_{\mathbb{R}^n \setminus 4B}(y).$$

A trivial argument involving the fact that T is bounded from $L^{r_1}(\mathbb{R}^n) \times ... \times L^{r_m}(\mathbb{R}^n)$ to $L^{r,\infty}(\mathbb{R}^n)$, and the argument used in the proof of Kolmogrov inequality, tells us that

$$\Big(\frac{1}{|B|}\int_B |T(f_1^1, ..., f_m^1)(y)|^\delta dy\Big)^{1/\delta} \lesssim \prod_{k=1}^m \Big(\frac{1}{|B|}\int_{4B} |f_k(y)|^{r_k} dy\Big)^{1/r_k} \lesssim \mathscr{M}_{\vec{r}}(f_1, ..., f_m)(x).$$

Now let $\{i_1, ..., i_m\} \subset \{1, 2\}$ such that $i_k = 2$ for some k with $1 \leqslant k \leqslant m$. Without loss of generality, we may assume that $i_k = 1$ when $1 \leqslant k \leqslant \ell$ and $i_k = 2$ when $\ell + 1 \leqslant k \leqslant m$. Then by assumption (ii), for any $y, y' \in B$,

$$|T(f_1^1, ..., f_l^1, f_{l+1}^2, ..., f_m^2)(y) - T(f_1^1, ..., f_l^1, f_{l+1}^2, ..., f_m^2)(y')|$$
$$\lesssim \sum_{j_{\ell+1}=1}^\infty \cdots \sum_{j_m=1}^\infty \prod_{k=1}^\ell \Big(\int_{4B} |f_k(y_k)|_k^r dy_k\Big)^{1/r_k} \prod_{l=\ell+1}^m \Big(\int_{S_{j_l}(4B)} |f_l(y_l)|^{r_l} dy_l\Big)^{1/r_l}$$
$$\Big(\int_{4B} \cdots \Big(\int_{4B} \Big(\int_{S_{j_{\ell+1}}(4B)} \cdots \int_{S_{j_m}(4B)} |V(y, y'; y_1, ..., y_m)|^{r_m'} dy_m\Big)^{\frac{r_{m-1}'}{r_m'}} \cdots dy_1\Big)^{\frac{1}{r_1'}}$$

$$\lesssim \sum_{j_{\ell+1}=1}^{\infty} \cdots \sum_{j_m=1}^{\infty} \prod_{k=1}^{\ell} \Big(\frac{1}{|4B|} \int_{4B} |f_k(y_k)|^{r_k} dy_k \Big)^{1/r_k}$$

$$\times \prod_{l=\ell+1}^{m} \Big(\frac{1}{|S_{j_l}(4B)|} \int_{S_{j_l}(4B)} |f_l(y_l)|^{r_l} dy_l \Big)^{1/r_l} \frac{|B|^{\rho/n}}{|S_{j^*}(4B)|^{1/r_1+\ldots+1/r_\ell+\rho/n}}$$

$$\lesssim \mathscr{M}_{\vec{r}}(f_1,\ldots,f_m)(x).$$

We can now conclude the proof of Theorem 3.1. By Theorem 2.1, we know that $\nu_w \in A_{p/r}(\mathbb{R}^n)$. If we choose δ small enough, we then know that $\nu_{\vec{w}} \in A_{p/\delta}(\mathbb{R}^n)$. Our desired conclusion then follows from (3.2), Lemma 2.1 and the relationship of sharp maximal operator and the Hardy-Littlewood maximal operator; see also [1]. □

To prove Theorem 1.1, we will also use some preliminary lemmas. For $\sigma \in L^\infty(\mathbb{R}^{mn})$, let σ_l be the same as in (1.3). Define

$$\|\sigma_\kappa\|_{W^{s_1,\ldots,s_m}(\mathbb{R}^{nm})} = \Big(\int_{\mathbb{R}^{nm}} \langle \xi_1 \rangle^{2s_1} \ldots \langle \xi_m \rangle^{2s_m} |\widehat{\sigma}_\kappa(\xi_1,\ldots,\xi_m)|^2 d\vec{\xi} \Big)^{1/2},$$

where $\langle \xi_k \rangle = (1+|\xi_k|^2)^{1/2}$.

LEMMA 3.1. *Let $q_1,\ldots,q_m \in [2,\infty)$, and $s_1,\ldots,s_m \geqslant 0$. Then*

$$\Big(\int_{\mathbb{R}^n} \cdots \Big(\int_{\mathbb{R}^n} |\widehat{\sigma}_\kappa(\xi_1,\ldots \xi_m)|^{q_1} \langle \xi_1 \rangle^{s_1} d\xi_1 \Big)^{q_2/q_1} \langle \xi_2 \rangle^{s_2} d\xi_2 \Big)^{q_3/q_2} \ldots \langle \xi_m \rangle^{s_m} d\xi_m \Big)^{1/q_m}$$
$$\lesssim \|\sigma_\kappa\|_{W^{s_1/q_1,\ldots,s_m/q_m}(\mathbb{R}^{mn})}.$$

For the proof of Lemma 3.1, see Appendix A in [4].

LEMMA 3.2. *Let $s_1,\ldots,s_m \in \mathbb{R}$, and $\alpha_1 \ldots, \alpha_m \in \mathbb{Z}_+^n$ be multi-indices. Set*

$$\zeta_\kappa^{\alpha_1,\ldots,\alpha_m}(\xi_1,\ldots,\xi_m) := \xi_1^{\alpha_1} \ldots \xi_m^{\alpha_m} \sigma_\kappa(\xi_1,\ldots,\xi_m).$$

Then

$$\|\zeta_\kappa^{\alpha_1,\ldots,\alpha_m}\|_{W^{s_1,\ldots,s_m}(\mathbb{R}^{mn})} \lesssim \sup_{l\in\mathbb{Z}} \|\sigma_l\|_{W^{s_1,\ldots,s_m}(\mathbb{R}^{mn})}.$$

This lemma was given in [9, Remark 2.5].

Let σ, $\Phi \in \mathscr{S}(\mathbb{R}^{nm})$ be the same as in Section 1. For $l \in \mathbb{Z}$, set

$$\widetilde{\sigma}_l(\xi_1,\ldots,\xi_m) = \sigma(\xi_1,\ldots,\xi_m)\Phi(2^{-l}\xi_1,\ldots,2^{-l}\xi_m),$$

For a positive integer N, let

$$\sigma^N(\xi_1,\ldots,\xi_m) = \sum_{|l|\leqslant N} \widetilde{\sigma}_l(\xi_1,\ldots,\xi_m)$$

and

$$K^N(x;y_1,\ldots,y_m) = \mathscr{F}^{-1}\sigma^N(x-y_1,\ldots,x-y_m). \tag{3.3}$$

For an integer k with $1 \leqslant k \leqslant m$ and $x, y_1, y_2, x' \in \mathbb{R}^n$, let

$$W^N(x,x';y_1,\ldots,y_m) = K^N(x;y_1,\ldots,y_m) - K^N(x';y_1,\ldots,y_m).$$

LEMMA 3.3. *Let m and k be positive integers with $1 \leqslant k \leqslant m$, σ be a multiplier which satisfies (1.4). Let $r_1, \ldots, r_m \in (1,2]$ such that $s \in (n/r_1+\ldots+n/r_m, n/r_1+\ldots+n/r_m+1)$. Then for any ball B with radial R, $x, x' \in \frac{1}{4}B$, integers $j_1, \ldots, j_m$ with $j^* = \max_{1\leqslant k\leqslant m} j_k \geqslant 2$,*

$$\Big(\int_{S_{j_1}(B)} \cdots \Big(\int_{S_{j_m}(B)} |W^N(x,x';y_1,\ldots,y_m)|^{r_m'} dy_m\Big)^{\frac{r_{m-1}'}{r_m'}} \ldots dy_1\Big)^{\frac{1}{r_1'}} \lesssim \frac{R^{s-n/r_1-\ldots-n/r_m}}{|2^{j^*}B|^{s/n}}. \tag{3.4}$$

Proof. For $l \in \mathbb{Z}$, set

$$W_l(x,x';y_1,\ldots,y_m) = \mathscr{F}^{-1}\widetilde{\sigma}_l(x-y_1,\ldots,x-y_m) - \mathscr{F}^{-1}\widetilde{\sigma}_l(x'-y_1,\ldots,x'-y_m),$$

and

$$J_{l;j_1\ldots,j_m} = \Big(\int_{S_{j_1}(B)} \cdots \Big(\int_{S_{j_2}(B)} |W_l(x,x';y_1,\ldots,y_m)|^{r_m'} dy_m\Big)^{\frac{r_{m-1}'}{r_m'}} \ldots dy_1\Big)^{\frac{1}{r_1'}}$$

where $\mathscr{F}^{-1}\widetilde{\sigma}_l$ denote the inverse Fourier transform of $\widetilde{\sigma}_l$. Let $j_1, \ldots, j_m$ be nonnegative integers such that $\max\{j_1,\ldots,j_m\} \geqslant 2$. Without loss of generality, we may assume that $j_1 = \max_{1\leqslant i\leqslant m} j_i$. We have by Lemma 3.1 that

$$\begin{aligned}
&\Big(\int_{S_{j_1}(B)} \cdots \Big(\int_{S_{j_m}(B)} |\mathscr{F}^{-1}\widetilde{\sigma}_l(x-y_1,\ldots,x-y_m)|^{r_m'} dy_m\Big)^{\frac{r_{m-1}'}{r_m'}} \ldots dy_1\Big)^{\frac{1}{r_1'}} \\
&\lesssim \Big(\int_{C_{j_1}} \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \cdots \Big(\int_{\mathbb{R}^n} |\mathscr{F}^{-1}\widetilde{\sigma}_l(z_1,\ldots,z_m)|^{r_m'} dz_m\Big)^{\frac{r_{m-1}'}{r_m'}} \ldots\Big)^{\frac{r_1'}{r_2'}} dz_1\Big)^{\frac{1}{r_1'}} \\
&\lesssim \Big(\int_{C_{j_1}} \Big(\cdots \Big(\int_{\mathbb{R}^n} |\mathscr{F}^{-1}\widetilde{\sigma}_l(z_1,\ldots,z_m)|^{r_m'} dz_m\Big)^{\frac{r_{m-1}'}{r_m'}} \ldots\Big)^{\frac{r_1'}{r_2'}} |z_1|^{r_1 s} dz_1\Big)^{\frac{1}{r_1'}} (2^{j_1}R)^{-s} \\
&\lesssim (2^{j_1}R)^{-s} 2^{-l(s-n/r_1-\ldots-n/r_m)},
\end{aligned}$$

where $C_{j_1} = \{x : 2^{j_1-2}R \leqslant |x| \leqslant 2^{j_1+2}R\}$. Similarly, we have that

$$\begin{aligned}
&\Big(\int_{S_{j_1}(B)} \cdots \Big(\int_{S_{j_m}(B)} |\mathscr{F}^{-1}\widetilde{\sigma}_l(x'-y_1,\ldots,x'-y_m)|^{r_m'} dy_m\Big)^{\frac{r_{m-1}'}{r_m'}} \ldots dy_1\Big)^{\frac{1}{r_1'}} \\
&\lesssim (2^{j_1}R)^{-s} 2^{-l(s-n/r_1-\ldots-n/r_m)}.
\end{aligned}$$

Therefore,

$$J_{l,j_1,\ldots,j_m} \lesssim (2^{j_1}R)^{-s} 2^{-l(s-n/r_1-\ldots-n/r_m)}. \tag{3.5}$$

Now write

$$\begin{aligned}
&\mathscr{F}^{-1}\widetilde{\sigma}_l(x-y_1,\ldots,x-y_m) - \mathscr{F}^{-1}\widetilde{\sigma}_l(x'-y_1,\ldots,x'-y_m) \\
&= \mathscr{F}^{-1}\widetilde{\sigma}_l(x-y_1,\ldots,x-y_m) - \mathscr{F}^{-1}\widetilde{\sigma}_l(x'-y_1,x-y_2,\ldots,x-y_m)
\end{aligned}$$

$$
\begin{aligned}
&+\mathscr{F}^{-1}\widetilde{\sigma}_l(x'-y_1,x-y_2,\dots,x-y_m)-\mathscr{F}^{-1}\widetilde{\sigma}_l(x'-y_1,x'-y_2,x-y_3,\dots,x-y_m)\\
&+\dots\\
&+\mathscr{F}^{-1}\widetilde{\sigma}_l(x'-y_1,\dots,x'-y_{m-1},x-y_m)-\mathscr{F}^{-1}\widetilde{\sigma}_l(x'-y_1,\dots,x'-y_m)\\
=&\sum_{\tau=1}^{m}L_\tau(x,y_1,\dots,y_2,x').
\end{aligned}
$$

A trivial computation leads to that

$$
\begin{aligned}
&|\mathscr{F}^{-1}\widetilde{\sigma}_l(z_1,\dots,z_m)-\mathscr{F}^{-1}\widetilde{\sigma}_l(z_1,\dots,z_{m_1},z_m+h)|\\
&\quad=2^{mnl}\left|\mathscr{F}^{-1}\sigma_l(2^lz_1,\dots,2^lz_m)-\mathscr{F}^{-1}\sigma_\kappa(2^lz_1,\dots,2^lz_{m-1},2^lz_m+2^lh)\right|\\
&\quad\leqslant 2^{mnl}2^lh\sum_{|\alpha|=1}\int_0^1|\partial^{0,\dots,0,\alpha}\mathscr{F}^{-1}\sigma_l(2^lz_1,\dots,2^lz_{m-1},2^lz_m+2^l\theta h)|d\theta
\end{aligned}
$$

Take $h=x-x'$. An application of Lemma 3.1, Lemma 3.2 then gives us that

$$
\begin{aligned}
&\Big(\int_{S_{j_1}(B)}\dots\Big(\int_{S_{jm}(B)}|L_m(x,y_1,\dots,y_m;x')|^{r'_m}dy_m\Big)^{\frac{r'_{m-1}}{r'_m}}\dots dy_1\Big)^{\frac{1}{r'_1}}\\
=&\Big(\int_{C_{j_1}}\Big(\int_{\mathbb{R}^n}\cdots\Big(\int_{\mathbb{R}^n}|\mathscr{F}^{-1}\widetilde{\sigma}_l(z_1,\dots,z_m)-\mathscr{F}^{-1}\widetilde{\sigma}_l(z_1,\dots,z_m+h)|^{r'_m}dz_m\Big)^{\frac{r'_{m-1}}{r'_m}}\dots\Big)dz_1\Big)^{\frac{1}{r'_1}}\\
\lesssim&\sum_{|\alpha|=1}\int_0^1\Big(\int_{C_{j_1}}\Big(\int_{\mathbb{R}^n}\cdots\Big(\int_{\mathbb{R}^n}\left|\partial^{0,\dots,0,\alpha}\mathscr{F}^{-1}\sigma_l(2^lz_1,\dots,2^lz_{m-1},2^lz_m+2^l\theta h)\right|^{r'_m}dz_m\Big)^{\frac{r'_{m-1}}{r'_m}}\\
&\dots dz_1\Big)^{1/r'_1}d\theta 2^{mnl}2^lR\\
\lesssim&\sum_{|\alpha|=1}\Big(\int_{C_{j_1}}\Big(\int_{\mathbb{R}^n}\cdots\Big(\int_{\mathbb{R}^n}\left|\partial^{0,\dots,0,\alpha}\mathscr{F}^{-1}\sigma_l(2^lz_1,\dots,2^lz_m)\right|^{r'_m}dz_m\Big)^{\frac{r'_{m-1}}{r'_2}}\dots dz_1\Big)^{1/r'_1}2^{mnl}2^lR\\
\lesssim&\sum_{|\alpha|=1}\Big(\int_{C_{j_1}}\Big(\int_{\mathbb{R}^n}\cdots\Big(\int_{\mathbb{R}^n}\left|\mathscr{F}^{-1}(\xi_m^\alpha\sigma_l)(2^lz_1,\dots,2^lz_m)\right|^{r'_m}dz_m\Big)^{\frac{r'_{m-1}}{r'_m}}\dots\Big)\\
&\times\langle z_1\rangle^{sr'_1}dz_1\Big)^{\frac{1}{r'_1}}2^lR(2^{j_1}R)^{-s}2^{-l(s-n/r_1-\dots-n/r_m)}\\
\lesssim&\, 2^lR(2^{j_1}R)^{-s}2^{-l(s-n/r_1-\dots-n/r_m)}.
\end{aligned}
$$

Similarly, we have the desired estimates for L_k with $1\leqslant k\leqslant m-1$. Thus,

$$
J_{l,j_1,\dots,j_m}\lesssim 2^lR(2^{j_1}R)^{-s}2^{-l(s-n/r_1-\dots-n/r_m)}. \tag{3.6}
$$

Combining the estimates (3.5) and (3.6) then yields

$$\Big(\int_{S_{j_1}(B)}\cdots\Big(\int_{S_{j_m}(B)}|W^N(x,x';y_1,\dots,y_m)|^{r_m'}dy_m\Big)^{\frac{r_{m-1}'}{r_m'}}\dots dy_1\Big)^{\frac{1}{r_1'}}$$

$$\lesssim \sum_{l:2^lR<1} J_{l,j_1,\dots,j_m}+\sum_{l:2^lR\geqslant 1} J_{l,j_1,\dots,j_m}\lesssim \frac{R^{s-n/r_1-\dots-n/r_m}}{|2^{j_1}B|^{s/n}},$$

and then completes the proof of Lemma 3.3. □

Proof of Theorem 1.1. Let $T_{\sigma,N}$ be the bilinear singular integral operator with associated kernel K^N defined by (3.3). Note that for $f_1,\dots,f_m\in\mathscr{S}(\mathbb{R}^n)$,

$$\|T_\sigma(f_1,\dots,f_m)-T_{\sigma,N}(f_1,\dots,f_m)\|_{L^\infty(\mathbb{R}^n)}\lesssim\|(\sigma-\sigma^N)\widehat{f_1}\dots\widehat{f_m}\|_{L^1(\mathbb{R}^n)}\to 0,\ N\to\infty.$$

Recall that $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^q(\mathbb{R}^n)$ for any $1\leqslant q<\infty$. By a standard density argument, it suffices to prove that the conclusion of Theorem 1.1 is true for $T_{\sigma,N}$, with bounded independent of N.

To prove Theorem 1.1, we first claim that if σ satisfies (1.4) for $s\in(mn/2,mn]$, $r_1,\dots,r_m\in(1,2]$ such that $1/r_1+\dots+1/r_m<s/n$, then for $p_1\in(r_1,\infty)$, ..., $p_m\in(r_m,\infty)$ and $\vec{w}=(w_1,\dots,w_m)\in A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$,

$$\|T(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})}\lesssim\prod_{k=1}^m\|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)}. \tag{3.7}$$

In fact, if σ satisfies (1.4) for some $s\in(mn/2,mn]$, $r_1,\dots,r_m\in(1,2]$ such that $1/r_1+\dots+1/r_m<s/n$, we can take $s_1,\dots,s_m>n/2$ such that $s_1+\dots+s_m\leqslant s$, $r_1>n/s_1,\dots,r_m>n/s_m$. This together with Theorem 6.1 in [4] states that $T_{\sigma,N}$ is bounded from $L^{p_1}(\mathbb{R}^n)\times\dots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with bounded independent of N. Thus by Theorem 3.1 and Lemma 3.3, we know that (3.7) holds.

We now prove Theorem 1.1 for the case that $t_1,\dots,t_m\in(1,2)$ such that $1/t_1+\dots+1/t_m=s/n$. For $p_1\in(t_1,\infty)$, ..., $p_m\in(t_m,\infty)$ and $\vec{w}\in A_{\vec{p}/\vec{t}}(\mathbb{R}^{mn})$, by Corollary 2.1, we can choose $\delta\in(0,1)$ which is close to 1, such that $t_1/\delta,\dots,t_m/\delta\in(1,2)$, $p_k>t_k/\delta$ for $k=1,\dots,m$, and $\vec{w}\in A_{\delta\vec{p}/\vec{t}}(\mathbb{R}^{mn})$. Note that $\delta/t_1+\dots+\delta/t_m<s/n$. Our desired conclusion then follows from (3.7).

We turn our attention to the case $t_1=1$ and $t_2,\dots,t_m>1$, $1/t_1+\dots+1/t_m=s/n$. For $p_1\in(t_1,\infty)$, $p_k\in(t_2,\infty)$ $(2\leqslant k\leqslant m)$ and $\vec{w}\in A_{\vec{p}/\vec{t}}(\mathbb{R}^{mn})$, we choose $\widetilde{t_1}\in(1,2]$ such that $p_1>\widetilde{t_1}>1$. Thus, $1/\widetilde{t_1}+1/t_2+\dots+1/t_m<s/n$, $\vec{w}\in A_{\vec{p}/\vec{\widetilde{t}}}(\mathbb{R}^{mn})$ with $\widetilde{t}=(\widetilde{t_1},t_2,\dots,t_m)$. We know from our claim (3.7) that (1.8) hold.

The case that $\min_{1\leqslant k\leqslant m}p_k=1$ can be considered in a way similar to the case that $t_1=1$ and $t_2,\dots,t_m>1$. This completes the proof of Theorem 1.1. □

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REFERENCES

[1] B. ANH AND X. T. DUONG, *Weighted norm inequalities for multilinear operators and applications to multilinear Fourier multipliers*, arXiv:1112.0823.

[2] R. R. COIFMAN AND Y. MEYER, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. 212 (1975), 315–331.

[3] R. R. COIFMAN AND Y. MEYER, *Nonlinear harmonic analysis, operator theory and P. D. E.*, in "Beijing Lectures in Harmonic Analysis" (Beijing, 1984), Ann. Math. Stud. 112, Princeton Univ. Press, Princeton, NJ, 1986, 3–45

[4] M. FUJITA AND N. TOMITA, *Weighted norm inequalities for multilinear Fourier multipliers*, Trans Amer Math Soc. 364 (2012), 6335–6353.

[5] L. GRAFAKOS, AND N. KALTON, *Multilinear Calderón-Zygmund operators on Hardy spaces*, Collect. Math. 52 (2001), 169–179.

[6] L. GRAFAKOS AND Z. SI, *The Hörmander multiplier theorem for multilinear operators*, J. Reine Angewandte Math. 668 (2012), 133–147.

[7] L. GRAFAKOS AND R. TORRES, *Multilinear Calderón-Zygmund theory*, Adv. Math., 165 (2002), 124–164.

[8] A. LERNER, S. OMBROSSI, C. PEREZ, R. H. TORRES AND R. TROJILLO-GONZALEZ, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theorey*, Adv. Math. 220 (2009), 1222–1264.

[9] A. MIYACHI AND N. TOMITA, *Minimal smoothness conditions for bilinear Fourier multiplier*, Rev Mat Iberomericana, to appear.

[10] N. TOMITA, *A Hörmander type multiplier theorem for multilinear operator*, J. Func. Anal. 259 (2010), 2028–2044.

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