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WEIGHTED COMPOSITION OPERATORS FROM THE LIPSCHITZ SPACE INTO THE ZYGMUND SPACE

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Abstract. In this work, we give several characterizations of the bounded and the compact weighted composition operators from the Lipschitz space into the Zygmund space.

1. Introduction

Let X and Y be Banach spaces of analytic functions on a domain Ω in $\mathbb{C}$, u an analytic function on Ω and let φ be an analytic function mapping Ω into itself. The *weighted composition operator with symbols* u *and* φ from X to Y is the operator uC_φ with range in Y defined by

$$uC_\varphi f = M_u C_\varphi f = u(f\circ\varphi),\ \text{for } f\in X,$$

where M_u is the multiplication operator with symbol u and C_φ is the composition operator with symbol φ.

We refer the interested reader to [7] and [24] for the theory of the composition operators.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $S(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$, and let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. Let $H^\infty = H^\infty(\mathbb{D})$ denote the space of bounded analytic functions f on $\mathbb{D}$ with norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$.

A well-known class of small spaces of analytic functions that will be considered in this work is the Lipschitz space Lip_α (with $0<\alpha<1$) of functions $f\in H(\mathbb{D})$ satisfying the Lipschitz condition of order α: there exists a constant $C>0$ such that

$$|f(z)-f(w)|\leqslant C|z-w|^\alpha,\ \text{ for all } z,w\in\mathbb{D}.$$

Such functions f extend continuously to the closure of the disk. Furthermore, the associated boundary function $f(e^{it})$, for $t\in\mathbb{R}$, satisfies the analogous condition

$$|f(e^{it})-f(e^{is})|\leqslant Ch^\alpha,\ \text{ for } |t-s|\leqslant h.$$

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The quantity

$$\|f\|_{Lip_\alpha} = |f(0)| + \sup\left\{\frac{|f(z)-f(w)|}{|z-w|^\alpha} : z,w \in \mathbb{D}, z \neq w\right\} < \infty$$

defines a norm on Lip_α.

Let $f \in Lip_\alpha$ and set $C = \sup\left\{\frac{|f(z)-f(w)|}{|z-w|^\alpha} : z,w \in \mathbb{D}, z \neq w\right\}$. Then, for $z \in \mathbb{D}$, we have

$$|f(z)| \leqslant |f(0)| + C|z|^\alpha \leqslant \|f\|_{Lip_\alpha}.$$

Thus, taking the supremum over all $z \in \mathbb{D}$, we obtain

$$\|f\|_\infty \leqslant \|f\|_{Lip_\alpha}. \tag{1}$$

By a theorem of Hardy and Littlewood [14], the elements of Lip_α are characterized by the following Bloch-type condition: A function $f \in H(\mathbb{D})$ belongs to Lip_α if and only if

$$\alpha(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^{1-\alpha}|f'(z)| < \infty. \tag{2}$$

Moreover, $\|f\|_{Lip_\alpha} \asymp |f(0)| + \alpha(f)$.

The Bloch space $\mathscr{B}$ is the Banach space consisting of the analytic functions f on $\mathbb{D}$ satisfying the condition

$$\beta_f = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The norm is defined as $\|f\|_{\mathscr{B}} = |f(0)| + \beta_f$.

In this work, we shall also consider the Zygmund space $\mathscr{Z}$ consisting of the functions $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\| = \sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $h > 0$. As a consequence of Theorem 5.3 of [10] and the Closed Graph Theorem, a function $f \in H(\mathbb{D})$ belongs to $\mathscr{Z}$ if and only if $f' \in \mathscr{B}$. Furthermore,

$$\|f\| \asymp \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|.$$

The quantity

$$\|f\|_{\mathscr{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|,$$

yields a Banach space structure on $\mathscr{Z}$.

The spaces Lip_α and the Zygmund space play an important role in connection to the theory of the H^p spaces when $0 < p < 1$. Indeed, in [11] it was shown that for appropriate choices of α, the spaces Lip_α and $\mathscr{Z}$, as well as the generalizations of such spaces where the function is replaced by its nth derivative, can be viewed as duals of certain H^p spaces. For more information on these and other facts regarding these spaces, we refer the interested reader to [10] and [11].

In recent years, considerable interest has emerged in the study of the weighted composition operators. A well-know theorem due to Banach states that for a compact metric space K, the onto linear isometries of $C(K)$ are of the form

$$Tf = u(f \circ \varphi),$$

where $|u(x)| = 1$ for all $x \in K$, and $\varphi : K \to K$ is a homeomorphism. Motivated by this theorem, active research on the description of the isometries of Banach spaces of analytic functions has confirmed that the weighted composition operators characterize the onto isometries on many Banach spaces of analytic functions, including the Hardy space H^p (for $1 \leqslant p \leqslant \infty$, $p \neq 2$) [9, 13], the weighted Bergman space [15], and the disk algebra [12].

Composition operators, weighted composition operators, and related operators between the Zygmund space and some other spaces of analytic functions have been studied in [2, 6, 16, 17, 18, 19, 25, 26, 27]. Composition operators, multiplication operators, and weighted composition operators on spaces of Lipschitz functions in various settings have been studied in [1, 4, 8, 20, 21, 22, 23].

An interesting question in operator theory is whether the boundedness (respectively, compactness) of a linear operator $T : X \to Y$ (with X and Y Banach spaces) can be characterized by means of the boundedness (respectively, convergence to 0, under the boundedness assumption on the operator) of some countable collection of functions in the range of T.

The existence of such a sequence to characterize compactness has been shown in the case of the composition operator on the Bloch space and BMOA [28], but no known examples exist for the weighted composition operators on spaces of analytic functions, such as the analytic Besov spaces B_p with $1 < p < \infty$, BMOA or the Bloch space itself.

On the other hand, in [3] and [5], it was shown that in the case of the weighted composition operator uC_φ mapping H^∞ or the minimal Möbius invariant space B_1 into $\mathscr{B}$, the sequence $\{\|u\varphi^k\|_{\mathscr{B}}\}$ can be used to characterize boundedness and compactness. In [6], we showed that the bounded and the compact weighted composition operator uC_φ mapping H^∞ into the Zygmund space can likewise be characterized in terms of the boundedness (respectively, convergence to 0) of the sequence $\{\|u\varphi^k\|_{\mathscr{Z}}\}$.

Motivated by these observations, in this work we seek analogous results for the operator uC_φ mapping the Lipschitz space to the Zygmund space. We show that, in fact, scaling the sequence $\{u\varphi^k\}$ by a factor $k^{-\alpha}$, similar characterizations of boundedness and compactness hold in this setting. Specifically, we show that $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ is bounded (respectively, compact) if and only if the sequence $\{\|k^{-\alpha} u\varphi^k\|_{\mathscr{Z}}\}$ is bounded (respectively, uC_φ is bounded and $\{\|k^{-\alpha} u\varphi^k\|_{\mathscr{Z}}\}$ converges to 0 as $k \to \infty$.) To prove these results, we shall make use of the following family of Lipschitz functions of order α:

$$\left\{ z \mapsto \frac{(1-|a|^2)^j}{(1-\overline{a}z)^{j-\alpha}} : a \in \mathbb{D},\ j = 1,2,3 \right\}.$$

Throughout this paper we shall adopt the convention of denoting by C a positive constant whose value may change at each occurrence.

2. Boundedness of $uC_\varphi : Lip_\alpha \to \mathscr{Z}$

In this section we characterize the bounded weighted composition operators from Lip_α to the Zygmund space. For a fixed $a \in \mathbb{D}$ and for $z \in \mathbb{D}$, set

$$f_{a,j}(z) = \frac{(1-|a|^2)^j}{(1-\overline{a}z)^{j-\alpha}}, \quad j = 1,2,3.$$

A calculation shows that these functions belong to the space Lip_α.

In order for the operator $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ to be bounded, it is clear that $u = uC_\varphi 1$ must belongs to $\mathscr{Z}$.

THEOREM 1. *Let $0 < \alpha < 1$, $u \in \mathscr{Z}$ and $\varphi \in S(\mathbb{D})$. Then the following conditions are equivalent:*

(a) *The operator $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ is bounded.*

(b) $\sup_{k \geqslant 1} \|k^{-\alpha} u\varphi^k\|_{\mathscr{Z}} < \infty$.

(c) *The following quantities are finite:*

$$N_1 := \sup_{z\in\mathbb{D}} (1-|z|^2)|u(z)\varphi'(z)^2|,$$
$$N_2 := \sup_{z\in\mathbb{D}} (1-|z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|,$$
$$A := \max_{j=1,2,3} \sup_{w\in\mathbb{D}} \|uC_\varphi f_{\varphi(w),j}\|_{\mathscr{Z}}.$$

(d) *The following quantities are finite:*

$$M_1 := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}},$$
$$M_2 := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2-\alpha}}.$$

Proof. (a) $\Rightarrow$ (b). For an integer $k \geqslant 1$ and $z \in \mathbb{D}$, let $p_k(z) = k^{-\alpha} z^k$. Since the sequence $\{p_k\}$ is bounded in Lip_α, if $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ is bounded, then for each integer $k \geqslant 1$, we have

$$\|k^{-\alpha} u\varphi^k\|_{\mathscr{Z}} = \|uC_\varphi p_k\|_{\mathscr{Z}} \leqslant C\|uC_\varphi\|.$$

Therefore, the supremum of $\|k^{-\alpha} u\varphi^k\|_{\mathscr{Z}}$ over all integers $k \geqslant 1$ is finite.

(b) $\Rightarrow$ (c). Suppose

$$M = \max\left\{ \|u\|_{\mathscr{Z}},\ \sup_{k\geqslant 1} \|k^{-\alpha} u\varphi^k\|_{\mathscr{Z}} \right\} < \infty.$$

By the formula $(u\varphi)'' = u''\varphi + 2u'\varphi' + u\varphi''$, for $z \in \mathbb{D}$, we obtain

$$|2u'(z)\varphi'(z) + u(z)\varphi''(z)| \leqslant |(u\varphi)''(z)| + |u''(z)\varphi(z)|,$$

which, together with the fact that $\|u\varphi\|_{\mathscr{Z}} = \|uC_\varphi p_1\|_{\mathscr{Z}} \leqslant M$, implies that

$$\begin{aligned} N_2 &\leqslant \sup_{z\in\mathbb{D}}(1-|z|^2)|(u\varphi)''(z)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|u''(z)| \\ &\leqslant \|u\varphi\|_{\mathscr{Z}} + \|u\|_{\mathscr{Z}} \leqslant 2M. \end{aligned} \tag{3}$$

On the other hand, from the formula

$$(u\varphi^2)'' = u''\varphi^2 + 4u'\varphi\varphi' + 2u\varphi'^2 + 2u\varphi\varphi'' = u''\varphi^2 + 2\varphi(2u'\varphi' + u\varphi'') + 2u\varphi'^2,$$

for $z \in \mathbb{D}$, we have

$$2|u(z)\varphi'(z)^2| \leqslant |(u\varphi^2)''(z)| + |u''(z)\varphi(z)^2| + 2|\varphi(z)||2u'(z)\varphi'(z) + u(z)\varphi''(z)|.$$

Therefore, since $|\varphi(z)| \leqslant 1$ and $u\varphi^2 \in \mathscr{Z}$, multiplying by $(1-|z|^2)$, taking the supremum over all $z \in \mathbb{D}$, and using (3), we obtain

$$\begin{aligned} 2N_1 &\leqslant \sup_{z\in\mathbb{D}}(1-|z|^2)|(u\varphi^2)''(z)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|u''(z)| \\ &\quad + 2\sup_{z\in\mathbb{D}}(1-|z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z)| \\ &\leqslant 2^\alpha\|2^{-\alpha}u\varphi^2\|_{\mathscr{Z}} + M + 4M \leqslant 7M. \end{aligned}$$

Next, by Stirling's formula, for $a \in \mathbb{D}$ and for $j = 1,2,3$, the power series representation of $f_{a,j}$ is given by

$$\begin{aligned} f_{a,j}(z) &= (1-|a|^2)^j \sum_{k=0}^{\infty} \frac{\Gamma(k+j-\alpha)}{k!\Gamma(j-\alpha)} \overline{a}^k z^k \\ &\asymp (1-|a|^2)^j \sum_{k=0}^{\infty} k^{j-\alpha-1} \overline{a}^k z^k. \end{aligned}$$

Hence for $w \in \mathbb{D}$, passing to the norm and using the assumption, we obtain

$$\begin{aligned} \|uC_\varphi f_{\varphi(w),j}\|_{\mathscr{Z}} &\leqslant C(1-|\varphi(w)|^2)^j \sum_{k=0}^{\infty} |\varphi(w)|^k k^{j-1} \|k^{-\alpha}u\varphi^k\|_{\mathscr{Z}} \\ &\leqslant CM, \quad \text{for } j = 1,2,3. \end{aligned}$$

(c) $\Rightarrow$ (d). Assume N_1, N_2 and A are finite. A direct calculation shows that

$$f_{a,j}(a) = (1-|a|^2)^\alpha, \qquad f'_{a,j}(a) = \frac{(j-\alpha)\overline{a}}{(1-|a|^2)^{1-\alpha}}$$

and

$$f''_{a,j}(a) = \frac{(j-\alpha)(j+1-\alpha)\overline{a}^2}{(1-|a|^2)^{2-\alpha}}, \qquad j = 1,2,3. \tag{4}$$

To simplify the notation, we set $v(z) = 2u'(z)\varphi'(z) + u(z)\varphi''(z)$. Then, for $w \in \mathbb{D}$, from (4), we obtain

$$(uC_\varphi f_{\varphi(w),1})''(w) = (1-|\varphi(w)|^2)^\alpha u''(w) + \frac{(1-\alpha)v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} + \frac{(1-\alpha)(2-\alpha)u(w)\varphi'(w)^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}}, \tag{5}$$

$$(uC_\varphi f_{\varphi(w),2})''(w) = (1-|\varphi(w)|^2)^\alpha u''(w) + \frac{(2-\alpha)v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} + \frac{(2-\alpha)(3-\alpha)u(w)\varphi'(w)^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}}, \tag{6}$$

and

$$(uC_\varphi f_{\varphi(w),3})''(w) = (1-|\varphi(w)|^2)^\alpha u''(w) + \frac{(3-\alpha)v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} + \frac{(3-\alpha)(4-\alpha)u(w)\varphi'(w)^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}}. \tag{7}$$

Subtracting (5) from (6), we get

$$\begin{aligned} &-(uC_\varphi f_{\varphi(w),1})''(w) + (uC_\varphi f_{\varphi(w),2})''(w) \\ &= \frac{v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} + \frac{(4-2\alpha)u(w)\varphi'(w)^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}}. \end{aligned} \tag{8}$$

On the other hand, subtracting (5) from (7), we obtain

$$\begin{aligned} &-(uC_\varphi f_{\varphi(w),1})''(w) + (uC_\varphi f_{\varphi(w),3})''(w) \\ &= \frac{2v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} + \frac{(10-4\alpha)u(w)\varphi'(w)^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}}. \end{aligned} \tag{9}$$

Subtracting from (9) twice (8), we obtain

$$\begin{aligned} &\frac{2u(w)\varphi'(w)^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}} \\ &= (uC_\varphi f_{\varphi(w),1})''(w) - 2(uC_\varphi f_{\varphi(w),2})''(w) + (uC_\varphi f_{\varphi(w),3})''(w), \end{aligned} \tag{10}$$

which implies that

$$
\begin{aligned}
&\frac{(1-|w|^2)|u(w)\varphi'(w)^2||\varphi(w)|^2}{(1-|\varphi(w)|^2)^{2-\alpha}} \\
&\leqslant \frac{1}{2}(1-|w|^2)|(uC_\varphi f_{\varphi(w),1})''(w)| + (1-|w|^2)|(uC_\varphi f_{\varphi(w),2})''(w)| \\
&\quad + \frac{1}{2}(1-|w|^2)|(uC_\varphi f_{\varphi(w),3})''(w)| \\
&\leqslant \frac{1}{2}\|uC_\varphi f_{\varphi(w),1}\|_{\mathscr{Z}} + \|uC_\varphi f_{\varphi(w),2}\|_{\mathscr{Z}} + \frac{1}{2}\|uC_\varphi f_{\varphi(w),3}\|_{\mathscr{Z}} \qquad (11) \\
&\leqslant 2A. \qquad (12)
\end{aligned}
$$

Moreover, from (9) and (10), we obtain

$$
\begin{aligned}
\frac{v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} &= (-3+\alpha)(uC_\varphi f_{\varphi(w),1})''(w) + (5-2\alpha)(uC_\varphi f_{\varphi(w),2})''(w) \\
&\quad -(2-\alpha)(uC_\varphi f_{\varphi(w),3})''(w),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{(1-|w|^2)|v(w)||\varphi(w)|}{(1-|\varphi(w)|^2)^{1-\alpha}} \\
&\leqslant (1-|w|^2)\Big((3-\alpha)|(uC_\varphi f_{\varphi(w),1})''(w)| + (5-2\alpha)|(uC_\varphi f_{\varphi(w),2})''(w)| \\
&\quad +(2-\alpha)|(uC_\varphi f_{\varphi(w),3})''(w)|\Big) \\
&\leqslant (3-\alpha)\|uC_\varphi f_{\varphi(w),1}\|_{\mathscr{Z}} + (5-2\alpha)\|uC_\varphi f_{\varphi(w),2}\|_{\mathscr{Z}} + (2-\alpha)\|uC_\varphi f_{\varphi(w),3}\|_{\mathscr{Z}} \qquad (13) \\
&\leqslant (10-4\alpha)A. \qquad (14)
\end{aligned}
$$

Fix $r \in (0,1)$. If $|\varphi(w)| > r$, then by (14), we have

$$\frac{(1-|w|^2)|v(w)|r}{(1-|\varphi(w)|^2)^{1-\alpha}} < (10-4\alpha)A,$$

so that

$$\frac{(1-|w|^2)|v(w)|}{(1-|\varphi(w)|^2)^{1-\alpha}} < \frac{(10-4\alpha)A}{r}. \qquad (15)$$

If $|\varphi(w)| \leqslant r$, then

$$\frac{(1-|w|^2)|v(w)|}{(1-|\varphi(w)|^2)^{1-\alpha}} \leqslant \frac{N_2}{(1-r^2)^{1-\alpha}},$$

which, combined with (15), implies that $M_1 < \infty$.

Arguing similarly, if $|\varphi(w)| > r$, then from (12) we obtain

$$\frac{(1-|w|^2)|u(w)\varphi'(w)^2|}{(1-|\varphi(w)|^2)^{2-\alpha}} \leqslant \frac{2A}{r^2}. \qquad (16)$$

On the other hand, if $|\varphi(w)| \leqslant r$, we get

$$\frac{(1-|w|^2)|u(w)\varphi'(w)^2|}{(1-|\varphi(w)|^2)^{2-\alpha}} \leqslant \frac{N_1}{(1-r^2)^{2-\alpha}}. \tag{17}$$

From (16) and (17) it follows that M_2 is finite.

(d) $\Rightarrow$ (a). Assume (d) holds. Since $Lip_\alpha \subset H^\infty$ and for $f \in Lip_\alpha$, by (1), $\|f\|_\infty \leqslant \|f\|_{Lip_\alpha}$, we have

$$|(uC_\varphi f)(0)| \leqslant |u(0)|\|f\|_{Lip_\alpha},$$

and

$$|(uC_\varphi f)'(0)| \leqslant |u'(0)|\|f\|_{Lip_\alpha} + \frac{C|u(0)\varphi'(0)|}{(1-|\varphi(0)|^2)^{1-\alpha}}\|f\|_{Lip_\alpha}. \tag{18}$$

In addition, for an arbitrary z in $\mathbb{D}$,

$$\begin{aligned}
&(1-|z|^2)|(uC_\varphi f)''(z)| \\
&= (1-|z|^2)\left|u''(z)f(\varphi(z)) + v(z)f'(\varphi(z)) + u(z)\varphi'(z)^2 f''(\varphi(z))\right| \\
&\leqslant (1-|z|^2)\left[|u''(z)||f(\varphi(z))| + |v(z)||f'(\varphi(z))| + |u(z)\varphi'(z)^2||f''(\varphi(z))|\right] \\
&\leqslant C\|f\|_{Lip_\alpha}\left[(1-|z|^2)|u''(z)| + \frac{(1-|z|^2)|v(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} + \frac{(1-|z|^2)|u(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{2-\alpha}}\right],
\end{aligned} \tag{19}$$

where in the last inequality we have used (2) and the following well-known characterization of Bloch-type functions (see [29]):

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{1-\alpha}|f'(z)| \asymp |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{2-\alpha}|f''(z)|.$$

The boundedness of $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ follows from (18) and (19) after taking the supremum over all $z \in \mathbb{D}$. The proof of the theorem is now complete.

3. Compactness of $uC_\varphi : Lip_\alpha \to \mathscr{Z}$

We begin the section with a useful compactness criterion which will be used to characterize the compact weighted composition operators from Lip_α to the Zygmund space. Its proof is based on a standard argument similar to that of the proof of Lemma 3.7 in [22] and will be omitted.

LEMMA 1. *Suppose that* $0 < \alpha < 1$, $\varphi \in S(\mathbb{D})$ *and* $u \in H(\mathbb{D})$. *The operator* $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ *is compact if and only if* $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ *is bounded and for any bounded sequence* $\{f_n\}_{n\in\mathbb{N}}$ *in* Lip_α *which converges to zero uniformly on* $\overline{\mathbb{D}}$, *we have* $\|uC_\varphi f_n\|_{\mathscr{Z}} \to 0$ *as* $n \to \infty$.

We now show that the compact weighted composition operators can be characterized in terms of the sequence $\|uC_\varphi p_n\|_{\mathscr{Z}}$, where we recall from Section 2 that $p_k(z) = k^{-\alpha}z^k$, for $k \in \mathbb{N}$ and $z \in \mathbb{D}$.

THEOREM 2. *Let* $0<\alpha<1$, $u\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$ *and suppose that* $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ *is bounded. Then the following conditions are equivalent:*

(a) *The operator* $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ *is compact.*

(b) $\lim_{k\to\infty} \|k^{-\alpha}u\varphi^k\|_{\mathscr{Z}} = 0.$

(c) $\lim_{|\varphi(w)|\to 1} \|uC_\varphi f_{\varphi(w),j}\|_{\mathscr{Z}} = 0, \quad j=1,2,3.$

(d) $\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2-\alpha}} = 0, \quad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} = 0.$

Proof. (a) $\Longrightarrow$ (b). Since the sequence $\{p_k\}$ is bounded in Lip_α and converges to zero uniformly on $\overline{\mathbb{D}}$, if $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ is compact, then by Lemma 1,

$$\|k^{-\alpha}u\varphi^k\|_{\mathscr{Z}} = \|uC_\varphi p_k\|_{\mathscr{Z}} \to 0 \quad \text{as} \quad k\to\infty.$$

(b) $\Longrightarrow$ (c). Suppose $\lim_{k\to\infty} \|k^{-\alpha}u\varphi^k\|_{\mathscr{Z}} = 0$. Let M be an upper bound for the sequence $\|k^{-\alpha}u\varphi^k\|_{\mathscr{Z}}$. Then, given $\varepsilon>0$, there exists a positive integer N such that

$$\|k^{-\alpha}u\varphi^k\|_{\mathscr{Z}} < \varepsilon \quad \text{for all} \ \ k\geqslant N.$$

From the power series representations of $f_{a,j}$ used in the proof of Theorem 1, for $w\in\mathbb{D}$, we have

$$\begin{aligned}\|uC_\varphi f_{\varphi(w),j}\|_{\mathscr{Z}} &\leqslant C(1-|\varphi(w)|^2)^j \sum_{k=0}^{N-1} |\varphi(w)|^k k^{j-1} \|k^{-\alpha}u\varphi^k\|_{\mathscr{Z}} \\ &\quad + C(1-|\varphi(w)|^2)^j \sum_{k=N}^{\infty} |\varphi(w)|^k k^{j-1} \|k^{-\alpha}u\varphi^k\|_{\mathscr{Z}} \\ &< (1-|\varphi(w)|^2)^j CN^jM + C\varepsilon.\end{aligned}$$

Letting $|\varphi(w)|\to 1$ and noting that ε is arbitrary, the desired result follows.

(c) $\Longrightarrow$ (d). Assume (c) holds. Using (13), we get

$$\begin{aligned}\frac{(1-|w|^2)|v(w)|}{(1-|\varphi(w)|^2)^{1-\alpha}} &\leqslant \frac{(3-\alpha)\|uC_\varphi f_{\varphi(w),1}\|_{\mathscr{Z}}}{|\varphi(w)|} + \frac{(5-2\alpha)\|uC_\varphi f_{\varphi(w),2}\|_{\mathscr{Z}}}{|\varphi(w)|} \\ &\quad + \frac{(2-\alpha)\|uC_\varphi f_{\varphi(w),3}\|_{\mathscr{Z}}}{|\varphi(w)|} \to 0,\end{aligned}$$

as $|\varphi(w)|\to 1$. Moreover, using (11), we obtain

$$\frac{(1-|w|^2)|u(w)\varphi'(w)^2|}{(1-|\varphi(w)|^2)^{2-\alpha}} \leqslant \frac{\|uC_\varphi f_{\varphi(w),1}\|_{\mathscr{Z}}}{2|\varphi(w)|^2} + \frac{\|uC_\varphi f_{\varphi(w),2}\|_{\mathscr{Z}}}{|\varphi(w)|^2} + \frac{\|uC_\varphi f_{\varphi(w),3}\|_{\mathscr{Z}}}{2|\varphi(w)|^2},$$

which approaches 0 as $|\varphi(w)|\to 1$. The desired result follows.

(d) $\Longrightarrow$ (a). Assume that (d) holds. By the boundedness of $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ and the proof of Theorem 1, we see that the quantities N_1 and N_2 are finite.

From (d), for any $\varepsilon > 0$, there is a number $\delta \in (0,1)$, such that

$$\frac{(1-|z|^2)|v(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} < \varepsilon, \quad \frac{(1-|z|^2)|u(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{2-\alpha}} < \varepsilon, \tag{20}$$

whenever $\delta < |\varphi(z)| < 1$.

Let $\{g_n\}$ be a bounded sequence in Lip_α converging to zero uniformly on $\overline{\mathbb{D}}$. Employing (20), we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)|(uC_\varphi g_n)''(z)| \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\left|u''(z)g_n(\varphi(z)) + u(z)\varphi'(z)^2 g_n''(\varphi(z)) + v(z)g_n'(\varphi(z))\right| \\
&\leqslant \sup_{z\in\mathbb{D}}(1-|z|^2)|u''(z)g_n(\varphi(z))| + \sup_{|\varphi(z)|\leqslant\delta}(1-|z|^2)|u(z)\varphi'(z)^2 g_n''(\varphi(z))| \\
&\quad + \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)|u(z)\varphi'(z)^2 g_n''(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|\leqslant\delta}(1-|z|^2)\left|v(z)g_n'(\varphi(z))\right| + \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)\left|v(z)g_n'(\varphi(z))\right| \\
&\leqslant \|u\|_{\mathscr{Z}}\sup_{z\in\mathbb{D}}|g_n(\varphi(z))| + \sup_{|\varphi(z)|\leqslant\delta}(1-|z|^2)|u(z)\varphi'(z)^2||g_n''(\varphi(z))| \\
&\quad + C\|g_n\|_{Lip_\alpha}\sup_{\delta<|\varphi(z)|<1}\frac{(1-|z|^2)|u(z)\varphi'(z)^2|}{(1-|\varphi(z)|^2)^{2-\alpha}} \\
&\quad + \sup_{|\varphi(z)|\leqslant\delta}(1-|z|^2)|v(z)||g_n'(\varphi(z))| + C\|g_n\|_{Lip_\alpha}\sup_{\delta<|\varphi(z)|<1}\frac{(1-|z|^2)|v(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} \\
&\leqslant \|u\|_{\mathscr{Z}}\sup_{w\in\overline{\mathbb{D}}}|g_n(w)| + N_1\sup_{|w|\leqslant\delta}|g_n''(w)| + N_2\sup_{|w|\leqslant\delta}|g_n'(w)| + \varepsilon C\|g_n\|_{Lip_\alpha}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\|uC_\varphi g_n\|_{\mathscr{Z}} &= |u(0)g_n(\varphi(0))| + |u'(0)g_n(\varphi(0)) + u(0)g_n'(\varphi(0))\varphi'(0)| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)|(uC_\varphi g_n)''(z)| \\
&\leqslant (|u(0)|+|u'(0)|)|g_n(\varphi(0))| + |u(0)g_n'(\varphi(0))\varphi'(0)| + \|u\|_{\mathscr{Z}}\sup_{w\in\overline{\mathbb{D}}}|g_n(w)| \\
&\quad + N_1\sup_{|w|\leqslant\delta}|g_n''(w)| + N_2\sup_{|w|\leqslant\delta}|g_n'(w)| + \varepsilon C\|g_n\|_{Lip_\alpha}.
\end{aligned} \tag{21}$$

From the uniform convergence to 0 of the sequence g_n on $\overline{\mathbb{D}}$, it follows that $g_n' \to 0$ and $g_n'' \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $n\to\infty$. Therefore, $(|u(0)| + |u'(0)|)|g_n(\varphi(0))| + |u(0)g_n'(\varphi(0))\varphi'(0)| \to 0$ and

$$\|u\|_{\mathscr{Z}}\sup_{w\in\overline{\mathbb{D}}}|g_n(w)| + N_1\sup_{|w|\leqslant\delta}|g_n''(w)| + N_2\sup_{|w|\leqslant\delta}|g_n'(w)| \to 0$$

as $n\to\infty$. Hence, letting $n\to\infty$ in (21) and noting that ε is an arbitrary positive number, we obtain $\lim_{n\to\infty}\|uC_\varphi g_n\|_{\mathscr{Z}} = 0$. By Lemma 1, it follows that $uC_\varphi : Lip_\alpha \to \mathscr{Z}$ is compact. The proof of the theorem is now complete.

4. Component operators

We conclude the paper by highlighting the results for the cases of the multiplication operator and the composition operator mapping the Lipschitz space into the Zygmund space, which, to the best of our knowledge, have not appeared in the literature.

COROLLARY 1. *Let* $u \in H(\mathbb{D})$ *and* $0 < \alpha < 1$. *The following statements are equivalent.*

(a) *The operator* $M_u : Lip_\alpha \to \mathscr{Z}$ *is bounded.*

(b) *The operator* $M_u : Lip_\alpha \to \mathscr{Z}$ *is compact.*

(c) *The symbol* u *is identically 0.*

Proof. From part (d) of Theorem 1, we see that if $M_u : Lip_\alpha \to \mathscr{Z}$ is bounded, then

$$\sup_{z\in\mathbb{D}} \frac{|u(z)|}{(1-|z|^2)^{1-\alpha}} < \infty.$$

This implies that $|u(z)| \to 0$ as $|z| \to 1$. Thus, u is the constant 0. The other implications are obvious.

Noting that $(\varphi^2)'' = 2\varphi'^2 + 2\varphi\varphi''$, for $\varphi \in \mathscr{Z}$, $\sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi'(z)|^2 < \infty$ is equivalent to $\varphi^2 \in \mathscr{Z}$. Thus, from Theorem 1, we obtain the following result.

COROLLARY 2. *Let* $0 < \alpha < 1$ *and* $\varphi \in S(\mathbb{D})$. *The following statements are equivalent:*

(a) *The operator* $C_\varphi : Lip_\alpha \to \mathscr{Z}$ *is bounded.*

(b) $\sup_{k\geqslant 1} \|k^{-\alpha}\varphi^k\|_{\mathscr{Z}} < \infty$.

(c) $\varphi, \varphi^2 \in \mathscr{Z}$, *and* $\max_{j=1,2,3} \sup_{w\in\mathbb{D}} \|C_\varphi f_{\varphi(w),j}\|_{\mathscr{Z}} < \infty$.

(d) $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2-\alpha}} < \infty$, $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\varphi''(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} < \infty$.

Lastly, from Theorem 2, we deduce the following characterization of the compact composition operators.

COROLLARY 3. *Let* $0 < \alpha < 1$, $\varphi \in S(\mathbb{D})$ *and suppose that* $C_\varphi : Lip_\alpha \to \mathscr{Z}$ *is bounded. Then the following conditions are equivalent:*

(a) *The operator* $C_\varphi : Lip_\alpha \to \mathscr{Z}$ *is compact.*

(b) $\lim_{k\to\infty} \|k^{-\alpha}\varphi^k\|_{\mathscr{Z}} = 0$.

(c) $\lim_{|\varphi(w)|\to 1} \|C_\varphi f_{\varphi(w),j}\|_{\mathscr{Z}} = 0, \quad j = 1,2,3.$

(d) $\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2-\alpha}} = 0$, $\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi''(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} = 0$.

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