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INEQUALITIES FOR THE p–CROSS–SECTION BODIES

WEIDONG WANG AND YANPING ZHOU

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Abstract. Gardner and Giannopoulos defined the p-cross-section body C_pK ($p > -1$) of convex body K. In this paper, some inequalities for the volumes of C_pK are shown. Further, the Shephard type problem and a monotony inequality of C_pK are obtained.

1. Introduction

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{E}^n$, for the set of convex bodies containing the origin in their interiors in $\mathbb{E}^n$ by $\mathscr{K}_o^n$. Write $\mathscr{S}_o^n$ for the set of star bodies (about the origin) in $\mathbb{E}^n$. For the n-dimensional volume and the $n-1$-dimensional volume of body K, denote by $V(K)$ and $V_{n-1}(K)$, respectively. Let S^{n-1} denote the unit sphere in $\mathbb{E}^n$, as for the standard unit ball B in $\mathbb{E}^n$, denote $V(B) = \omega_n$.

In mid 1990s, Lutwak ([10, 11]) showed that the Firey sum ([2]) of convex bodies led to the Brunn-Minkowski theory for each $p \geqslant 1$, and established an embryonic L_p-Brunn-Minkowski theory. This theory has expanded rapidly (see [4, 5, 7–9, 12–18, 22–28]). Similar to the Firey sum of convex bodies, Uhrin in [21] introduced general convex combination of two sets. Associated with this general combination, he established l_p-form of the Brunn-Minkowski-Lusternik inequality.

In 1999, Gardner and Giannopoulos in [4] showed the notion of p-cross-section body as follows:

For $K \in \mathscr{K}^n$, the p-cross-section body C_pK of K is defined for nonzero $p > -1$ by

$$\rho_{C_pK}(u) = \left(\frac{1}{V(K)}\int_K V_{n-1}(K \cap (u^\perp + x))^p dx\right)^{\frac{1}{p}}, \tag{1.1}$$

for each $u \in S^{n-1}$; They also defined that for each $u \in S^{n-1}$,

$$\rho_{C_0K}(u) = \exp\left(\frac{1}{V(K)}\int_K \log V_{n-1}(K \cap (u^\perp + x))dx\right)$$

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and

$$\rho_{C_\infty K}(u) = \max_{x\in K} V_{n-1}(K\cap(u^\perp + x)).$$

Recall that the classical cross-section body CK of $K\in\mathscr{K}^n$ is defined by (see [3])

$$\rho_{CK}(u) = \max_{x\in K} V_{n-1}(K\cap(u^\perp + x)),$$

for each $u\in S^{n-1}$.

Compare to above definitions of CK and $C_\infty K$, obviously,

$$C_\infty K = CK. \tag{1.2}$$

For the classical cross-section body, Busemann's theorem shows that if K is centrally symmetric with center x, then CK is convex. Meyer (see [19]) proved that CK is convex when $n=3$, but Brehm in [1] showed that when $n\geqslant 4$, CK is not convex when K is a simplex.

From the definition of p-cross-section body, Gardner and Giannopoulos in [4] pointed that ρ_{C_pK} is continuous for $K\in\mathscr{K}^n$. Further, they (see [4]) showed that C_1K is convex, and C_pK is convex when $n=2$ and $p>0$ or $n=3$ and $p=\infty$.

The reports of p-cross-section bodies are few since this notion was introduced. In this paper, we continuously research the p-cross-section bodies by L_p-dual mixed volume. Our works belong to the L_p-Brunn-Minkowski theory. First, we establish inequalities for the volumes of C_pK and intersection body IK as follows:

THEOREM 1.1. *If $K\in\mathscr{K}^n$, $p>-1$, then exists $x_0\in K$ such that for $-1<p\leqslant n$,*

$$V(C_pK)\leqslant V(I(K-x_0)), \tag{1.3}$$

for $p\geqslant n$,

$$V(C_pK)\geqslant V(I(K-x_0)). \tag{1.4}$$

In every inequality with equality if and only if $p=n$ or $p\neq n$ and $C_pK=I(K-x_0)$.

As the application of (1.3), we obtain the following inequality for the volume of p-cross-section body C_pK.

THEOREM 1.2. *If $K\in\mathscr{K}_o^n$, $-1<p\leqslant n$, then*

$$V(C_pK)\leqslant \frac{\omega_{n-1}^n}{\omega_n^{n-2}}V(K)^{n-1}, \tag{1.5}$$

with equality if and only if K is an ellipsoid.

Further, we give the Shephard type problem for the p-cross-section bodies as follows:

THEOREM 1.3. *For $K,L\in\mathscr{K}^n$, nonzero $p>-1$, if $C_pK\subseteq C_pL$, then there exist $x_0\in K$ and $y_0\in L$ such that*

$$V(I(K-x_0))\leqslant V(I(L-y_0)), \tag{1.6}$$

with equality if and only if $p=n$ and $C_pK=C_pL$ or $p\neq n$ and $C_pK=C_pL$ and $I(K-x_0)=I(L-y_0)$.

From Theorem 1.3, let $p\to\infty$ and together with (1.2), we easily get that

COROLLARY 1.1. *If $K,L \in \mathscr{K}^n$, and $CK \subseteq CL$, then there exist $x_0 \in K$ and $y_0 \in L$ such that*

$$V(I(K-x_0)) \leqslant V(I(L-y_0)),$$

with equality if and only if $CK = CL$ and $I(K-x_0) = I(L-y_0)$.

Finally, a monotony inequality for the p-cross-section bodies is obtained:

THEOREM 1.4. *For $K,L \in \mathscr{K}^n$, $p > 0$, if $K \subseteq L$, then*

$$V(K)^{\frac{n}{p}} V(C_pK) \leqslant V(L)^{\frac{n}{p}} V(C_pL), \tag{1.7}$$

with equality if and only if $K = L$.

Let $p \to \infty$ in Theorem 1.4 and use (1.2), we also can get that

COROLLARY 1.2. *For $K,L \in \mathscr{K}^n$, if $K \subseteq L$, then*

$$V(CK) \leqslant V(CL),$$

with equality if and only if $K = L$.

2. Preliminaries

2.1. Radial function

If K is a compact star-shaped (about the origin) in $\mathbb{E}^n$, its radial function, $\rho_K = \rho(K,\cdot)$, is defined by (see [3, 20])

$$\rho(K,u) = \max\{\lambda \geqslant 0 : \lambda u \in K\}, \tag{2.1}$$

for all $u \in S^{n-1}$. If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathscr{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{E}^n$. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If K is a compact star-shaped with respect to $x \in \mathbb{E}^n$, its radial function $\rho_K(x,\cdot)$ respect to x is defined, for all $u \in S^{n-1}$ such that the line through x parallel to u intersects K, by (see [5])

$$\rho_K(x,u) = \max\{\lambda \geqslant 0 : x + \lambda u \in K\}. \tag{2.2}$$

From (2.1) and (2.2), we easily know

$$\rho_K(x,u) = \rho_{K-x}(u), \tag{2.3}$$

for $u \in S^{n-1}$. We call that $\rho_K(x,\cdot)$ is the extended radial function of K with respect to x. If x is the origin o, then $\rho_K(x,u) = \rho_K(u)$ for all $u \in S^{n-1}$.

2.2. Intersection body

For $K \in \mathscr{S}_o^n$, the intersection body IK of K is a centered body whose radial function is defined by (see [3])

$$\rho_{IK}(u) = V_{n-1}(K \cap u^{\perp}) = \frac{1}{n-1} \int_{S^{n-1} \bigcap u^{\perp}} \rho_K(v)^{n-1} dv, \tag{2.4}$$

for all $u \in S^{n-1}$.

From definition (2.4), because of ρ_K is continuous, so ρ_{IK} is also continuous, i.e., $IK \in \mathscr{S}_o^n$. Intersection body IK of K is not generally convex even K is a convex body. The fact is that if $K \in \mathscr{K}^n$, then there is a translate $L \in \mathscr{K}_o^n$ of K, such that IL is not convex. But if K is a centered convex body, then IK is also a centered convex body (see [3]).

For the intersection bodies, the Busemann intersection inequality can be stated as follows (see [3]): *If $K \in \mathscr{K}_o^n$, then*

$$V(IK) \leqslant \frac{\omega_{n-1}^n}{\omega_n^{n-2}} V(K)^{n-1}, \tag{2.5}$$

with equality if and only if K is a centered ellipsoid.

2.3. L_p-dual mixed volume

The notion of L_p-dual mixed volume was introduced by Grinberg and Zhang (see [6]). For $K, L \in \mathscr{S}_o^n$ and any real p, the L_p-dual mixed volume, $\widetilde{V}_p(K,L)$, of K and L is defined by

$$\widetilde{V}_p(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-p} \rho(L,u)^p du. \tag{2.6}$$

From (2.6), we easily know that

$$\widetilde{V}_p(K,K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^n du = V(K), \tag{2.7}$$

The Minkowski's inequality of L_p-dual mixed volume is that (see [6])
If $K, L \in \mathscr{S}_o^n$, p is any real, then for $0 \leqslant p \leqslant n$,

$$\widetilde{V}_p(K,L) \leqslant V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}; \tag{2.8}$$

for $p < 0$ or $p \geqslant n$,

$$\widetilde{V}_p(K,L) \geqslant V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}. \tag{2.9}$$

In every case, equality holds if and only if $p = n$ or $p \neq n$ and K is a dilatate of L.

3. Proofs of the Theorems

LEMMA 3.1. *If $K,L \in \mathscr{K}^n$, $p > -1$, then for any $Q \in \mathscr{S}_o^n$,*

$$\widetilde{V}_p(Q,C_pK) = \frac{1}{V(K)}\int_K \widetilde{V}_p(Q,I(K-x))dx. \tag{3.1}$$

Proof. From (2.3) and (1.1), then for $p > -1$ (see [4]),

$$\begin{aligned}\rho(C_pK,u) &= \left(\frac{1}{V(K)}\int_K V_{n-1}((K-x)\cap u^{\perp})^p dx\right)^{\frac{1}{p}},\\ &= \left(\frac{1}{V(K)}\int_K \rho_{I(K-x)}(u)^p dx\right)^{\frac{1}{p}},\end{aligned} \tag{3.2}$$

Using (2.6) and (3.2), then for any $Q \in \mathscr{S}_o^n$ and $p > -1$, we have that

$$\begin{aligned}\widetilde{V}_p(Q,C_pK) &= \frac{1}{n}\int_{S^{n-1}} \rho_Q(u)^{n-p}\rho_{C_pK}(u)^p du\\ &= \frac{1}{nV(K)}\int_{S^{n-1}}\int_K \rho_Q(u)^{n-p}\rho_{I(K-x)}(u)^p dxdu\\ &= \frac{1}{V(K)}\int_K [\frac{1}{n}\int_{S^{n-1}} \rho_Q(u)^{n-p}\rho_{I(K-x)}(u)^p du]dx\\ &= \frac{1}{V(K)}\int_K \widetilde{V}_p(Q,I(K-x))dx. \quad \square\end{aligned}$$

Proof of Theorem 1.1. Form (3.1) and together with the integral mean value theorem, then exists $x_o \in K$ such that

$$\widetilde{V}_p(Q,C_pK) = \frac{1}{V(K)}\widetilde{V}_p(Q,I(K-x_o))\int_K dx = \widetilde{V}_p(Q,I(K-x_o)). \tag{3.3}$$

Let $Q = C_pK$ in (3.3) and use (2.7), we have that

$$V(C_pK) = \widetilde{V}_p(C_pK,I(K-x_o)). \tag{3.4}$$

Hence for $0 \leqslant p \leqslant n$, from (3.4) and (2.8), we get that

$$V(C_pK) \leqslant V(C_pK)^{\frac{n-p}{n}} V(I(K-x_o))^{\frac{p}{n}},$$

i.e.,

$$V(C_pK)^{\frac{p}{n}} \leqslant V(I(K-x_o))^{\frac{p}{n}},$$

this yields inequality (1.3).

For $-1 < p < 0$ or $p \geqslant n$, associated with (3.4) and inequality (2.9), then

$$V(C_pK) \geqslant V(C_pK)^{\frac{n-p}{n}} V(I(K-x_o))^{\frac{p}{n}},$$

i.e.,

$$V(C_pK)^{\frac{p}{n}} \geqslant V(I(K-x_o))^{\frac{p}{n}}.$$

Thus for $-1<p<0$,

$$V(C_pK) \leqslant V(I(K-x_o));$$

for $p \geqslant n$,

$$V(C_pK) \geqslant V(I(K-x_o)).$$

From this, inequalities (1.3) and (1.4) are obtained, respectively.

According to the conditions of equality hold in inequalities (2.8) and (2.9), we see that equality hold in (1.3) and (1.4) if and only if $p=n$ or $p \neq n$ and C_pK is a dilatate of $I(K-x_o)$. But C_pK is a dilatate of $I(K-x_o)$ and $V(C_pK)=V(I(K-x_o))$ imply $C_pK=I(K-x_o)$. Hence, equality hold in (1.3) and (1.4) if and only if $p=n$ or $p\neq n$ and $C_pK=I(K-x_o)$. □

Proof of Theorem 1.2. From (1.3), (2.5) and notice that $V(K-x_0)=V(K)$, we have that

$$\begin{aligned} V(C_pK) &\leqslant V(I(K-x_o)) \\ &\leqslant \frac{\omega_{n-1}^n}{\omega_n^{n-2}} V(K-x_0)^{n-1} \\ &= \frac{\omega_{n-1}^n}{\omega_n^{n-2}} V(K)^{n-1}. \end{aligned}$$

Combining with the cases of equality hold in (2.5) and (1.3), we see that equality holds in (1.5) if and only if $p=n$ and $K-x_0$ is a centered ellipsoid or $p\neq n$ and $K-x_0$ is a centered ellipsoid and $C_pK=I(K-x_o)$.

Since if $K-x_0$ is a centered ellipsoid E then (see [3])

$$I(K-x_0)=\frac{\omega_{n-1}}{\omega_n}V(E)E^*=C_pK,$$

thus $K-x_0$ is a centered ellipsoid with $x_0\in K$ implies $C_pK=I(K-x_o)$, this mean K is an ellipsoid with $o\in intK$. To sum up, we know that equality hold in (1.5) if and only if K is an ellipsoid with $o\in intK$, i.e., K is an ellipsoid and $K\in\mathscr{K}_o^n$. □

Proof of Theorem 1.3. Since $C_pK\subseteq C_pL$, by (2.6) then for $p>0$ and any $Q\in\mathscr{S}_o^n$, we have that

$$\widetilde{V}_p(Q,C_pK)\leqslant \widetilde{V}_p(Q,C_pL). \tag{3.5}$$

with equality if and only if $C_pK=C_pL$. Using (3.3), there exist $x_0\in K$ and $y_0\in L$ such that

$$\widetilde{V}_p(Q,I(K-x_0))\leqslant \widetilde{V}_p(Q,I(L-y_0)). \tag{3.6}$$

For $0<p\leqslant n$, let $Q=I(K-x_0)$ in (3.6) where $x_0\in K$, this together with (2.7) and (2.8), then

$$\begin{aligned} V(I(K-x_0)) &\leqslant \widetilde{V}_p(I(K-x_0),I(L-y_0)) \\ &\leqslant V(I(K-x_0))^{\frac{n-p}{n}} V(I(L-y_0))^{\frac{p}{n}}. \end{aligned}$$

Hence,

$$V(I(K-x_0))^{\frac{p}{n}} \leqslant V(I(L-y_0))^{\frac{p}{n}}.$$

This gives inequality (1.6).

For $p \geqslant n$, taking $Q = I(L-y_0)$ in (3.6) where $y_0 \in L$, and using (2.7) and inequality (2.9), we have that

$$\begin{aligned} V(I(L-y_0)) &\geqslant \widetilde{V}_p(I(L-y_0), I(K-x_0)) \\ &\geqslant V(I(L-y_0))^{\frac{n-p}{n}} V(I(K-x_0))^{\frac{p}{n}}. \end{aligned}$$

This also yields inequality (1.6).

For $-1 < p < 0$, from $C_pK \subseteq C_pL$ and (2.6), we know that inequality (3.5) is reverse. Using (3.3), there exist $x_0 \in K$ and $y_0 \in L$ such that

$$\widetilde{V}_p(Q, I(K-x_0)) \geqslant \widetilde{V}_p(Q, I(L-y_0)). \tag{3.7}$$

Let $Q = I(K-x_0)$ in (3.7) where $x_0 \in K$, and according to (2.9), we have that

$$V(I(K-x_0))^{\frac{p}{n}} \geqslant V(I(L-y_0))^{\frac{p}{n}}.$$

Since $-1 < p < 0$, thus

$$V(I(K-x_0)) \leqslant V(I(L-y_0)).$$

This is just inequality (1.6).

From the conditions of equality hold in inequalities (3.5) and (2.8) (or (2.9)), we know that with equality in (1.6) if and only if $p = n$ and $C_pK = C_pL$ or $p \neq n$ and $C_pK = C_pL$ and $I(K-x_0) = I(L-y_0)$. □

Proof of Theorem 1.4. Since $K \subseteq L$, then $K - x \subseteq L - x$ where $x \in K$. According to the definition (2.4) of intersection body, we have that

$$\rho_{I(K-x)}(u) = V_{n-1}((K-x) \cap u^{\perp}) \leqslant V_{n-1}((L-x) \cap u^{\perp}) = \rho_{I(L-x)}(u),$$

for all $u \in S^{n-1}$. Thus $I(K-x) \subseteq I(L-x)$ with $x \in K$. This together with (2.6), we know that for $p > 0$,

$$\widetilde{V}_p(Q, I(K-x)) \leqslant \widetilde{V}_p(Q, I(L-x)),$$

hence

$$\int_K \widetilde{V}_p(Q, I(K-x))dx \leqslant \int_K \widetilde{V}_p(Q, I(L-x))dx \leqslant \int_L \widetilde{V}_p(Q, I(L-x))dx,$$

using (3.1), then

$$\widetilde{V}_p(Q, C_pK)V(K) \leqslant \widetilde{V}_p(Q, C_pL)V(L). \tag{3.8}$$

For $0 < p \leqslant n$, let $Q = C_pK$ in (3.8) and together (2.7) with (2.8), we get that

$$V(K)V(C_pK) \leqslant V(L)\widetilde{V}_p(C_pK, C_pL) \leqslant V(L)V(C_pK)^{\frac{n-p}{n}} V(C_pL)^{\frac{p}{n}},$$

this yields the result (1.7).

For $p > n$, let $Q = C_pK$ in (3.8) and together with (2.7) and (2.9), we obtain that

$$V(L)V(C_pL) \geqslant V(K)\widetilde{V}_p(C_pL, C_pK) \geqslant V(K)V(C_pL)^{\frac{n-p}{n}}V(C_pK)^{\frac{p}{n}},$$

this gives the desired result.

Obviously, from the equality condition of Theorem 1.4, we see that equality holds in inequality (1.7) if and only if $K = L$. □

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Wang Weidong
Department of Mathematics
China Three Gorges University
Yichang, 443002
P.R.China
e-mail: wdwxh722@163.com

Zhou Yanping
Department of Mathematics
China Three Gorges University
Yichang, 443002
P.R.China