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SOME RESULTS RELATED WITH BEREZIN SYMBOLS AND TOEPLITZ OPERATORS

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Abstract. We investigate some problems related with Berezin symbols of operators on Hardy and Bergman spaces and their applications in summability theory and in solution of Beurling problem. We also study boundedness and invertibility of some Toeplitz products on the Hardy and Bergman spaces.

1. Introduction and background

A reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathscr{H} = \mathscr{H}(Q)$ of complex-valued functions on a (nonempty) set Q, which has the property that point evaluations $f \to f(\lambda)$ are continuous in $\mathscr{H}$ for all $\lambda \in Q$. Then the classical Riesz representation theorem guarantees that for every $\lambda \in Q$ there is a unique element $k_{\mathscr{H},\lambda} \in \mathscr{H}$ such that $f(\lambda) = \langle f, k_{\mathscr{H},\lambda} \rangle$ for all $f \in \mathscr{H}$. The function $k_{\mathscr{H},\lambda}$ is called the reproducing kernel of $\mathscr{H}$. It is known that (see Aronzajn [1]) the reproducing kernel $k_{\mathscr{H},\lambda}$ of $\mathscr{H}$ is represented by

$$k_{\mathscr{H},\lambda} = \sum_n \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $(e_n)_{n\geqslant 1}$ of $\mathscr{H}$ (Note that it is well-known that every RKHS is separable). Let $\widehat{k}_{\mathscr{H},\lambda} = \frac{k_{\mathscr{H},\lambda}}{\|k_{\mathscr{H},\lambda}\|}$ be the normalized reproducing kernel of $\mathscr{H}$. Berezin symbol $\widetilde{T}$ of a bounded linear operator on $\mathscr{H}$ (shortly, $T \in \mathscr{B}(\mathscr{H})$) is defined by (see Zhu [8])

$$\widetilde{T}(\lambda) := \left\langle T\widehat{k}_{\mathscr{H},\lambda}, \widehat{k}_{\mathscr{H},\lambda} \right\rangle \ (\lambda \in Q).$$

Berezin set and Berezin number of operator T are defined by (see Karaev [2])

$$Ber(T) := Range\left(\widetilde{T}\right) = \left\{\widetilde{T}(\lambda) : \lambda \in Q\right\}$$

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and

$$ber(T) := \sup\left\{\left|\widetilde{T}(\lambda)\right| : \lambda \in Q\right\},$$

respectively.

Recall that the sequence $\{a_n\}_{n\geqslant 0}$ of the complex numbers a_n is Abel convergent to a if the limit

$$\lim_{t\to 1^-}(1-t)\sum_{n=0}^{\infty} a_n t^n$$

exists. The series $\sum_{n=0}^{\infty} a_n$ is Abel convergent to ℓ if the series $\sum_{n=0}^{\infty} a_n t^n$ is convergent for all $t \in (0,1)$ and $\lim_{t\to 1^-} \sum_{n=0}^{\infty} a_n t^n = \ell$ exists.

In this paper we study some problems related with Berezin symbols of operators on Hardy and Bergman spaces of analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and their applications. Namely, we prove criteria for Abel convergence of some sequences and series of complex numbers. We also give a particular solution of Beurling problem in terms of Berezin symbols. Moreover, we study invertible Toeplitz products on the Hardy and Bergman spaces.

Let $\mathbb{T} = \left\{e^{it} : t \in [0,2\pi)\right\}$ denote the unit circle and $dm(z) = (2\pi)^{-1} dt = (2\pi i z)^{-1} dz$ the normalized *arc* length measure on it. The Hardy space H^2 of the disc is defined as the set of all functions analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which

$$\|f\|_{H^2} := \sup_{0\leqslant r<1}\left(\int_0^{2\pi}\left|f\left(re^{it}\right)\right|^2\frac{dt}{2\pi}\right)^{1/2} < \infty.$$

All such functions possess radial limits

$$f\left(e^{it}\right) = \lim_{r\to 1^-} f\left(re^{it}\right)$$

for almost all $t \in [0,2\pi)$ and $H^2 = H^2(\mathbb{D})$ can be identied with a closed subspace of the Lebesgue space $L^2(\mathbb{T}, dm)$. Thus, H^2 is a Hilbert space when equipped with the inner product

$$\langle f, g\rangle_{H2} = \int_{\mathbb{T}} f\overline{g}\,dm = \int_{\mathbb{T}} f\left(e^{it}\right)\overline{g\left(e^{it}\right)}\frac{dt}{2\pi} = \sum_{n=0}^{\infty} f(n)\overline{g(n)},$$

where $f(z) = \sum_n \widehat{f}(n) z^n$ and $g(z) = \sum_n \widehat{g}(n) z^n$ are the Taylor series in the unit disc $\mathbb{D}$ of f and g, respectively. H^2 has a reproducing kernel

$$k_{H^2,\lambda}(z) := \frac{1}{1-\overline{\lambda}z},\ \lambda, z \in \mathbb{D},$$

called also the Szegö kernel.

Let $L_a^2 = L_a^2(\mathbb{D})$ denote the Bergman space of all analytic functions on $\mathbb{D}$ satisfying

$$\|f\|_{L_a^2}^2 := \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty,$$

where $dA(z)$ denote Lebesgue area measure on the unit disc $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. It is well known (see Zhu [8]) that $k_{L_a^2,\lambda}(z) = \frac{1}{(1-\overline{\lambda}z)^2}$, $\lambda, z \in \mathbb{D}$. A standard calculation involving the Taylor coefficients shows that $\|f\|_{L_a^2}^2 \leqslant \|f\|_{H^2}$, which implies that $H^2 \subset L_a^2$.

2. Berezin symbols and Abel convergence of sequences and series

In this section we give in terms of Berezin symbols of weighted shift operator T_Λ,

$$T_\Lambda \sqrt{n+1} z^n = \lambda_n \sqrt{n+2} z^{n+1}, \; n \geqslant 0, \tag{1}$$

on the Bergman space L_a^2 some criterion for Abel convergence of sequences and series of complex numbers. Recall that $\left\{\sqrt{n+1}z^n\right\}_{n\geqslant 0}$ is an orthonormal basis of the Bergman space L_a^2.

THEOREM 1. *Let $\{a_n\}_{n\geqslant 0}$ be a bounded sequence of complex numbers, and T_a be an associated weighted shift operator acting on the Bergman space L_a^2 by the formula (1). Then*

(a) the series $\sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} a_n$ is Abel convergent if and only if

$$\frac{\left|\widetilde{T_a}\right|(\sqrt{t})}{t} = O\left((1-t)^2\right) \text{ as } t \to 1^-;$$

(b) the sequence $\left(\sqrt{(n+1)(n+2)} a_n\right)_{n\geqslant 0}$ is Abel convergent if and only if

$$\frac{\left|\widetilde{T_a}\right|(t)}{\sqrt{t}(1-t)} = O(1-t) \text{ as } t \to 1^-.$$

First let us give the following well-known lemma for completeness we provide its proof here.

LEMMA 1. *For any operator $T \in \mathscr{B}(L_a^2)$ we have*

$$\widetilde{T}(\lambda) = \left(1-|\lambda|^2\right)^2 \sum_{n,m=0}^{\infty} \sqrt{(n+1)(m+1)} \langle Te_n, e_m \rangle \overline{\lambda}^n \lambda^m \tag{2}$$

for $\lambda \in \mathbb{D}$, where $e_n(z) = \left\{\sqrt{n+1}z^n\right\}_{n\geqslant 0}$ is the canonical basis for the space L_a^2.

Proof. Indeed,

$$\begin{aligned}
\widetilde{T}(\lambda) &= \left\langle T\widehat{k}_{L_a^2,\lambda}, \widehat{k}_{L_a^2,\lambda}\right\rangle = \left\langle T\frac{k_{L_a^2,\lambda}}{\left\|k_{L_a^2,\lambda}\right\|}, \frac{k_{L_a^2,\lambda}}{\left\|k_{L_a^2,\lambda}\right\|}\right\rangle \\
&= \left(1-|\lambda|^2\right)^2 \left\langle \sum_{n\geqslant 0} \overline{e_n(\lambda)} Te_n(z), \sum_{n\geqslant 0} \overline{e_n(\lambda)} e_n(z)\right\rangle \\
&= \left(1-|\lambda|^2\right)^2 \sum_{n,m=0}^{\infty} \overline{e_n(\lambda)} e_m(\lambda) \langle Te_n, e_m\rangle \\
&= \left(1-|\lambda|^2\right)^2 \sum_{n,m=0}^{\infty} \sqrt{(n+1)}\overline{\lambda}^n \sqrt{(m+1)}\lambda^m \langle Te_n, e_m\rangle \\
&= \left(1-|\lambda|^2\right)^2 \sum_{n,m=0}^{\infty} \sqrt{(n+1)(m+1)} \langle Te_n, e_m\rangle \overline{\lambda}^n \lambda^m,
\end{aligned}$$

which proves (2). □

Proof of Theorem 1. Let T_a be a weighted shift operator on L_a^2. Then it follows from formula (2) that

$$\begin{aligned}
\widetilde{T_a}(\lambda) &= \left(1-|\lambda|^2\right)^2 \sum_{n,m=0}^{\infty} \sqrt{(n+1)(m+1)} \langle a_n e_{n+1}, e_m\rangle \overline{\lambda}^n \lambda^m \\
&= \lambda\left(1-|\lambda|^2\right)^2 \sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} a_n |\lambda|^{2n},
\end{aligned}$$

or

$$\widetilde{T_a}(\lambda) = \lambda\left(1-|\lambda|^2\right)^2 \sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} a_n |\lambda|^{2n} \quad (\lambda \in \mathbb{D}). \tag{3}$$

Consequently,

$$\left|\frac{\widetilde{T_a}(\lambda)}{\lambda\left(1-|\lambda|^2\right)^2}\right| = \left|\sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} a_n |\lambda|^{2n}\right| \tag{4}$$

and

$$\left|\frac{\widetilde{T_a}(\lambda)}{\lambda\left(1-|\lambda|^2\right)}\right| = \left|\left(1-|\lambda|^2\right) \sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} a_n |\lambda|^{2n}\right| \tag{5}$$

for all $\lambda \in \mathbb{D}$. Let $t = |\lambda|^2$. Then (4) and (5) has the form

$$\frac{\left|\widetilde{T_a}\right|(t)}{\sqrt{t}(1-t)^2} = \left|\sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} a_n t^n\right| \tag{6}$$

and

$$\frac{\left|\widetilde{T_a}\right|(t)}{\sqrt{t}\,(1-t)} = \left|(1-t)\sum_{n=0}^{\infty}\sqrt{(n+1)(n+2)}a_n t^n\right| \tag{7}$$

for all $t,\ 0 \leqslant t < 1$. It follows now immediately from (6) and (7) the assertions (a) and (b) of the theorem. □

COROLLARY 1. *We have:*
(a) $Ber(T_a) = \left\{\lambda\left(1-|\lambda|^2\right)^2 \sum_{n=0}^{\infty}\sqrt{(n+1)(n+2)}a_n|\lambda|^{2n} : \lambda \in \mathbb{D}\right\}$
(b) $ber(T_a) = \sup\left\{|\lambda|\left(1-|\lambda|^2\right)^2\left|\sum_{n=0}^{\infty}\sqrt{(n+1)(n+2)}a_n|\lambda|^{2n}\right| : \lambda \in \mathbb{D}\right\}$
(c) If $\sup_{n\geqslant 0}\sqrt{(n+1)(n+2)}\,|a_n| < +\infty$, *then*
$ber(T_a) \leqslant \frac{2}{3\sqrt{3}}\sup_{n\geqslant 0}\sqrt{(n+1)(n+2)}\,|a_n|$.

Proof. The proofs of (a) and (b) are immediate from the formula (3). Let us prove (c). Indeed, it follows from the condition $\sup_{n\geqslant 0}\sqrt{(n+1)(n+2)}\,|a_n| < +\infty$ that

$$\begin{aligned} ber(T_a) &\leqslant \sup_{\lambda\in\mathbb{D}}\left(|\lambda|\left(1-|\lambda|^2\right)^2\sum_{n=0}^{\infty}|\lambda|^{2n}\right)\sup_{n\geqslant 0}\sqrt{(n+1)(n+2)}\,|a_n| \\ &\leqslant \sup_{n\geqslant 0}\sqrt{(n+1)(n+2)}\,|a_n|\sup_{\lambda\in\mathbb{D}}\left(|\lambda|\left(1-|\lambda|^2\right)^2\frac{1}{1-|\lambda|^2}\right) \\ &= \sup_{n\geqslant 0}\sqrt{(n+1)(n+2)}\,|a_n|\sup_{\lambda\in\mathbb{D}}\left(|\lambda|\left(1-|\lambda|^2\right)\right). \end{aligned}$$

But, it is easy to verify that if $f(x) := x\left(1-x^2\right),\ 0 \leqslant x < 1$, then $\sup_{0\leqslant x<1} f(x) = \frac{2}{3\sqrt{3}}$. This proves (c). □

3. On the solutions of the Beurling problem

It is easy to see from the Beurling theorem on shift-invariant subspaces that an equivalent statement of Beurling's theorem is the following: if $SE \subset E$, then $P_E = T_\theta T_\theta^*$ for some inner function θ. Therefore, if $T \in \mathscr{B}\left(H^2\right)$ be any operator, then it is natural to ask: when $T = T_\varphi T_\varphi^*$ for some function $\varphi \in H^\infty$?

In this section, by using the unicity theorem for Berezin symbols (see, for example Zhu [9, Prop.6.2]) we solve this problem in terms of Berezin symbols. Note that more general problem (namely, Abstract Beurling Problem) is completely solved by Rosenblum and Rovnyak in [4, Problem 1].

Before formulating the Abstract Beurling Problem, let us introduce some necessary notations. Let $\mathscr{H}$ be a Hilbert space. An operator S in $\mathscr{B}(H)$ is a shift operator if S is an isometry and $S^{*n} \to 0$ strongly, that is, $\lim_{n\to\infty}\|S^{*n}f\| = 0$ for all f in $\mathscr{H}$. The operator multiplication by z on $H^2(\mathbb{D})$, defined by $S : f(z) \to zf(z)$ for all $f(z)$

in $H^2(\mathbb{D})$, is a shift operator with adjoint $S^*: f(z) \to [f(z) - f(0)]/z$. For any Hilbert space $\mathcal{G}$, the operator

$$S: (c_0, c_1, c_2, \ldots) \to (0, c_0, c_1, \ldots)$$

on $\ell^2_{\mathcal{G}} = \mathcal{G} \oplus \mathcal{G} \oplus \ldots$ is a shift operator. Its adjoint is $S^*: (c_0, c_1, c_2, \ldots) \to (c_1, c_2, c_3, \ldots)$. An operator $A \in \mathcal{B}(H)$ is S-analytic if $AS = SA$ (see [4]). Let $S \in \mathcal{B}(H)$ be a shift operator.

PROBLEM 1. (Abstract Beurling Problem) Characterize all products AA^*, where A is an S-analytic operator on $\mathcal{H}$.

As we mentioned above, the solution of this problem is contained in [4]. In [4], the authors apply the solution to prove the famous Beurling-Lax theorem. This leads to an inner-outer factorization theory for analytic operators, and hence to an inner-outer factorization theory for operator-valued holomorphic functions (more details can be found in [4]).

The following result was proved in [4].

THEOREM 2. (Solution of Problem 1) *Let $S \in \mathcal{B}(H)$ be a shift operator, and let $K = \ker S^*$. If $T \in \mathcal{B}(H)$, then the following are equivalent:*

(i) $T = AA^*$ for some S-analytic operator $A \in \mathcal{B}(H)$;

(ii) $T - STS^* = J^*J$ for some operator $J \in \mathcal{B}(H, K)$.

Now we give our result.

THEOREM 3. *Let $S \in \mathcal{B}\left(H^2\right)$ be a shift operator, and let $K = \ker S^* = \left(zH^2\right)^{\perp} = H^2 \Theta z H^2$. If $T \in \mathcal{B}\left(H^2\right)$, then the following are equivalent:*

(i) $T = T_{\varphi} T_{\varphi}^*$ for some analytic Toeplitz operator $T_{\varphi} \in \mathcal{B}\left(H^2\right)$;

(ii) $\widetilde{T}(\lambda) = |\varphi(\lambda)|^2$ $(\forall \lambda \in \mathbb{D})$ for some function $\varphi \in H^{\infty}$;

(iii) $\widetilde{T}(\lambda) = \frac{\|J\widehat{k}_{\lambda}\|^2}{1 - |\lambda|^2}$ $(\forall \lambda \in \mathbb{D})$ for some operator $J \in \mathcal{B}\left(H^2, \left(zH^2\right)^{\perp}\right)$.

Proof. Suppose that $T = T_{\varphi} T_{\varphi}^*$ for some $\varphi \in H^{\infty}$. Then, by the unicity theorem for the Berezin symbols (see [9, Prop. 6.2]), representation $T = T_{\varphi} T_{\varphi}^*$ is equivalent to $\widetilde{T} = \widetilde{T_{\varphi} T_{\varphi}^*}$, which means that

$$\begin{aligned}\widetilde{T}(\lambda) &= \left\langle T_{\varphi} T_{\varphi}^* \widehat{k}_{\lambda}, \widehat{k}_{\lambda} \right\rangle = \left\langle T_{\varphi}^* \widehat{k}_{\lambda}, T_{\varphi}^* \widehat{k}_{\lambda} \right\rangle \\ &= \left\langle \overline{\varphi(\lambda)} \widehat{k}_{\lambda}, \overline{\varphi(\lambda)} \widehat{k}_{\lambda} \right\rangle = |\varphi(\lambda)|^2\end{aligned}$$

for all $\lambda \in \mathbb{D}$. Further, $T = T_{\varphi} T_{\varphi}^*$ for some $\varphi \in H^{\infty}$ if and only if $\widetilde{T} - \widetilde{STS^*} = \widetilde{J^*J}$ for some operator $J \in \mathcal{B}\left(H^2, \left(zH^2\right)^{\perp}\right)$. The last equality means that $\widetilde{T}(\lambda) - \widetilde{STS^*}(\lambda) = \left\|J\widehat{k}_{\lambda}\right\|^2$, or

$$\begin{aligned}\left\|J\widehat{k}_\lambda\right\|^2 &= \widetilde{T}(\lambda) - \left\langle STS^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\ &= \widetilde{T}(\lambda) - \left\langle TS^*\widehat{k}_\lambda, S^*\widehat{k}_\lambda\right\rangle \\ &= \widetilde{T}(\lambda) - \left\langle T\overline{\lambda}\widehat{k}_\lambda, \overline{\lambda}\widehat{k}_\lambda\right\rangle \\ &= \widetilde{T}(\lambda) - |\lambda|^2\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\ &= \widetilde{T}(\lambda) - |\lambda|^2\widetilde{T}(\lambda) = \left(1-|\lambda|^2\right)\widetilde{T}(\lambda)\end{aligned}$$

for all $\lambda \in \mathbb{D}$. Thus, $\widetilde{T}(\lambda) = \frac{\|J\widehat{k}_\lambda\|^2}{1-|\lambda|^2}$ for all $\lambda \in \mathbb{D}$. Therefore $(i) \Leftrightarrow (iii)$. By considering that $(i) \Leftrightarrow (ii)$, we assert that $(ii) \Leftrightarrow (iii)$. Hence $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$, as desired. □

The following result, which is an immediate corollary of Theorem 3, apparently has an independent interest.

PROPOSITION 1. *For every function $f \in H^\infty$ there exists a 1-dimensional operator (compact operator) $J_f : H^2 \to \left(zH^2\right)^\perp$ such that:*

(i) $|f(\lambda)| = \frac{\|J_f\widehat{k}_\lambda\|^2}{1-|\lambda|^2}$ *for all* $\lambda \in \mathbb{D}$;

(ii) $\|f\|_{H^\infty} = \sup_{\lambda\in\mathbb{D}} \frac{\|J_f\widehat{k}_\lambda\|^2}{1-|\lambda|^2}$.

Proof. Indeed, for every $f \in H^\infty$, let us denote $J_f := (I-SS^*)T_f^*$. Since $I-SS^* : H^2 \to \left(zH^2\right)^\perp$ is a 1-dimensional operator, $T_f^*\widehat{k}_\lambda = \overline{f(\lambda)}\widehat{k}_\lambda$ and $\left\|(I-SS^*)\widehat{k}_\lambda\right\| = \left(1-|\lambda|^2\right)^{1/2}$ for all $\lambda \in \mathbb{D}$, formula (i) follows immediately. It is obviously that $(i) \Rightarrow (ii)$, which completes the proof. □

4. Bounded and Invertible Toeplitz product

Let $u \in L^\infty = L^\infty(\mathbb{T}, dm)$. The Toeplitz operator T_u with symbol u on the Hardy space H^2 is defined as

$$T_u f = P_+(uf),$$

where $P_+ : L^2(\mathbb{T}) \to H^2$ denotes the orthogonal Riesz projection. Basic properties of Toeplitz operators are that $T_u^* = T_{\overline{u}}$ for any $u \in L^\infty$ and $T_vT_uT_w = T_{vuw}$ if and only if $v \in \overline{H^\infty}$ and $w \in H^\infty$.

Let $h \in L^2(\mathbb{T})$, define the Toeplitz operator T_h on H^2 by $T_hP = P_+(hp)$ for polynomials P. It is well-known that T_h is bounded on H^2 if and only if $h \in L^\infty(\mathbb{T}, dm)$. However, Sarason [5, 6] found examples f and g in H^2 such that the product $T_fT_{\overline{g}}$ is actually a bounded operator on H^2, though neither T_f nor T_g is bounded (see for more informations Sarason [6] and Stroethoff and Zheng [7]).

In this section we give some results concerning the boundedness of Berezin numbers and boundedness below of Toeplitz products on the Hardy space.For more results about boundedness and invertibility of operators $T_f T_{\overline{g}}$ in L_a^2 and H^2, see [7] and its references.

For any $u \in L^1(\mathbb{T})$, $\widehat{u}(\lambda)$ denotes the Poisson extension of u over the unit disc $\mathbb{D}$:

$$\widehat{u}(\lambda) = \int_{\mathbb{T}} u(\xi) \frac{1-|\lambda|^2}{\left|1-\overline{\lambda}\xi\right|^2} dm(\xi).$$

THEOREM 4. *Let $f, g \in H^2$. Then we have:*
(a) if $\sup_{\lambda \in \mathbb{D}} \left|\widetilde{T_f T_{\overline{g}}}(\lambda)\right| < +\infty$, (i.e., $ber\left(T_f T_{\overline{g}}\right) < +\infty$), then

$$\sup_{\lambda \in \mathbb{D}} |f(\lambda)| \, |g(\lambda)| \leqslant ber\left(T_f T_{\overline{g}}\right);$$

(b) if $T_f T_{\overline{g}}$ is a bounded and bounded below operator on H^2, then there exist $C_1, C_2 > 0$ such that

$$C_1 \leqslant |g(\lambda)|^2 \widehat{|f|^2}(\lambda) \leqslant C_2$$

for all $\lambda \in \mathbb{D}$.

Proof. (a) Since $k_{H^2,\lambda}(z) = \frac{1}{1-\overline{\lambda}z} \in H^\infty$ for each $\lambda \in \mathbb{D}$, the Berezin symbol $\widetilde{T_f T_{\overline{g}}}$ of the Toeplitz product $T_f T_{\overline{g}}$ is well defined. Then we have:

$$\begin{aligned}
\widetilde{T_f T_{\overline{g}}}(\lambda) &= \left\langle T_f T_{\overline{g}} \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle = \left\langle T_f \overline{g}(\lambda) \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle \\
&= \overline{g(\lambda)} \left\langle T_f \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle = \overline{g(\lambda)} \left\langle f(z) \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle \\
&= \overline{g(\lambda)} f(\lambda) \frac{k_{H^2,\lambda}(\lambda)}{\left\|k_{H^2,\lambda}(\lambda)\right\|^2} \\
&= \overline{g(\lambda)} f(\lambda) \frac{\left\|k_{H^2,\lambda}(\lambda)\right\|^2}{\left\|k_{H^2,\lambda}(\lambda)\right\|^2} = \overline{g(\lambda)} f(\lambda)
\end{aligned}$$

for all $\lambda \in \mathbb{D}$. So,

$$\widetilde{T_f T_{\overline{g}}}(\lambda) = f(\lambda) \overline{g(\lambda)} \ (\lambda \in \mathbb{D}),$$

which implies that if $ber\left(T_f T_{\overline{g}}\right) < +\infty$, then

$$\left| f(\lambda) \overline{g(\lambda)} \right| = |f(\lambda)| \, |g(\lambda)| = \left|\widetilde{T_f T_{\overline{g}}}(\lambda)\right| \leqslant ber\left(T_f T_{\overline{g}}\right) < +\infty$$

for all $\lambda \in \mathbb{D}$, which means that

$$\sup_{\lambda \in \mathbb{D}} |f(\lambda)| \, |g(\lambda)| \leqslant ber\left(T_f T_{\overline{g}}\right) < +\infty,$$

as desired.

(b) Since $T_f T_{\overline{g}}$ is bounded on H^2, $\sup_{\|h\|=1} \left\| T_f T_{\overline{g}} h \right\| < +\infty$. In particular, $\sup_{\lambda \in \mathbb{D}} \left\| T_f T_{\overline{g}} \widehat{k}_{H^2,\lambda} \right\| < +\infty$. On the other hand,

$$\begin{aligned}
\left\| T_f T_{\overline{g}} \widehat{k}_{H^2,\lambda} \right\|^2 &= \left\langle T_f T_{\overline{g}} \widehat{k}_{H^2,\lambda}, T_f T_{\overline{g}} \widehat{k}_{H^2,\lambda} \right\rangle \\
&= \left\langle T_{\overline{f}} T_f T_{\overline{g}} \widehat{k}_{H^2,\lambda}, T_{\overline{g}} \widehat{k}_{H^2,\lambda} \right\rangle \\
&= g(\lambda) \left\langle T_{|f|^2} \overline{g(\lambda)} \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle \\
&= |g(\lambda)|^2 \left\langle T_{|f|^2} \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle \\
&= |g(\lambda)|^2 \int_{\mathbb{T}} \frac{1-|\lambda|^2}{\left|1-\overline{\lambda}\xi\right|^2} |f(\xi)|^2 \, dm(\xi) = |g(\lambda)|^2 \widehat{|f|^2}(\lambda),
\end{aligned}$$

for all $\lambda \in \mathbb{D}$. Thus

$$\left\| T_f T_{\overline{g}} \widehat{k}_{H^2,\lambda} \right\|^2 = |g(\lambda)|^2 \widehat{|f|^2}(\lambda) \ \ (\lambda \in \mathbb{D}),$$

from which we assert that

$$\sup_{\lambda \in \mathbb{D}} |g(\lambda)|^2 \widehat{|f|^2}(\lambda) < +\infty. \tag{8}$$

On the other hand, since by condition $T_f T_{\overline{g}}$ is bounded below, we have that there exists $C > 0$ such that $\left\| T_f T_{\overline{g}} h \right\| \geqslant C \|h\|$ for each $h \in H^2$. In particular, putting $h = \widehat{k}_{H^2,\lambda}$, we have

$$\left\| T_f T_{\overline{g}} \widehat{k}_{H^2,\lambda} \right\|^2 \geqslant C \left\| \widehat{k}_{H^2,\lambda} \right\|^2 = C$$

or

$$\begin{aligned}
C &\leqslant \left\langle T_f T_{\overline{g}} \widehat{k}_{H^2,\lambda}, T_f T_{\overline{g}} \widehat{k}_{H^2,\lambda} \right\rangle = \left\langle T_f \overline{g(\lambda)} \widehat{k}_{H^2,\lambda}, T_f \overline{g(\lambda)} \widehat{k}_{H^2,\lambda} \right\rangle \\
&= |g(\lambda)|^2 \left\langle f \widehat{k}_{H^2,\lambda}, f \widehat{k}_{H^2,\lambda} \right\rangle = |g(\lambda)|^2 \widehat{|f|^2}(\lambda)
\end{aligned}$$

for all $\lambda \in \mathbb{D}$. So,

$$C \leqslant |g(\lambda)|^2 \widehat{|f|^2}(\lambda) \ \ (\lambda \in \mathbb{D}). \tag{9}$$

It follows from (8) and (9) the desired result, which proves (b). The theorem is proved. □

COROLLARY 2. *If $T_f T_{\overline{g}}$ is bounded in H^2 and invertible in H^2, then*

$$\sup_{\lambda \in \mathbb{D}} |g(\lambda)|^2 \widehat{|f|^2}(\lambda) < +\infty$$

and

$$\inf_{\lambda \in \mathbb{D}} |g(\lambda)|^2 \widehat{|f|^2}(\lambda) > 0.$$

Before stating our next results, we need some auxiliary results related with so-called atomic decomposition in Bergman spaces $L_a^2(\mathbb{D})$ (for more details see Zhu [8]).

Recall that for $\lambda \in \mathbb{D}$, φ_λ is the innovative Möbius transformation of $\mathbb{D}$ which interchanges the origin and λ, namely,

$$\varphi_\lambda(z) := \frac{\lambda - z}{1 - \overline{\lambda} z},\ z \in \mathbb{D}.$$

The pseudo-hyperbolic distance on $\mathbb{D}$ is defined by

$$\rho(\lambda, z) = |\varphi_\lambda(z)| = \left| \frac{\lambda - z}{1 - \overline{\lambda} z} \right|,\ \lambda, z \in \mathbb{D}.$$

The most important property of the pseudo-hyperbolic distance is that it is Möbius invariant, that is,

$$\rho(\varphi(\lambda), \varphi(z)) = \rho(\lambda, z)$$

for all $\varphi \in Aut(\mathbb{D})$, the Möbius group of $\mathbb{D}$, and all $\lambda, z \in \mathbb{D}$.

The Bergman metric on $\mathbb{D}$, also called the hyperbolic metric, is given by $\sqrt{H(\lambda)}ds$, where ds is the Euclidean length element and

$$H(\lambda) = \frac{1}{2} \frac{\partial^2}{\partial \lambda \partial \overline{\lambda}} \log K(\lambda, \lambda) = \frac{1}{\left(1 - |\lambda|^2\right)^2},$$

here $K(z, \lambda) = \frac{1}{1 - \lambda \overline{z}}$.

It is well known that the induced distance on $\mathbb{D}$ is given by

$$\beta = \frac{1}{2} \log \frac{1 + \rho(\lambda, z)}{1 - \rho(\lambda, z)},\ \lambda, z \in \mathbb{D}.$$

The Bergman metric is also Möbius invariant

$$\beta(\varphi(\lambda), \varphi(z)) = \beta(\lambda, z)$$

for all $\varphi \in Aut(\mathbb{D})$ and all $\lambda, z \in \mathbb{D}$.

For any $z \in \mathbb{D}$ and $r > 0$, let

$$\mathbb{D}(\lambda, r) := \{z \in \mathbb{D} : \beta(\lambda, z) < r\}$$

be the Bergman metric disc with "center" λ and "radius" r.

LEMMA 2. *There is a positive integer N such that for any $r \leqslant 1$, there exists a sequence $\{\lambda_n\}$ in $\mathbb{D}$ satisfying the following conditions:*

(i_1) $\mathbb{D} = \cup_{n=1}^{\infty} \mathbb{D}(\lambda_n, r)$;

(i_2) $\mathbb{D}\left(\lambda_n, \frac{r}{4}\right) \cap \mathbb{D}\left(\lambda_m, \frac{r}{4}\right) = \emptyset$ *if* $n \neq m$;

(i_3) *Any point in $\mathbb{D}$ belongs to at most N of the sets $\mathbb{D}(\lambda_n, 2r)$.*

For any $0 < r \leqslant 1$, we will fix a sequence $\{\lambda_n\}$ in $\mathbb{D}$ with the properties stated in Lemma 2.

LEMMA 3. *For any $r > 0$, there is a constant $C > 0$ (depending on r) such that*

$$\sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right)^2|f(\lambda_n)|^p \leqslant C\int_{\mathbb{D}}|f(z)|^p\,dA(z)$$

for all analytic f and $p \geqslant 1$.

LEMMA 4. *Let $r \leqslant 1$, $s = \tanh r$, and $p \geqslant 1$; then there exists a constant $C > 0$ (independent of r) such that*

$$\sum_{n=1}^{\infty}\int_{\mathbb{D}_n}|f(z)-f(\lambda_n)|^p\,dA(z) \leqslant Cs^p\int_{\mathbb{D}}|f(z)|^p\,dA(z)$$

for all $f \in L_a^2$.

The following atomic decomposition is proved by using Lemmas 2-4 (see Zhu [8]).

THEOREM 5. *If $p > 1$, and $\frac{1}{p}+\frac{1}{q} = 1$, then there exists a sequence $\{\lambda_n\}$ in $\mathbb{D}$ and a constant $C > 0$ with the following properties:*

(i_1) *For any $\{a_n\}$ in ℓ^p, the function*

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{\left(1-|\lambda_n|^2\right)^{\frac{2}{q}}}{\left(1-\overline{\lambda}_n z\right)^2}$$

is in $L_a^p(\mathbb{D})$ with

$$\|f\|_{L_a^p} \leqslant C\,\|\{a_n\}\|_{\ell^p};$$

(i_2) *If $f \in L_a^p$, then there is $\{a_n\}$ in ℓ^p such that*

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{\left(1-|\lambda_n|^2\right)^{\frac{2}{q}}}{\left(1-\overline{\lambda}_n z\right)^2}$$

and

$$\|\{a_n\}\|_{\ell^p} \leqslant C\,\|f\|_{L_a^p}.$$

Now we state our next result.

THEOREM 6. *Let* f *and* g *be in* L_a^2. *Let* $\Lambda := \{\lambda_n\}_{n\geqslant 1}$ *and* $C>0$ *are the same as in Theorem 5. Let* $T_f T_{\overline{g}}$ *be a bounded Toeplitz product on* L_a^2 *satisfying*

(i_1) $\nu := \sum_{n=1}^{\infty} |g(\lambda_n)|^2 \left\| (f - f(\lambda_n)) \widehat{k}_{L_a^2,\lambda_n} \right\|^2 < +\infty;$

(i_2) $\nu^* := \sum_{n=1}^{\infty} |f(\lambda_n)|^2 \left\| (g - g(\lambda_n)) \widehat{k}_{L_a^2,\lambda_n} \right\|^2 < +\infty;$

If there exists $\delta > 0$ *such that*

$$|f(\lambda)|\,|g(\lambda)| \geqslant \delta > \max\left(C^3\nu, C^3\nu^*\right)$$

for all $\lambda \in \mathbb{D}$, *then* $T_f T_{\overline{g}}$ *is invertible in* L_a^2 *and*

$$\left\| \left(T_f T_{\overline{g}}\right)^{-1} \right\| \leqslant \frac{C^2}{\delta - C^3\nu}.$$

Proof. Since $\widetilde{T_f T_{\overline{g}}}(\lambda) = f(\lambda)\overline{g(\lambda)}$ and $\widetilde{\left(T_f T_{\overline{g}}\right)^*}(\lambda) = \widetilde{T_g T_{\overline{f}}}(\lambda) = g(\lambda)\overline{f(\lambda)}$ for all $\lambda \in \mathbb{D}$, by using (i_1) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\| T_f T_{\overline{g}} \widehat{k}_{L_a^2,\lambda_n} - \widetilde{T_f T_{\overline{g}}}(\lambda_n) \widehat{k}_{L_a^2,\lambda_n} \right\|^2 \\
&= \sum_{n=1}^{\infty} \left\| T_f \overline{g(\lambda_n)} \widehat{k}_{L_a^2,\lambda_n} - f(\lambda_n)\overline{g(\lambda_n)} \widehat{k}_{L_a^2,\lambda_n} \right\|^2 \\
&= \sum_{n=1}^{\infty} |g(\lambda_n)|^2 \left\| T_f \widehat{k}_{L_a^2,\lambda_n} - f(\lambda_n) \widehat{k}_{L_a^2,\lambda_n} \right\| \\
&= \sum_{n=1}^{\infty} |g(\lambda_n)|^2 \left\| (f - f(\lambda_n)) \widehat{k}_{L_a^2,\lambda_n} \right\|^2 = \nu < +\infty.
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\| T_g T_{\overline{f}} \widehat{k}_{L_a^2,\lambda_n} - \widetilde{T_g T_{\overline{f}}}(\lambda_n) \widehat{k}_{L_a^2,\lambda_n} \right\|^2 \\
&= \sum_{n=1}^{\infty} |f(\lambda_n)|^2 \left\| (g - g(\lambda_n)) \widehat{k}_{L_a^2,\lambda_n} \right\|^2 = \nu^* < +\infty.
\end{aligned}$$

Now by using the condition of theorem and Theorem 3.3 in [3], we deduce invertibility of the operator $T_f T_{\overline{g}}$ in L_a^2, and also the inequality

$$\left\| \left(T_f T_{\overline{g}}\right)^{-1} \right\| \leqslant \frac{C^2}{\delta - C^3\nu}. \quad \square$$

COROLLARY 3. *Let* $g \in L_a^2$ *and* $f \in L_a^4(\mathbb{D})$ *be functions such that* $T_f T_{\overline{g}}$ *is bounded in* L_a^2 *and*

$$\tau := \sum_{n=1}^{\infty} \frac{|g(\lambda_n)|^2}{1 - |\lambda_n|^2} \left(\int_{\mathbb{D}} |f(z) - f(\lambda_n)|^4 \, dA(z) \right)^{1/2} < +\infty.$$

If there exists $\delta > 0$ *such that*

$$|f(\lambda)||g(\lambda)| \geqslant \delta > C^3\tau$$

for all $\lambda \in \mathbb{D}$, *then* $T_f T_{\overline{g}}$ *is a bounded below operator on the Bergman space* $L_a^2(\mathbb{D})$.

Proof. It suffices to show that condition (i_1) of Theorem 6 is satisfied. Indeed, first note that the norm of the reproducing kernel $k_{L_a^2,\lambda}(z) = \frac{1}{(1-\overline{\lambda}z)^2}$ in L_a^p is comparable to $\left(1-|\lambda|^2\right)^{-2/q}$, where $\frac{1}{p}+\frac{1}{q}=1$. Thus, for any $\lambda \in \mathbb{D}$, the function $\left(1-|\lambda|^2\right)^{2/q}\left(1-\overline{\lambda}z\right)^{-2}$ is comparable to a unit vector in L_a^p. By considering this and using that $q=\frac{4}{3}$ for $p=4$, we have:

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z)-f(\lambda_n)|^2 \left|\widehat{k}_{L_a^2,\lambda_n}(z)\right|^2 dA(z) \\
&\leqslant \left(\int_{\mathbb{D}} \left(|f(z)-f(\lambda_n)|^2\right)^2 dA(z)\right)^{1/2} \left(\int_{\mathbb{D}} \left(\left|\widehat{k}_{L_a^2,\lambda_n}(z)\right|^2\right)^2 dA(z)\right)^{1/2} \\
&= \left(\int_{\mathbb{D}} |f(z)-f(\lambda_n)|^4 dA(z)\right)^{1/2} \left(\int_{\mathbb{D}} \frac{\left(1-|\lambda_n|^2\right)^4}{\left|1-\overline{\lambda}_n z\right|^8} dA(z)\right)^{1/2} \\
&= \left(\int_{\mathbb{D}} |f(z)-f(\lambda_n)|^4 dA(z)\right)^{1/2} \left(\int_{\mathbb{D}} \frac{\left(1-|\lambda_n|^2\right)^{\frac{2}{\frac{4}{3}}\cdot 4} \frac{\left(1-|\lambda_n|^2\right)^4}{\left(1-|\lambda_n|^2\right)^6}}{\left|1-\overline{\lambda}_n z\right|^{2\cdot 4}} dA(z)\right)^{1/2} \\
&= \frac{1}{1-|\lambda_n|^2} \left(\int_{\mathbb{D}} |f(z)-f(\lambda_n)|^4 dA(z)\right)^{1/2} \left(\left(\int_{\mathbb{D}} \left|\frac{\left(1-|\lambda_n|^2\right)^{\frac{3}{2}}}{\left|1-\overline{\lambda}_n z\right|^2}\right|^4 dA(z)\right)^{1/4}\right)^2.
\end{aligned}$$

Now by considering that

$$\left(\int_{\mathbb{D}} \left|\frac{\left(1-|\lambda_n|^2\right)^{\frac{3}{2}}}{\left|1-\overline{\lambda}_n z\right|^2}\right|^4 dA(z)\right)^{1/4} \asymp 1,$$

we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} |g(\lambda_n)|^2 \left\|(f-f(\lambda_n))\widehat{k}_{L_a^2,\lambda_n}\right\|^2 \\
&\leqslant \sum_{n=1}^{\infty} \frac{|g(\lambda_n)|^2}{1-|\lambda_n|^2} \left(\int_{\mathbb{D}} |f(z)-f(\lambda_n)|^4 dA(z)\right)^{1/2} = \tau < +\infty,
\end{aligned}$$

and hence, as in the proof of Theorem 4, it follows from the proof of Theorem 3.3 in [3] that

$$\left\|T_f T_{\overline{g}} h\right\|_{L_a^2} \geqslant \left(\frac{\delta}{C^2} - C\tau\right) \|h\|_{L_a^2}$$

for all $h \in L_a^2$, which proves the corollary. □

COROLLARY 4. *Let $g \in L_a^2$ and $f \in L_a^2$ be functions such that there exists $M_1, M_2 > 0$ such that*

$$\int_{\mathbb{D}} |f(z) - f(\lambda_n)|^2 \left|\widehat{k}_{L_a^2,\lambda_n}(z)\right|^2 dA(z) \leqslant M_1 \left(1 - |\lambda_n|^2\right)^2$$
$$\int_{\mathbb{D}} |g(z) - g(\lambda_n)|^2 \left|\widehat{k}_{L_a^2,\lambda_n}(z)\right|^2 dA(z) \leqslant M_2 \left(1 - |\lambda_n|^2\right)^2$$

for all $n \geqslant 1$. If $|f(\lambda)||g(\lambda)| \geqslant \delta > \max\left(C^3\nu, C^3\nu^*\right)$ and $T_f T_{\overline{g}}$ is bounded in L_a^2, then $T_f T_{\overline{g}}$ is invertible in L_a^2.

Proof. Indeed, by using Lemma 3 for $r = 1$ and $p = 2$, we have:

$$\begin{aligned}
&\sum_{n=1}^{\infty} |g(\lambda_n)|^2 \left\|(f(z) - f(\lambda_n))\widehat{k}_{L_a^2,\lambda_n}\right\|^2 \\
&\leqslant \sum_{n=1}^{\infty} |g(\lambda_n)|^2 \int_{\mathbb{D}} |f(z) - f(\lambda_n)|^2 \left|\widehat{k}_{L_a^2,\lambda_n}(z)\right|^2 dA(z) \\
&\leqslant M_1 \sum_{n=1}^{\infty} \left(1 - |\lambda_n|^2\right)^2 |g(\lambda_n)|^2 < +\infty,
\end{aligned}$$

and analogously,

$$\begin{aligned}
&\sum_{n=1}^{\infty} |f(\lambda_n)|^2 \left\|(g(z) - g(\lambda_n))\widehat{k}_{L_a^2,\lambda_n}\right\|^2 \\
&\leqslant M_2 \sum_{n=1}^{\infty} \left(1 - |\lambda_n|^2\right)^2 |f(\lambda_n)|^2 < +\infty,
\end{aligned}$$

and hence Theorem 6 works. □

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