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THE λ–INTERSECTION BODIES AND AN ANALYTIC GENERALIZED BUSEMANN–PETTY PROBLEM

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Abstract. In this paper, we obtain an extension of connections between an analytic generalization of the Busemann-Petty problem and positive definite distributions. We show that the class of the λ-intersection bodies is closely related to the analytic generalized Busemann-Petty problem.

1. Introduction

We call a compact set K with non-empty interior in $\mathbb{R}^n$, $n \geqslant 2$, a star body if $tK \subseteq K$, $\forall t \in [0,1]$, and the radial function $\rho_K(\theta) = \sup\{\lambda \geqslant 0 : \lambda\theta \in K\}$ is continuous on the unit sphere S^{n-1}. A compact, convex set in $\mathbb{R}^n$ is said to be a convex body if it has non empty interior. To make the presentation simple, we introduce some symbols. We denote by $\mathscr{S}^n$ the set of all origin-symmetric star bodies in $\mathbb{R}^n$, and by $G_{n,i}$ the Grassmann manifold of i-dimensional linear subspaces of $\mathbb{R}^n$. We shall use $\mathtt{vol}_i(\cdot)$ to denote the i-dimensional volume function. Instead of $\mathtt{vol}_n(\cdot)$ we usually write $V(\cdot)$. The volume of the standard unit ball B_n and unit sphere S^{n-1} in $\mathbb{R}^n$ are denoted by ω_n and σ_{n-1}, respectively. The Minkowski functional of a body $K \in \mathscr{S}^n$ is defined by $\|x\|_K = \min\{a \geqslant 0 : x \in aK\}$, so that $\|\theta\|_K = \rho_K^{-1}(\theta)$, $\theta \in S^{n-1}$.

The class of intersection bodies plays an important role in solving the Busemann-Petty problem that was first posed in 1956 (see [4]): Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$\mathtt{vol}_{n-1}(K \cap \theta^\perp) \leqslant \mathtt{vol}_{n-1}(L \cap \theta^\perp), \ \ \forall\, \theta \in S^{n-1},$$

where $\theta^\perp = \{x \in \mathbb{R}^n : \langle x, \theta \rangle = 0\}$ is the central hyperplane orthogonal to θ. Does it follow that

$$V(K) \leqslant V(L)?$$

Much work has been devoted to the study of the problem ([1, 2, 5, 6, 7, 8, 10, 14, 15, 16, 19, 20]; see [15] for the history of the solution). The problem was completely solved at the end of 1990's, and the answer is affirmative if $n \leqslant 4$ and negative

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if $n \geqslant 5$. One of the main ingredients of the solution was a connection between intersection bodies and the Busemann-Petty problem established by Lutwak [15]: if K is an intersection body, then the answer to the problem is affirmative for any star body L. On the other hand, if L is a symmetric convex body but not an intersection body, then one can construct K and L in $\mathbb{R}^n$ providing a counterexample.

We say that $K \in \mathscr{S}^n$ is the intersection body of $L \in \mathscr{S}^n$ if $\rho_K(\theta) = \mathtt{vol}_{n-1}(L \cap \theta^\perp)$ for every $\theta \in S^{n-1}$. A body $K \in \mathscr{S}^n$ is said to be an intersection body if K is the limit in the radial metric of intersection bodies of star bodies, i.e., there exists a sequence of bodies $\{K_i\}_{i\in\mathbb{N}}$ satisfying $\lim_{j\to\infty} \|\rho_{K_j} - \rho_K\|_{C(S^{n-1})} = 0$, where K_i is the intersection body of $L_i \in \mathscr{S}^n$ for all $i \in \mathbb{N}$.

In [21], Zhang first considered the Generalized Busemann-Petty problem: For each fixed $1 \leqslant i < n$, let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ satisfying

$$\mathtt{vol}_i(K \cap \xi) \leqslant \mathtt{vol}_i(L \cap \xi), \quad \forall \xi \in G_{n,i}.$$

Does it follow that

$$V(K) \leqslant V(L)?$$

Zhang showed that the Generalized Busemann-Petty problem is related to Zhang's class which is a generalization of the class of intersection bodies (see [21]). It was showed in [3] that the Generalized Busemann-Petty problem has a negative answer when $3 < i < n$ (see also [11, 18]). For $i = 2$ and $i = 3$ $(n \geqslant 5)$ the Generalized Busemann-Petty problem is still open.

Another generalization of the concept of the intersection body was suggested in [12] and described in detail in [11]. Namely, a body $K \in \mathscr{S}^n$ is a k-intersection body of $L \in \mathscr{S}^n$ (we write $K = \mathscr{I}\mathscr{B}_k(L)$) if

$$\mathtt{vol}_k(K \cap \xi) = \mathtt{vol}_{n-k}(L \cap \xi^\perp) \;\; \forall \xi \in G_{n,k}. \tag{1.1}$$

We denote by $\mathscr{I}\mathscr{B}_{k,n}$ the set of all bodies $K \in \mathscr{S}^n$ satisfying (1.1) for some $L \in \mathscr{S}^n$.

An origin symmetric star body K is said to be a k-intersection body if K is the limit in the radial metric of k-intersection bodies $\{K_i\}$ of star bodies $\{L_i\}$. The class of k-intersection bodies in $\mathbb{R}^n$ is denoted by $\mathscr{I}_k^n$.

LEMMA 1.1. (see [11] Theorem 4.8) *A body $K \in \mathscr{S}^n$ is a k-intersection body if and only if $\|\cdot\|_K^{-k}$ represents a positive definite tempered distribution on $\mathbb{R}^n$, that is, the Fourier transform $(\|\cdot\|_K^{-k})^\wedge$ is a positive tempered distribution on $\mathbb{R}^n$.*

By using the concept of k-intersection body, Koldobsky [12] generalized Lutwak's connections as follows:

THEOREM A. *Let $1 \leqslant k < n$, and let K, L be origin-symmetric $(k-1)$-smooth star bodies in $\mathbb{R}^n$ if k is odd and k-smooth if k is even. Suppose that the functions*

$$\|\cdot\|_K^{-k} \;\; and \;\; \|\cdot\|_L^{-n+k} - \|\cdot\|_K^{-n+k}$$

represent positive definite distributions in $\mathbb{R}^n \setminus \{0\}$. Then

$$V(K) \leqslant V(L).$$

THEOREM B. *Let $1 \leqslant k < n$, and suppose there exists an origin-symmetric convex body D in $\mathbb{R}^n$ for which $\|\cdot\|_D^{-k}$ is not a positive definite distribution. Then there exist origin-symmetric infinitely smooth convex bodies K and L so that $\|\cdot\|_L^{-n+k} - \|\cdot\|_K^{-n+k}$ is a positive definite distribution in $\mathbb{R}^n \setminus \{0\}$, but*

$$V(K) > V(L).$$

Note that the condition that $\|\cdot\|_K^{-k}$ is a positive definition distribution in $\mathbb{R}^n \setminus \{0\}$ can be replaced by $K \in \mathscr{I}_k^n$, and the condition that $\|\cdot\|_L^{-n+k} - \|\cdot\|_K^{-n+k}$ is a positive definite distribution in $\mathbb{R}^n \setminus \{0\}$ is equivalent to that the Fourier transform of $\|\cdot\|_L^{-n+k} - \|\cdot\|_K^{-n+k}$ is a positive distribution in $\mathbb{R}^n \setminus \{0\}$.

In [17], Rubin introduced a more general class of intersection bodies than the one of the k-intersection bodies, namely λ-intersection bodies.

Let λ be a real number,

$$s_\lambda = \begin{cases} 1, & \text{if } \lambda > 0, \ \ \lambda \neq n, n+2, n+4, \cdots; \\ \Gamma(\lambda/2), & \text{if } \lambda < 0, \ \lambda \neq -2, -4, -6, -8, \cdots. \end{cases}$$

DEFINITION 1. For $\lambda < n$ and $\lambda \neq 0$, the body $K \in \mathscr{S}^n$ is called a λ-intersection body if there is a measure $\mu \in \mathscr{M}_{e+}(S^{n-1})$ such that $s_\lambda \rho_K^\lambda = M^{1-\lambda}\mu$ if $\lambda \neq -2l, l \in \mathbb{N}$, and $\rho_K^\lambda = \tilde{M}^{1-\lambda}\mu$, otherwise. The class of λ-intersection bodies is denoted by $\mathscr{I}_\lambda^n$.

The equality $s_\lambda \rho_K^\lambda = M^{1-\lambda}\mu$ means that for any $\varphi \in \mathscr{D}(S^{n-1})$,

$$s_\lambda \int_{S^{n-1}} \rho_K^\lambda(\theta)\varphi(\theta)d\theta = \int_{S^{n-1}} (M^{1-\lambda}\varphi)(\theta)d\mu(\theta),$$

where for $\lambda \geqslant 1$, $(M^{1-\lambda}\varphi)(\theta)$ is understood in the sense of analytic continuation.

REMARK 1. (1). If $\lambda = k \in \{1, 2, \cdots, n-1\}$, then $\mathscr{I}_\lambda^n$ coincides with the class of k-intersection bodies which follows from Lemma 1.1 and Lemma 2.2(b) (see Section 2).

(2). The case $\lambda > n$ is not so interesting, because by Lemma 2.2(c), $\mathscr{I}_\lambda^n$ is either empty (if $\Gamma((n-\lambda)/2) < 0$) or coincides with the whole class S^n (if $\Gamma((n-\lambda)/2) > 0$).

The generalized Minkowski-Funk transforms denoted by $M^\alpha f$ ($\tilde{M}^\alpha f$) were also introduced by Rubin [17] (see Section 2 for precise definitions).

In this paper, we give an analogue of Theorem A and Theorem B for the class of λ-intersection bodies. Despite the fact that the geometric models of λ-intersection bodies are clear (see [17]), our results are analytic ones. We extend the condition $K \in \mathscr{I}_k^n, k \in \{1, 2, \cdots, n-1\}$ to $K \in \mathscr{I}_\lambda^n$, $\lambda < n$ and $\lambda \neq 0$, and replace the condition that the Fourier transform of $\|\cdot\|_L^{-n+k} - \|\cdot\|_K^{-n+k}$ is a positive distribution by the condition that the generalized Minkowski-Funk transform of $\|\cdot\|_L^{-n+k} - \|\cdot\|_K^{-n+k}$ is a positive distribution.

We now describe our main results.

THEOREM 1. *Let K, L be origin-symmetric infinitely smooth star bodies in $\mathbb{R}^n$.*
(1). *Let $0<\lambda<n$, $K \in \mathscr{I}_\lambda^n$. Suppose that the function*

$$M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - M^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$$

represents a positive distribution in $\mathbb{R}^n \setminus \{0\}$. Then

$$V(K) \leqslant V(L).$$

(2). *Let $\lambda<0$, $\lambda \neq -2l$, $l \in \mathbb{N}$, $L \in \mathscr{I}_\lambda^n$. Suppose that the function*

$$s_\lambda M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - s_\lambda M^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$$

represents a positive distribution in $\mathbb{R}^n \setminus \{0\}$. Then

$$V(K) \leqslant V(L).$$

(3). *Let $\lambda = -2l$, $l \in \mathbb{N}$, and $L \in \mathscr{I}_\lambda^n$. Suppose that the function*

$$\tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - \tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$$

represents a positive distribution in $\mathbb{R}^n \setminus \{0\}$. Then

$$V(K) \leqslant V(L).$$

If $K \notin \mathscr{I}_\lambda^n$ (or $L \notin \mathscr{I}_\lambda^n$), the solution of Theorem 1 does not hold. In those cases, we construct a counterexample to the Analytic Busemann-Petty problem.

THEOREM 2. (1). *Let $0<\lambda<n$, $L \notin \mathscr{I}_\lambda^n$. There are origin-symmetric star bodies K, L in $\mathbb{R}^n$ such that*

$$M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - M^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$$

is a positive distribution in $\mathbb{R}^n \setminus \{0\}$, but

$$V(K) > V(L).$$

(2). *Let $\lambda<0$, $\lambda \neq -2l$, $l \in \mathbb{N}$, $K \notin \mathscr{I}_\lambda^n$. There are origin-symmetric star bodies K, L in $\mathbb{R}^n$ such that*

$$s_\lambda M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - s_\lambda M^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$$

is a positive distribution in $\mathbb{R}^n \setminus \{0\}$, but

$$V(K) > V(L).$$

(3). *Let $\lambda = -2l$, $l \in \mathbb{N}$, $K \notin \mathscr{I}_\lambda^n$. There are origin-symmetric star bodies K, L in $\mathbb{R}^n$, such that*

$$\tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - \tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$$

is a positive distribution in $\mathbb{R}^n \setminus \{0\}$, but

$$V(K) > V(L).$$

REMARK 2. The classical Lutwak's connection is the special case $\lambda = 1$ in Theorem 1 and Theorem 2. In fact, $\lambda = 1$, $\mathscr{I}_\lambda^n$ will be the class of intersection bodies. From the properties of generalized Minkowski-Funk transforms (2.3), it follows:

$$\begin{aligned}\lim_{\lambda\to 1} M^{1-\lambda}\|\theta\|_K^{-n+\lambda} &= C_{n-1}M\|\theta\|_K^{-n+1}\\ &= C_{n-1}\int_{S^{n-1}\cap\theta^\perp}\|u\|_K^{-n+1}d_\theta u,\\ &= \frac{(n-1)C_{n-1}}{\sigma_{n-1}}\int_{S^{n-1}\cap\theta^\perp}\|u\|_K^{-n+1}du,\\ &= \frac{(n-1)C_{n-1}}{\sigma_{n-1}}\mathtt{vol}_{n-1}(K\cap\theta^\perp),\end{aligned}$$

where $C_{n-1} = \frac{\sigma_{n-2}}{2\pi^{(n-2)/2}}$. Therefore, if $\lambda = 1$, Theorem 1 and Theorem 2 will be two parts of Lutwak's connection.

In this paper, we consider all cases $\lambda < n$, $\lambda \neq 0$ for the connection between λ-intersection bodies and Analytic Busemann-Petty problem. The classical Lutwak's connection is only the case $\lambda = 1$ in our results. Koldobsky's extension, Theorem A and Theorem B, only considered the cases $\lambda \in \{1,2,\cdots,n-1\}$. Therefore, our results are more general. We have not considered the case $\lambda > n$, because by Remark 1 (2), $\mathscr{I}_\lambda^n(\lambda > n)$ is not so useful.

In the proof of theorem 1 and theorem 2, the first step is different. In the proof of Theorem 1, from the meaning of the positive distribution in the condition, we should choose a proper $\phi(x)$ at first. In the proof of Theorem 2, we construct K (or L) from $L \notin \mathscr{I}_\lambda^n$ (or $K \notin \mathscr{I}_\lambda^n$) satisfying the conditions of Theorem 2. We use the same method in the second step. Using the properties of generalized Minkowski-Funk transforms and Hölder integral inequality, we obtain the results of Theorem 1 and Theorem 2.

We "split" three cases in the proofs of Theorem 1 and Theorem 2. Since the generalized Minkowski-Funk transforms are defined as M^α in the case $\lambda \neq -2l$, $l \in \mathbb{N}$, and as $\tilde{M}^\alpha$ in the case $\lambda = -2l$, $l \in \mathbb{N}$, we consider them separately. Since we need to choose different $\phi(x)$ in the cases $0 < \lambda < n$ and $\lambda < 0$ in the proof of Theorem 1, and since we need to construct K from $L \notin \mathscr{I}_\lambda^n$ for $0 < \lambda < n$, but L from $K \notin \mathscr{I}_\lambda^n$ for $\lambda < 0$ in the proof of Theorem 2, we prove the case $\lambda \neq -2l$, $l \in \mathbb{N}$ by considering two situations $0 < \lambda < n$ and $\lambda < 0$. Therefore, we have three cases: (1) $0 < \lambda < n$, (2) $\lambda < 0$, $\lambda \neq -2l$, $l \in \mathbb{N}$, (3) $\lambda = -2l$, $l \in \mathbb{N}$.

We now describe the structure of this paper. We give some notation and preliminaries in Section 2. In Section 3, we will establish Theorem 1 and Theorem 2. From Theorem 1 and Lemma 2.3, we obtain Corollary 3.1, which is the affirmative answer to the analytic generalized Busemann-Petty problem. Furthermore, we obtain Corollary 3.2 as a direct consequence of Theorem 1 and Theorem 2.

2. Notation and preliminaries

$\mathscr{D}(S^{n-1})$ stands for the space of C^∞-functions on S^{n-1} with the standard topology, and $\mathscr{D}'(S^{n-1})$ stands for the corresponding dual space of distribution. The subspaces of even test functions (distributions) are denoted by $\mathscr{D}_e(S^{n-1})$ $(\mathscr{D}'_e(S^{n-1}))$; $\mathscr{D}(G_{n,i})$ is the space of infinitely differentiable functions on $G_{n,i}$.

We write $\mathscr{M}(S^{n-1})$ and $\mathscr{M}(G_{n,i})$ for the spaces of finite Borel measures on S^{n-1} and $G_{n,i}$; $\mathscr{M}_+(S^{n-1})$ and $\mathscr{M}_+(G_{n,i})$ are the spaces of non-negative measures; $\mathscr{M}_{e+}(S^{n-1})$ and $\mathscr{M}_{e+}(G_{n,i})$ are the spaces of even measures $\mu \in \mathscr{M}_+(S^{n-1})$.

DEFINITION 2. For an integrable function f on S^{n-1}, the Minkowski-Funk transform $Mf(u)$, $\forall u \in S^{n-1}$ is defined by

$$Mf(u) = \int_{\{\theta:\theta\cdot u=0\}} f(\theta)d_u\theta, \quad u \in S^{n-1},$$

where $d_u\theta$ denotes the probability measure on the manifolds $S^{n-1}\cap u^\perp$.

This transform is a number of the analytic family of the generalized Minkowski-Funk transforms

$$M^\alpha f(u) = \gamma_n(\alpha)\int_{S^{n-1}} f(\theta)|\theta\cdot u|^{\alpha-1}d\theta, \tag{2.1}$$

$$\gamma_n(\alpha) = \frac{\sigma_{n-1}\Gamma((1-\alpha)/2)}{2\pi^{(n-1)/2}\Gamma(\alpha/2)} \;\; Re\alpha > 0, \;\; \alpha \neq 1,3,5,\cdots.$$

To include the poles $\alpha = 1,3,5,\cdots$ into consideration, we set

$$\tilde{M}^\alpha f(u) = \int_{S^{n-1}} f(\theta)|\theta\cdot u|^{\alpha-1}d\theta. \tag{2.2}$$

It was showed that $Mf(u)$ is the limit of $M^\alpha f(u)$ in [21]:

$$\lim_{\alpha\to 0} M^\alpha f = M^0 f = C_{n-1}Mf, \quad C_{n-1} = \frac{\sigma_{n-2}}{2\pi^{(n-2)/2}}. \tag{2.3}$$

LEMMA 2.1. (see [17]) *Let* $\alpha,\beta \in \mathbb{C}$, $\alpha,\beta \neq 1,3,5\cdots$ *If* $\alpha+\beta = 2-n$ *and* $f \in \mathscr{D}_e(S^{n-1})$ *then*

$$M^\alpha M^\beta f = f. \tag{2.4}$$

If $\alpha, 2-n-\alpha \neq 1,3,5\cdots$, *then* M^α *is automorphism of the spaces* $\mathscr{D}(S^{n-1})$ *and* $\mathscr{D}'(S^{n-1})$.

Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing C^∞-function on $\mathbb{R}^n$ and $\mathscr{S}'(\mathbb{R}^n)$ its dual. A distribution $f \in \mathscr{S}'(\mathbb{R}^n)$ is positive if $\langle f,\phi\rangle \geqslant 0$ for all non-negative $\phi \in \mathscr{S}(\mathbb{R}^n)$. A distribution $f \in \mathscr{S}'(\mathbb{R}^n)$ is positive definite if $\hat{f}$ is positive.

LEMMA 2.2. (see [17]) *For* $\lambda \in \mathbb{R}$, $\lambda \notin \{n, n+2, n+4,\cdots\}\cup\{0,-2,-4,\cdots\}$, *the following statements are equivalent:*

(a) $K \in \mathscr{I}^n_\lambda$,

(b) The Fourier transform $[s_\lambda\|\cdot\|_K^{-\lambda}]^\wedge$ *is a positeve distribution on* $\mathbb{R}^n\setminus\{0\}$,

(c) $s_\lambda M^{1+\lambda-n}\rho_K^\lambda \in \mathscr{M}_{e+}(S^{n-1})$.

LEMMA 2.3. (see [17]) *Let* $p > -n$, $p \neq 0$. *Then* $(\mathbb{R}^n, \|\cdot\|_K)$ *embeds isometrically in* L_p *if and only if* $K \in \mathscr{I}^n_{-p}$.

LEMMA 2.4. (Hölder's integral inequality, see [9]) *Suppose that* f *and* g *are Borel measurable functions on* X, *let* p,q *be nonzero real numbers with* $\frac{1}{p} + \frac{1}{q} = 1$.

(1) If $p > 1$ *then*

$$\int_X fgdx \leqslant \Big(\int_X f^p dx\Big)^{\frac{1}{p}} \Big(\int_X g^q dx\Big)^{\frac{1}{q}},$$

with equality if and only if there are constants A, B *not both zero, such that* $A|f|^p = B|g|^q$.

(2) If $p < 0$ *or* $0 < p < 1$ *then*

$$\int_X fgdx \geqslant \Big(\int_X f^p dx\Big)^{\frac{1}{p}} \Big(\int_X g^q dx\Big)^{\frac{1}{q}},$$

with equality if and only if either (a) there are constants A, B *not both zero, such that* $A|f|^p = B|g|^q$, *or (b)* fg *is null.*

Throughout this paper, different notations $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ are used for distributions on R^n and S^{n-1}, respectively.

3. Proofs of our main results

In this section, we will prove Theorem 1 and Theorem 2, that is, we will give an solution of Analytic General Busemann-Petty problem.

Proof of Theorem 1. We prove our results by considering three cases using the properties of generalized Minkowski-Funk transforms and Hölder's integral inequality.

(1). The first case is $0 < \lambda < n$. Since $K \in \mathscr{I}^n_\lambda$, we have $s_\lambda M^{1+\lambda-n}\|\cdot\|_K^{-\lambda} \in \mathscr{M}_{e+}(S^{n-1})$. Let $M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - M^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$ be a positive distribution. It means that for every $\phi \in \mathscr{S}(\mathbb{R}^n)$ such that $\phi \geqslant 0$ and $0 \notin \mathrm{supp}\phi$,

$$\langle M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - M^{1-\lambda}\|\cdot\|_K^{-n+\lambda}, \phi\rangle \geqslant 0.$$

We choose $\phi(x) = \psi(|x|)M^{1+\lambda-n}\|\cdot\|_K^{-\lambda}(x/|x|)$, where ψ is a smooth non-negative function such that $\int_0^\infty r^{n-\lambda-1}\psi(r)dr = 1$ and $0 \notin \mathrm{suppt}\phi$. From (2.1), we get

$$M^{1-\lambda}(\|r\theta\|_L^{-n+\lambda} - \|r\theta\|_K^{-n+\lambda}) = r^{-\lambda}M^{1-\lambda}(\|\theta\|_L^{-n+\lambda} - \|\theta\|_K^{-n+\lambda}).$$

Then,

$$\begin{aligned}
&\langle M^{1-\lambda}\|\cdot\|_L^{-n+\lambda}-M^{1-\lambda}\|\cdot\|_K^{-n+\lambda},\phi(x)\rangle\\
&=\langle M^{1-\lambda}(\|\cdot\|_L^{-n+\lambda}-\|\cdot\|_K^{-n+\lambda}),\psi(|x|)M^{1+\lambda-n}\|\cdot\|_K^{-\lambda}(x/|x|)\rangle\\
&=\int_{R^n}M^{1-\lambda}(\|x\|_L^{-n+\lambda}-\|x\|_K^{-n+\lambda})\psi(|x|)M^{1+\lambda-n}\|\frac{x}{|x|}\|_K^{-\lambda}dx\\
&=\int_{S^{n-1}}\int_0^\infty M^{1-\lambda}(\|r\theta\|_L^{-n+\lambda}-\|r\theta\|_K^{-n+\lambda})\psi(r)M^{1+\lambda-n}\|\theta\|_K^{-\lambda}r^{n-1}drd\theta\\
&=\int_{S^{n-1}}\int_0^\infty r^{-\lambda}M^{1-\lambda}(\|\theta\|_L^{-n+\lambda}-\|\theta\|_K^{-n+\lambda})\psi(r)M^{1+\lambda-n}\|\theta\|_K^{-\lambda}r^{n-1}drd\theta\\
&=\int_{S^{n-1}}\left(\int_0^\infty r^{n-\lambda-1}\psi(r)dr\right)M^{1-\lambda}(\|\theta\|_L^{-n+\lambda}-\|\theta\|_K^{-n+\lambda})M^{1+\lambda-n}\|\theta\|_K^{-\lambda}d\theta\\
&=\int_{S^{n-1}}M^{1-\lambda}(\|\theta\|_L^{-n+\lambda}-\|\theta\|_K^{-n+\lambda})M^{1+\lambda-n}\|\theta\|_K^{-\lambda}d\theta\\
&=(M^{1-\lambda}(\|\cdot\|_L^{-n+\lambda}-\|\cdot\|_K^{-n+\lambda}),M^{1+\lambda-n}\|\cdot\|_K^{-\lambda})\geqslant 0.
\end{aligned}$$

Note that $1-\lambda,1+\lambda-n\neq 1,3,5\cdots,$ from Lemma 2.1, we get

$$\begin{aligned}
&(M^{1-\lambda}(\|\cdot\|_L^{-n+\lambda}-\|\cdot\|_K^{-n+\lambda}),M^{1+\lambda-n}\|\cdot\|_K^{-\lambda})\\
&=(\|\cdot\|_L^{-n+\lambda}-\|\cdot\|_K^{-n+\lambda},M^{1-\lambda}M^{1+\lambda-n}\|\cdot\|_K^{-\lambda})\\
&=(\|\cdot\|_L^{-n+\lambda}-\|\cdot\|_K^{-n+\lambda},\|\cdot\|_K^{-\lambda})\geqslant 0.
\end{aligned}$$

Therefore,

$$(\|\cdot\|_L^{-n+\lambda},\|\cdot\|_K^{-\lambda})\geqslant(\|\cdot\|_K^{-n+\lambda},\|\cdot\|_K^{-\lambda}).$$

Using Hölder's integral inequality ($0<\lambda<n$), we obtain that

$$\begin{aligned}
V(K)&=\frac{1}{n}\int_{S^{n-1}}\|\theta\|_K^{-n+\lambda}\|\theta\|_K^{-\lambda}d\theta\\
&\leqslant\frac{1}{n}\int_{S^{n-1}}\|\theta\|_L^{-n+\lambda}\|\theta\|_K^{-\lambda}d\theta\\
&\leqslant\left(\frac{1}{n}\int_{S^{n-1}}\|\theta\|_L^{-n}d\theta\right)^{\frac{n-\lambda}{n}}\left(\frac{1}{n}\int_{S^{n-1}}\|\theta\|_K^{-n}d\theta\right)^{\frac{\lambda}{n}}\\
&=V(L)^{\frac{n-\lambda}{n}}V(K)^{\frac{\lambda}{n}}.
\end{aligned}$$

This clearly yields $V(K)\leqslant V(L)$.

(2). The second case is $\lambda<0$, $\lambda\neq -2l$, $l\in\mathbb{N}$. Since $L\in\mathscr{I}_\lambda^n$, we have $s_\lambda M^{1+\lambda-n}\|\cdot\|_L^{-\lambda}\in\mathscr{M}_{e+}(S^{n-1})$.

Let $s_\lambda M^{1-\lambda}\|\cdot\|_L^{-n+\lambda}-s_\lambda M^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$ be a positive distribution. It means that for every $\phi\in\mathscr{S}(\mathbb{R}^n)$ such that $\phi\geqslant 0$ and $0\notin\operatorname{supp}\phi$,

$$\langle s_\lambda M^{1-\lambda}\|\cdot\|_L^{-n+\lambda}-s_\lambda M^{1-\lambda}\|\cdot\|_K^{-n+\lambda},\phi\rangle\geqslant 0.$$

We choose $\phi(x)=\psi(|x|)s_\lambda M^{1+\lambda-n}\|\cdot\|_L^{-\lambda}(x/|x|)$, where ψ is a smooth non-negative function such that $\int_0^\infty r^{n-\lambda-1}\psi(r)dr=1$ and $0\notin \text{suppt}\phi$. Then

$$\begin{aligned}
&\langle s_\lambda M^{1-\lambda}\|\cdot\|_L^{-n+\lambda}-s_\lambda M^{1-\lambda}\|\cdot\|_K^{-n+\lambda},\phi(x)\rangle\\
&=\langle s_\lambda M^{1-\lambda}(\|\cdot\|_L^{-n+\lambda}-\|\cdot\|_K^{-n+\lambda}),s_\lambda\psi(x)M^{1+\lambda-n}\|\cdot\|_L^{-\lambda}(x/|x|)\rangle\\
&=(s_\lambda M^{1-\lambda}(\|\cdot\|_L^{-n+\lambda}-\|\cdot\|_K^{-n+\lambda}),s_\lambda M^{1+\lambda-n}\|\cdot\|_L^{-\lambda})\\
&=(M^{1-\lambda}(\|\cdot\|_L^{-n+\lambda}-\|\cdot\|_K^{-n+\lambda}),M^{1+\lambda-n}\|\cdot\|_L^{-\lambda})\geqslant 0.
\end{aligned}$$

Note that $1-\lambda,1+\lambda-n\neq 1,3,5\cdots$, it follows that

$$(\|\cdot\|_L^{-n+\lambda}-\|\cdot\|_K^{-n+\lambda},\|\cdot\|_L^{-\lambda})\geqslant 0,$$

$$(\|\cdot\|_L^{-n+\lambda},\|\cdot\|_L^{-\lambda})\geqslant(\|\cdot\|_K^{-n+\lambda},\|\cdot\|_L^{-\lambda}).$$

Using Hölder's integral inequality ($\lambda<0$), we obtain that

$$\begin{aligned}
V(L)&=\frac{1}{n}\int_{S^{n-1}}\|\theta\|_L^{-n+\lambda}\|\theta\|_L^{-\lambda}d\theta\\
&\geqslant\frac{1}{n}\int_{S^{n-1}}\|\theta\|_K^{-n+\lambda}\|\theta\|_L^{-\lambda}d\theta\\
&\geqslant\left(\frac{1}{n}\int_{S^{n-1}}\|\theta\|_K^{-n}d\theta\right)^{\frac{n-\lambda}{n}}\left(\frac{1}{n}\int_{S^{n-1}}\|\theta\|_L^{-n}d\theta\right)^{\frac{\lambda}{n}}\\
&=V(K)^{\frac{n-\lambda}{n}}V(L)^{\frac{\lambda}{n}}.
\end{aligned}$$

This clearly yields $V(K)\leqslant V(L)$.

(3). The third case is $\lambda=-2l$, $l\in\mathbb{N}$. Since $L\in\mathscr{I}_\lambda^n$, there is a measure $\mu\in\mathscr{M}_{e+}(S^{n-1})$ such that $\|\cdot\|_L^{-\lambda}=\tilde{M}^{1-\lambda}\mu$. That is: for any $\varphi\in\mathscr{D}(S^{n-1})$,

$$\int_{S^{n-1}}\|\cdot\|_L^{-\lambda}\varphi(\theta)d\theta=\int_{S^{n-1}}\tilde{M}^{1-\lambda}\varphi(\theta)d\mu(\theta).$$

Let $\tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda}-\tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$ be a positive distribution. It means that for every $\phi\in\mathscr{S}(\mathbb{R}^n)$ such that $\phi\geqslant 0$ and $0\notin\text{supp}\phi$,

$$\langle\tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda}-\tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda},\phi\rangle\geqslant 0,$$

where ϕ is a smooth non-negative function such that $\int_0^\infty r^{n-\lambda-1}\phi(r)dr=1$ and $0\notin\text{suppt}\phi$. Then

$$\begin{aligned}
&\langle\tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda}-\tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda},\phi(x)\rangle\\
&=\langle\tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda}-\tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda},\phi(|x|)\rangle\\
&=(\tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda}-\tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda},1)\geqslant 0.
\end{aligned}$$

Therefore, we obtain

$$(\tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda}-\tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda},1)\geqslant 0,$$

$$(\tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda},1) \geqslant (\tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda},1).$$

Both sides' integrals of the last inequality over S^{n-1} is about measure μ, we get

$$\int_{S^{n-1}} \tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda} d\mu(\theta) \geqslant \int_{S^{n-1}} \tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda} d\mu(\theta),$$

$$\int_{S^{n-1}} \|\cdot\|_L^{-\lambda}\|\cdot\|_L^{-n+\lambda} d\theta \geqslant \int_{S^{n-1}} \|\cdot\|_L^{-\lambda}\|\cdot\|_K^{-n+\lambda} d\theta.$$

From Hölder's integral inequality ($\lambda < 0$), we obtain that

$$\begin{aligned} V(L) &= \frac{1}{n}\int_{S^{n-1}} \|\theta\|_L^{-\lambda}\|\theta\|_L^{-n+\lambda} d\theta \\ &\geqslant \frac{1}{n}\int_{S^{n-1}} \|\theta\|_L^{-\lambda}\|\theta\|_K^{-n+\lambda} d\theta \\ &\geqslant \left(\frac{1}{n}\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta\right)^{\frac{\lambda}{n}} \left(\frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta\right)^{\frac{n-\lambda}{n}} \\ &= V(L)^{\frac{\lambda}{n}} V(K)^{\frac{n-\lambda}{n}}. \end{aligned}$$

Therefore, $V(K) \leqslant V(L)$.

Above all, Theorem 1 is proved. □

Since all 2-dimensional space embed in L_1, and therefore in L_p with $-2 < p < 1$ (see e.g. [11] Chapter 6), and all 3-dimensional space embed in L_0, and therefore in L_p with $-3 < p < 0$ (see e.g. [13]). From Lemma 2.3, we have the following

COROLLARY 3.1. *If $n = 2,3$, $0 < \lambda < n$. K, L are origin-symmetric infinitely smooth star bodies in $\mathbb{R}^n$. The function $M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - M^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$ is a positive distribution in $\mathbb{R}^n \setminus \{0\}$, then*

$$V(K) \leqslant V(L).$$

Proof of Theorem 2. **(1).** The first case is $0 < \lambda < n$. Suppose that L is an origin-symmetric convex body in $\mathbb{R}^n$, so that $\|\cdot\|_L^{-\lambda}$ is infinitely smooth, the boundary of L has a positive curvature and $L \notin \mathscr{I}_\lambda^n$. Therefore, there is a function $\mu \in D_e(S^{n-1})$, which is negative on some open origin-symmetric set $\Omega \subset S^{n-1}$ and such that $\|\cdot\|_L^{-\lambda} = M^{1-\lambda}\mu$.

We choose a function $\upsilon \in D_e(S^{n-1})$ so that $\upsilon(\theta) \geqslant 0$ and $\exists\ \theta$ such that $\upsilon \neq 0$ if $\theta \in \Omega$, $\upsilon(\theta) \equiv 0$ otherwise. We define an origin-symmetric smooth body K by $\|\cdot\|_K^{-n+\lambda} = \|\cdot\|_L^{-n+\lambda} - \varepsilon M^{1+\lambda-n}\upsilon$, $\varepsilon > 0$. Since $1+\lambda-n, -n+\lambda \neq 1,3,5,\cdots$,

$$M^{1-\lambda}M^{1-n+\lambda}\upsilon = \upsilon \geqslant 0,$$

we have

$$M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - M^{1-\lambda}\|\cdot\|_K^{-n+\lambda} = \varepsilon M^{1-\lambda}M^{1-n+\lambda}\upsilon = \varepsilon\upsilon \geqslant 0.$$

On the other hand,

$$\begin{aligned}(\|\cdot\|_L^{-\lambda}, \|\cdot\|_L^{-n+\lambda} - \|\cdot\|_K^{-n+\lambda}) &= (M^{1-\lambda}\mu, \varepsilon M^{1+\lambda-n}\upsilon)\\ &= \varepsilon(\mu,\nu) < 0,\end{aligned}$$

i.e., $(\|\cdot\|_L^{-\lambda}, \|\cdot\|_L^{-n+\lambda}) < (\|\cdot\|_L^{-\lambda}, |\cdot\|_K^{-n+\lambda})$.

By Hölder's inequality,

$$\begin{aligned}V(L) &= \frac{1}{n}\int_{S^{n-1}} \|\theta\|_L^{-\lambda}\|\theta\|_L^{-n+\lambda} d\theta\\ &< \frac{1}{n}\int_{S^{n-1}} \|\theta\|_L^{-\lambda}\|\theta\|_K^{-n+\lambda} d\theta\\ &\leqslant \left(\frac{1}{n}\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta\right)^{\frac{\lambda}{n}} \left(\frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta\right)^{\frac{n-\lambda}{n}}\\ &= V(L)^{\frac{\lambda}{n}} V(K)^{\frac{n-\lambda}{n}}.\end{aligned}$$

Therefore $V(K) > V(L)$, as desired.

(2). The second case is $\lambda < 0$, $\lambda \neq -2l$, $l \in \mathbb{N}$. Suppose that K is an origin-symmetric convex body in $\mathbb{R}^n$, so that $\|\cdot\|_K^{-\lambda}$ is infinitely smooth, the boundary of K has a positive curvature and $K \notin \mathscr{I}_\lambda^n$. Therefore, there is a function $\mu \in D_e(S^{n-1})$, which is negative on some open origin-symmetric set $\Omega \subset S^{n-1}$ and such that $s_\lambda \|\cdot\|_K^{-\lambda} = M^{1-\lambda}\mu$.

We choose a function $\upsilon \in D_e(S^{n-1})$ so that $\upsilon(\theta) \geqslant 0$ and $\exists\ \theta$ such that $\upsilon \neq 0$ if $\theta \in \Omega$, $\upsilon(\theta) \equiv 0$ otherwise. We define an origin-symmetric smooth body L by $s_\lambda \|\cdot\|_L^{-n+\lambda} = s_\lambda \|\cdot\|_K^{-n+\lambda} + \varepsilon M^{1+\lambda-n}\upsilon$, $\varepsilon > 0$. Since $1+\lambda-n, -n+\lambda \neq 1,3,5,\cdots$,

$$M^{1-\lambda}M^{1-n+\lambda}\upsilon = \upsilon \geqslant 0,$$

we have

$$s_\lambda M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - s_\lambda M^{1-\lambda}\|\cdot\|_K^{-n+\lambda} = \varepsilon M^{1-\lambda}M^{1-n+\lambda}\upsilon = \varepsilon\upsilon \geqslant 0.$$

On the other hand,

$$\begin{aligned}(\|\cdot\|_K^{-\lambda}, \|\cdot\|_L^{-n+\lambda} - \|\cdot\|_K^{-n+\lambda}) &= (M^{1-\lambda}\mu, \varepsilon M^{1+\lambda-n}\nu)\\ &= \varepsilon(\mu,\nu) < 0,\end{aligned}$$

i.e., $(\|\cdot\|_K^{-\lambda}, \|\cdot\|_L^{-n+\lambda}) < (\|\cdot\|_K^{-\lambda}, |\cdot\|_K^{-n+\lambda})$.

By Hölder's inequality,

$$\begin{aligned}V(K) &= \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-\lambda}\|\theta\|_K^{-n+\lambda} d\theta\\ &> \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-\lambda}\|\theta\|_L^{-n+\lambda} d\theta\\ &\geqslant \left(\frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta\right)^{\frac{\lambda}{n}} \left(\frac{1}{n}\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta\right)^{\frac{n-\lambda}{n}}\\ &= V(K)^{\frac{\lambda}{n}} V(L)^{\frac{n-\lambda}{n}}.\end{aligned}$$

Therefore $V(K) > V(L)$.

(3). The third case is $\lambda = -2l$, $l \in \mathbb{N}$. Suppose that K is an origin-symmetric convex body in $\mathbb{R}^n$, so that $\|\cdot\|_K^{-\lambda}$ is infinitely smooth, the boundary of K has a positive curvature and $K \notin \mathscr{I}_\lambda^n$. Therefore, there is a function $\mu \in D_e(S^{n-1})$, which is negative on some open origin-symmetric set $\Omega \subset S^{n-1}$ and such that $s_\lambda \|\cdot\|_K^{-\lambda} = \tilde{M}^{1-\lambda}\mu$. We choose a non-zero continuous function $g \in D_e(S^{n-1})$ so that

$$\tilde{M}^{1-\lambda} g(\theta) = \int_{S^{n-1}} g(u)|u\cdot\theta|^{-\lambda} du = 0 \quad \forall \theta \in S^{n-1}.$$

We consider the smooth origin-symmetric star body L, and suppose that

$$\int_{S^{n-1}} \|\cdot\|_K^{-\lambda} g(\theta) d\theta = \int_{S^{n-1}} g(\theta)\tilde{M}^{1-\lambda}\mu d\theta \geqslant 0,$$

(otherwise, consider $-g(\theta)$ instead of $g(\theta)$). We choose $\varepsilon \geqslant 0$ such that

$$\|\theta\|_K^{n-\lambda} - \varepsilon g(\theta) > 0 \quad \forall \theta \in S^{n-1},$$

We define an origin-symmetric smooth body L in $\mathbb{R}^n$ by

$$\|\theta\|_L^{n-\lambda} = \|\theta\|_K^{n-\lambda} - \varepsilon g(\theta) > 0 \quad \forall \theta \in S^{n-1}.$$

Therefore

$$\tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda} - \tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda} = \varepsilon \tilde{M}^{1-\lambda} g(\theta) = 0.$$

On the other hand,

$$\begin{aligned}(\|\cdot\|_K^{-\lambda}, \|\cdot\|_K^{-n+\lambda} - \|\cdot\|_L^{-n+\lambda}) &= (\tilde{M}^{1-\lambda}\mu, \varepsilon g(\theta)) \\ &= \varepsilon \int_{S^{n-1}} g(\theta)\tilde{M}^{1-\lambda}\mu d\theta \geqslant 0,\end{aligned}$$

i.e., $(\|\cdot\|_K^{-\lambda}, \|\cdot\|_K^{-n+\lambda}) \geqslant (\|\cdot\|_K^{-\lambda}, \|\cdot\|_L^{-n+\lambda})$.

By Hölder's inequality,

$$\begin{aligned} V(K) &= \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-\lambda}\|\theta\|_K^{-n+\lambda} d\theta \\ &\geqslant \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-\lambda}\|\theta\|_L^{-n+\lambda} d\theta \\ &\geqslant \left(\frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta\right)^{\frac{\lambda}{n}} \left(\frac{1}{n}\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta\right)^{\frac{n-\lambda}{n}} \\ &= V(K)^{\frac{\lambda}{n}} V(L)^{\frac{n-\lambda}{n}}. \end{aligned}$$

Therefore $V(K) \geqslant V(L)$.

If $V(L) = V(K)$, it follows from the equality condition that $\|\cdot\|_K = \|\cdot\|_L$, i.e. $K = L$, which contradicts $\|\theta\|_L^{n-\lambda} = \|\theta\|_K^{n-\lambda} - \varepsilon g(\theta) > 0$. Therefore $V(K) > V(L)$, as desired.

Above all, Theorem 2 is proved. □

A direct consequence of Theorem 1 and Theorem 2 is:

COROLLARY 3.2. *K, L are origin-symmetric infinitely smooth star bodies in $\mathbb{R}^n$. Each case in the following holds if and only if every origin-symmetric star body $\mathbb{R}^n$ is a λ-intersection body:*

(1) Let $0 < \lambda < n$. The positive distribution $M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - M^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$ in $\mathbb{R}^n \setminus \{0\}$ implies $V(K) \leqslant V(L)$.

(2) Let $\lambda < 0$, $\lambda \neq -2l$, $l \in \mathbb{N}$. The positive distribution $s_\lambda M^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - s_\lambda M^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$ in $\mathbb{R}^n \setminus \{0\}$ implies $V(K) \leqslant V(L)$.

(3) Let $\lambda = -2l$, $l \in \mathbb{N}$. The positive distribution $\tilde{M}^{1-\lambda}\|\cdot\|_L^{-n+\lambda} - \tilde{M}^{1-\lambda}\|\cdot\|_K^{-n+\lambda}$ in $\mathbb{R}^n \setminus \{0\}$ implies $V(K) \leqslant V(L)$.

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