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ON A QUESTION FOR DUAL QUERMASSINTEGRALS

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Abstract. The well-known question for dual quermassintegrals is following: For which values of $i \in \mathbb{N}$ and every pair of star bodies K and L, is it true that

$$\frac{\tilde{W}_i(K\tilde{+}L)}{\tilde{W}_{i+1}(K\tilde{+}L)} \leqslant \frac{\tilde{W}_i(K)}{\tilde{W}_{i+1}(K)} + \frac{\tilde{W}_i(L)}{\tilde{W}_{i+1}(L)} \ ?$$

In 2005, the inequality was proved if $i=n-1$ or $i=n-2$. But, now the question is not solved completely for $0 \leqslant i \leqslant n-3$ and every pair of star bodies K and L. In the paper, we will give a completely answer for the dual question.

1. Notations and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n.

We use $V(K)$ for the n-dimensional volume of convex body K. Let $h(K,\cdot)$: $S^{n-1} \to \mathbb{R}$, denote the support function of $K \in \mathscr{K}^n$; i.e. for $u \in S^{n-1}$

$$h(K,u) = Max\{u \cdot x : x \in K\},$$

where $u \cdot x$ denotes the usual inner product u and x in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathscr{K}^n$, i.e., for $K,L \in \mathscr{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K,u) = Max\{\lambda \geqslant 0 : \lambda u \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathscr{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K,L \in \mathscr{S}^n$, then $\tilde{\delta}(K,L) = |\rho_K - \rho_L|_\infty$.

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1.1. Mixed volumes

If $K_i \in \mathscr{K}^n$ $(i = 1,2,\ldots,r)$ and λ_i $(i = 1,2,\ldots,r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^{r} \lambda_i K_i$ is a homogeneous polynomial in the variables λ_i given by (see e.g. [1] or [2])

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{1.1}$$

where the sum is taken over all n-tuples $(i_1,\ldots,i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1\ldots i_n}$ depends only on the bodies $K_{i_1},\ldots,K_{i_n}$ and is uniquely determined by (1.1), it is called the mixed volume of $K_{i_1},\ldots,K_{i_n}$, and is written as $V(K_{i_1},\ldots,K_{i_n})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1 \ldots K_n)$ is written as $V_i(K,L)$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$, then the mixed volumes $V_i(K,B)$ is written as $W_i(K)$ and is called the quermassintegral of convex body K.

It is convenient to write relation (1.1) in the form (see [2, p. 137])

$$V(\lambda_1 K_1 + \cdots + \lambda_s K_s)$$
$$= \sum_{p_1+\cdots+p_r=n} \sum_{1\leqslant i_1<\cdots<i_r\leqslant s} \frac{n!}{p_1!\cdots p_r!} \lambda_{i_1}^{p_1} \cdots \lambda_{i_r}^{p_r} V(\underbrace{K_{i_1},\ldots,K_{i_1}}_{p_1},\ldots,\underbrace{K_{i_r},\ldots,K_{i_r}}_{p_r}). \tag{1.2}$$

Let $s = 2$, $\lambda_1 = 1$, $K_1 = K$, $K_2 = B$, we have

$$V(K+\lambda B) = \sum_{i=0}^{n} \binom{n}{i} \lambda^i W_i(K),$$

known as formula "Steiner decomposition".

On the other hand, for convex bodies K and L, (1.2) can show the following special case:

$$W_i(K+\lambda L) = \sum_{j=0}^{n-i} \binom{n-i}{j} \lambda^j V(\underbrace{K,\ldots,K}_{n-i-j},\underbrace{B,\ldots,B}_{i},\underbrace{L,\ldots,L}_{j}).$$

1.2. Dual mixed volumes

If $x_i \in \mathbb{R}^n$, $1 \leqslant i \leqslant r$, then $x_1 \tilde{+} \cdots \tilde{+} x_r$ is defined to be the usual vector sum of the points x_i, if all of them are contained in a line through o, and o otherwise. Let $K_1,\ldots,K_r \in \mathscr{S}^n$ with $o \in K_i$ and $\lambda_i \geqslant 0$, $1 \leqslant i \leqslant r$, then

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}$$

is called a *radial Minkowski linear combination* (see [3]).

It has the following important property (see [4]):

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K,\cdot) + \mu \rho(L,\cdot),$$

for $K,L\in\mathscr{S}^n$ and λ, $\mu\geqslant 0$.

For $K_1,\dots,K_r\in\mathscr{S}^n$ and $\lambda_1,\dots,\lambda_r\geqslant 0$, the volume of the radial Minkowski linear combination $\lambda_1K_1\tilde{+}\cdots\tilde{+}\lambda_rK_r$ is a homogeneous polynomial of degree n in the variables λ_i, given by (see [4] and [5]),

$$V(\lambda_1K_1\tilde{+}\cdots\tilde{+}\lambda_rK_r)=\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}\tilde{V}_{i_1,\dots,i_n}\lambda_{i_1}\cdots\lambda_{i_n}, \tag{1.3}$$

If we require the coefficients of the polynomial in (1.3) to be symmetric in their argument, then they are uniquely determined. The coefficient $\tilde{V}_{i_1,\dots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1},\dots,K_{i_n}$. Here we denote $\tilde{V}_{i_1,\dots,i_n}$ to $\tilde{V}(K_{i_1},\dots,K_{i_n})$ and is called the dual mixed volume of star bodies $K_{i_1},\dots,K_{i_n}$. If $K_1=\cdots=K_{n-i}=K$, $K_{n-i+1}=\cdots=K_n=L$, the dual mixed volume $\tilde{V}(K_1,\dots,K_n)$ is written as $\tilde{V}_i(K,L)$. If $L=B$, the dual mixed volume $\tilde{V}_i(K,L)=\tilde{V}_i(K,B)$ is written as $\tilde{W}_i(K)$ and is called the dual quermassintegral of star body K.

By the way of similar to the quermassintegral of convex body, for star bodies $K_1,\dots,K_s$, from (1.3), we obtain

$$V(\lambda_1K_1\tilde{+}\cdots\tilde{+}\lambda_sK_s)$$

$$=\sum_{p_1+\cdots+p_r=n}\sum_{1\leqslant i_1<\cdots<i_r\leqslant s}\frac{n!}{p_1!\cdots p_r!}\lambda_{i_1}^{p_1}\cdots\lambda_{i_r}^{p_r}\tilde{V}(\underbrace{K_{i_1},\dots,K_{i_1}}_{p_1},\dots,\underbrace{K_{i_r},\dots,K_{i_r}}_{p_r}).$$

Let $s=2$, $\lambda_1=1$, $K_1=K$, $K_2=B$, we have

$$\tilde{V}(K\tilde{+}\lambda B)=\sum_{i=0}^{n}\binom{n}{i}\lambda^i\tilde{W}_i(K).$$

Moreover, we have

$$\tilde{W}_i(K\tilde{+}\lambda L)=\sum_{j=0}^{n-i}\binom{n-i}{j}\lambda^j\tilde{V}(\underbrace{K,\dots,K}_{n-i-j},\underbrace{B,\dots,B}_{i},\underbrace{L,\dots,L}_{j}). \tag{1.4}$$

2. Introduction and main result

The origin of this work is an interesting inequality of Marcus and Lopes [6]. We write $E_i(x)$, $0\leqslant i\leqslant n$, for the i-th elementary symmetric function of an n-tuple $x=(x_1,\dots,x_n)$ of positive real numbers. This is defined by $E_0(x)=1$ and

$$E_i(x)=\sum_{1\leqslant j_1<\cdots<j_i\leqslant n}x_{j_1}x_{j_2}\cdots x_{j_i},\qquad 1\leqslant i\leqslant n.$$

In particular, $E_1(x)=x_1+\cdots+x_n$, $E_2(x)=\sum_{i\neq j}x_ix_j$, $\dots$, $E_n(x)=x_1x_2\cdots x_n$.

The Marcus-Lopes inequality (see also [7, p. 33]) states that

$$\frac{E_i(x+y)}{E_{i-1}(x+y)}\geqslant\frac{E_i(x)}{E_{i-1}(x)}+\frac{E_i(y)}{E_{i-1}(y)} \tag{2.1}$$

for every pair of positive n-tuples x and y. This is a refinement of a further result concerning the symmetric functions, namely,

$$[E_i(x+y)]^{1/i} \geqslant [E_i(x)]^{1/i} + [E_i(y)]^{1/i}. \tag{2.2}$$

A discussion of the cases of equality is contained in the reference [6].

A matrix analogue of (2.1) is the following result of Bergstrom [8] (see also the article [9] and [10, p. 67] for an interesting proof): if K and L are positive definite matrices, and if K_i and L_i denote the submatrices obtained by deleting their i-th row and column, then

$$\frac{\det(K+L)}{\det(K_i+L_i)} \geqslant \frac{\det(K)}{\det(K_i)} + \frac{\det(L)}{\det(L_i)}. \tag{2.3}$$

The following generalization of (2.3) was established by Ky Fan [10]:

$$\left(\frac{\det(K+L)}{\det(K_i+L_i)}\right)^{1/k} \geqslant \left(\frac{\det(K)}{\det(K_i)}\right)^{1/k} + \left(\frac{\det(L)}{\det(L_i)}\right)^{1/k}. \tag{2.4}$$

The proof is based on a minimum principle; see also Ky Fan [11] and Mirsky [12].

There is a remarkable similarity between inequalities about symmetric functions (or determinants of symmetric matrices) and inequalities about the mixed volumes of convex bodies. For example, the analogues of (2.2) in the Brunn-Minkowski and the dual Brunn-Minkowski theory are the following:

(i) If K and L are convex bodies in $\mathbb{R}^n$ and if $0 \leqslant i \leqslant n-1$, then

$$W_i(K+L)^{1/(n-i)} \geqslant W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \tag{2.5}$$

with equality if and only if K and L are homothetic, where $W_i(K)$ is the i-th quermassintegral of K.

(ii) If K and L are star bodies in $\mathbb{R}^n$ and if $0 \leqslant i \leqslant n-1$, then

$$\tilde{W}_i(K\tilde{+}L)^{1/(n-i)} \leqslant \tilde{W}_i(K)^{1/(n-i)} + \tilde{W}_i(L)^{1/(n-i)}, \tag{2.6}$$

with equality if and only if K and L are dilates, where $\tilde{W}_i(K)$ is the i-th dual quermassintegral of K and $\tilde{+}$ is the radial sum.

In view of these analogies, V. Milman asked if there exists a version of (2.1) or (2.3) in the theory of mixed volumes (see [13], [14]).

OPEN QUESTION. *For which values of* $0 \leqslant i \leqslant n-1$, $i \in \mathbb{N}$ *is it true that, for every pair of convex bodies* K *and* L *in* $\mathbb{R}^n$ *one has*

$$\frac{W_i(K+L)}{W_{i+1}(K+L)} \geqslant \frac{W_i(K)}{W_{i+1}(K)} + \frac{W_i(L)}{W_{i+1}(L)}? \tag{2.7}$$

In 1991, the special case $i=0$ had been stated also in [15] as an open question. In the same paper it was also mentioned that (2.7) follows directly from the Aleksandrov-Fenchel inequality when $i=0$ and L is a ball.

In 2002, it was proved in [14] that (2.7) is true for all $i=1,\dots,n-1$ in the case where L is a ball.

In 2003, it was proved in [13] that (2.7) holds true for every pair of convex bodies K and L in $\mathbb{R}^n$ if and only if $i=n-2$ or $i=n-1$.

The dual question is now naturally formulated as follows:

DUAL QUESTION. *For which values of* $0 \leqslant i \leqslant n-1$, $i \in \mathbb{N}$ *is it true that, for every pair of star bodies* K *and* L *in* $\mathbb{R}^n$ *one has*

$$\frac{\widetilde{W}_i(K\tilde{+}L)}{\widetilde{W}_{i+1}(K\tilde{+}L)} \leqslant \frac{\widetilde{W}_i(K)}{\widetilde{W}_{i+1}(K)} + \frac{\widetilde{W}_i(L)}{\widetilde{W}_{i+1}(L)}\,? \tag{2.8}$$

In 2005, Li and Leng [16] proved this dual inequality in the special case where L is a ball:

THEOREM A. *If* K *be a star body and* B *be a ball in* $\mathbb{R}^n$, *then for* $0 \leqslant i \leqslant n-1$, $i \in \mathbb{N}$

$$\frac{\widetilde{W}_i(K\tilde{+}B)}{\widetilde{W}_{i+1}(K\tilde{+}B)} \leqslant \frac{\widetilde{W}_i(K)}{\widetilde{W}_{i+1}(K)} + \frac{\widetilde{W}_i(B)}{\widetilde{W}_{i+1}(B)}. \tag{2.9}$$

It can also be proved that (2.8) is true for all pairs of star bodies L and K if $i = n-2$ or $i = n-1$ (see [16]).

THEOREM B. *Let* K *and* L *be star bodies in* $\mathbb{R}^n$. *If* $i = n-1$ *or* $i = n-2$, *then*

$$\frac{\widetilde{W}_i(K\tilde{+}L)}{\widetilde{W}_{i+1}(K\tilde{+}L)} \leqslant \frac{\widetilde{W}_i(K)}{\widetilde{W}_{i+1}(K)} + \frac{\widetilde{W}_i(L)}{\widetilde{W}_{i+1}(L)}.$$

But, the dual question for $0 \leqslant i \leqslant n-3$ and all pairs of star bodies L and K, is not solved yet. In 2004, an open problem was point out by Li [17].

QUESTION. *For which values of* $0 \leqslant i \leqslant n-3$, $i \in \mathbb{N}$ *is it true that, for every pair of convex bodies* K *and* L *in* $\mathbb{R}^n$ *one has*

$$\frac{\widetilde{W}_i(K\tilde{+}L)}{\widetilde{W}_{i+1}(K\tilde{+}L)} \leqslant \frac{\widetilde{W}_i(K)}{\widetilde{W}_{i+1}(K)} + \frac{\widetilde{W}_i(L)}{\widetilde{W}_{i+1}(L)}\,?$$

In this paper, we give a positive answer as follows:

THEOREM. *Let* K *and* L *be star bodies in* $\mathbb{R}^n$. *If* $0 \leqslant i \leqslant n-3$, $i \in \mathbb{N}$, *then*

$$\frac{\widetilde{W}_i(K\tilde{+}L)}{\widetilde{W}_{i+1}(K\tilde{+}L)} \leqslant \frac{\widetilde{W}_i(K)}{\widetilde{W}_{i+1}(K)} + \frac{\widetilde{W}_i(L)}{\widetilde{W}_{i+1}(L)}$$

is not true.

3. Proof of main result

THEOREM 3.1. *Let* K *and* L *be star bodies in* $\mathbb{R}^n$. *If* $0 \leqslant i \leqslant n-3$, $i \in \mathbb{N}$, *then*

$$\frac{\widetilde{W}_i(K\tilde{+}L)}{\widetilde{W}_{i+1}(K\tilde{+}L)} \leqslant \frac{\widetilde{W}_i(K)}{\widetilde{W}_{i+1}(K)} + \frac{\widetilde{W}_i(L)}{\widetilde{W}_{i+1}(L)} \tag{3.1}$$

is not true.

In order to prove theorem 3.1, we need the following Lemmas 3.1–3.3.

LEMMA 3.1. (The dual Aleksandrov-Fenchel inequality) *If* $K_1,\dots,K_n$ *are star bodies in* $\mathbb{R}^n$. *Let* $1<r\leqslant n-1$, *then*

$$\tilde{V}(K_1,\dots,K_n)^r\leqslant\prod_{j=1}^{r}\tilde{V}(\underbrace{K_j,\dots,K_j}_{r},K_{r+1},\dots,K_n),$$

with equality if and only if $K_1,\dots,K_r$ *are all dilations.*

A special case of the dual Aleksandrov-Fenchel inequality is the following:
If K is a star body and $0\leqslant i<j<k\leqslant n$, then

$$\tilde{W}_j(K)^{k-i}\leqslant\tilde{W}_i(K)^{k-j}\tilde{W}_j(K)^{j-i}. \tag{3.2}$$

LEMMA 3.2. *Let* $0\leqslant i\leqslant n-1$, $i\in\mathbb{N}$. *Assume that (3.1) is true for all star bodies* K *and* L *in* $\mathbb{R}^n$. *Then, the function*

$$g(t)=\frac{\tilde{W}_i(K\tilde{+}tL)}{\tilde{W}_{i+1}(K\tilde{+}tL)}$$

is concave function on $[0,+\infty)$ *for every* K *and* L.

Proof. In view of the following fact: If K and L are star bodies and α, $\beta\geqslant 0$, then

$$\alpha(K\tilde{+}L)=\alpha K\tilde{+}\alpha L\quad\text{and}\quad(\alpha+\beta)K=(\alpha K\tilde{+}\beta L).$$

Assume that (3.1) is true, for s, $t\in[0,\infty)$ we obtain

$$\begin{aligned}
g\left(\frac{t+s}{2}\right)&=\frac{\tilde{W}_i\left(K\tilde{+}\frac{t+s}{2}L\right)}{\tilde{W}_{i+1}\left(K\tilde{+}\frac{t+s}{2}L\right)}\\
&=\frac{\tilde{W}_i\left(\frac{1}{2}K\tilde{+}\frac{1}{2}K\tilde{+}\frac{t}{2}L\tilde{+}\frac{s}{2}L\right)}{\tilde{W}_{i+1}\left(\frac{1}{2}K\tilde{+}\frac{1}{2}K\tilde{+}\frac{t}{2}L\tilde{+}\frac{s}{2}L\right)}\\
&=\frac{\tilde{W}_i\left(\frac{K\tilde{+}tL}{2}\tilde{+}\frac{K\tilde{+}sL}{2}\right)}{\tilde{W}_{i+1}\left(\frac{K\tilde{+}tL}{2}\tilde{+}\frac{K\tilde{+}sL}{2}\right)}\\
&\leqslant\frac{\tilde{W}_i\left(\frac{K\tilde{+}tL}{2}\right)}{\tilde{W}_{i+1}\left(\frac{K\tilde{+}tL}{2}\right)}+\frac{\tilde{W}_i\left(\frac{K\tilde{+}sL}{2}\right)}{\tilde{W}_{i+1}\left(\frac{K\tilde{+}sL}{2}\right)}\\
&=\frac{1}{2}(g(t)+g(s)).
\end{aligned}$$

Hence the function $g(t)$ is concave function on $[0,+\infty)$ for every star bodies K and L. □

LEMMA 3.3. *Assume (3.1) is true. If* $0 \leqslant i \leqslant n-3$, $i \in \mathbb{N}$, *for all star body* K *in* $\mathbb{R}^n$. *Then*

$$(n-i)\tilde{W}_{i+2}(K)\Big(\tilde{W}_{i+1}(K)^2 - \tilde{W}_i(K)\tilde{W}_{i+2}(K)\Big)$$
$$\leqslant (n-i-2)\tilde{W}_i(K)\Big(\tilde{W}_{i+2}^2(K) - \tilde{W}_{i+1}(K)\tilde{W}_{i+3}(K)\Big). \tag{3.3}$$

Proof. Let K be a convex body in $\mathbb{R}^n$. For every $0 \leqslant i \leqslant n-1$, we set

$$f_i(t) = \tilde{W}_i(K+tB),$$

then

$$f_i(t+\varepsilon) = \tilde{W}_i\Big((K+tB)+\varepsilon B\Big) = \sum_{j=0}^{n-i}\tbinom{n-i}{j}\varepsilon^j \tilde{W}_{i+j}(K+tB)$$
$$= f_i(t) + \varepsilon(n-i)f_{i+1}(t) + O(\varepsilon^2).$$

Therefore

$$f_i'(t) = (n-i)f_{i+1}(t).$$

The derivative of the function

$$g_i(t) = \frac{f_i(t)}{f_{i+1}(t)} = \frac{\tilde{W}_i(K+tB)}{\tilde{W}_{i+1}(K+tB)}$$

is thus given by

$$g_i'(t) = (n-i) - (n-i-1)\frac{f_i(t)f_{i+2}(t)}{f_{i+1}^2(t)}. \tag{3.4}$$

From the dual Aleksandrov-Fenchel inequality (3.2), we have

$$f_{i+1}^2(t) \leqslant f_i(t)f_{i+2}(t),$$

for all $0 \leqslant i \leqslant n-2$. Hence $g_i'(s) \geqslant 1$.

Since $g_i(s)$ is a concave function. This implies that $\frac{f_i(t)f_{i+2}(t)}{f_{i+1}^2(t)}$ is an decreasing function, and differentiating the both sides of (3.4) again, we obtain

$$(n-i)f_{i+2}(t)f_{i+1}^2(t) + (n-i-2)f_i(t)f_{i+1}(t)f_{i+3}(t) - 2(n-i-1)f_i(t)f_{i+2}^2(t) \leqslant 0,$$

for $t \in (0,+\infty)$.

This can be equivalently written in the form

$$(n-i)f_{i+2}(t)\Big(f_{i+1}^2(t) - f_i(t)f_{i+2}(t)\Big) \leqslant (n-i-2)f_i(t)\Big(f_{i+2}^2(t) - f_{i+1}(t)f_{i+3}(t)\Big).$$

Hence

$$(n-i)\tilde{W}_{i+2}(K\tilde{+}tB)\Big(\tilde{W}_{i+1}(K\tilde{+}tB)^2 - \tilde{W}_i(K\tilde{+}tB)\tilde{W}_{i+2}(K\tilde{+}tB)\Big)$$
$$\leqslant (n-i-2)\tilde{W}_i(K\tilde{+}tB)\Big(\tilde{W}_{i+2}(K\tilde{+}tB)^2 - \tilde{W}_{i+1}(K\tilde{+}tB)\tilde{W}_{i+3}(K\tilde{+}tB)\Big). \tag{3.5}$$

Letting $t \to 0^+$ in (3.5), we conclude the Lemma 3.3.

Assume that $\tilde{W}_{i+1}(K)^2 - \tilde{W}_i(K)\tilde{W}_{i+2}(K) = 0$. From Lemma 3.3, we have $\tilde{W}_{i+2}(K)^2 - \tilde{W}_{i+1}(K)\tilde{W}_{i+3}(K) = 0$. This leads to a contradiction. Hence Theorem 3.1 is ture. □

Combine Theorems B and Theorem 3.1 to obtain that

THEOREM 3.2. *Let K and L be star bodies in $\mathbb{R}^n$, then*

$$\frac{\tilde{W}_i(K\tilde{+}L)}{\tilde{W}_{i+1}(K\tilde{+}L)} \leqslant \frac{\tilde{W}_i(K)}{\tilde{W}_{i+1}(K)} + \frac{\tilde{W}_i(L)}{\tilde{W}_{i+1}(L)}$$

is true if and only if $i = n-1$ or $i = n-2$.

An interesting special case is when $n = 3$ and $i = n-2$. If S and ω denote surface area and mean width respectively, we obtain the inequality

$$\frac{S(K\tilde{+}L)}{\omega(K\tilde{+}L)} \leqslant \frac{S(K)}{\omega(K)} + \frac{S(L)}{\omega(L)},$$

for all star bodies K and L in $\mathbb{R}^3$.

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REFERENCES

[1] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge Univ. Press, 1993.
[2] Y. D. BURAGO, V. A. ZALGALLER, *Geometric Inequalities*, Springer-Verlag, Berlin-New york, 1988.
[3] E. LUTWAK, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.
[4] R. J. GARDNER, *Geometric Tomography*, Cambridge Univ. Press, New York, 1996.
[5] E. LUTWAK, *Dual mixed volumes*, Pacific J. Math. **58** (1975), 531–538.
[6] M. MARCUS, I. LOPES, *Inequalities for symmetric functions and Hermitian matrices*, Canad J. Math. **8** (1956), 524–531.
[7] E. F. BECHENBACH, R. BELLMAN, *Inequalities*, Springer-Verlag, Berlin-Göttingen, Heidelberg, 1961.
[8] H. BERGSTROM, *A triangle inequality for matrices. Den Elfte Skandinaviski Matematiker-kongress*, Trondheim, 1949, Oslo: John Grundt Tanums Forlag, 1952.
[9] R. BELLMAN, *Notes on matrix theory-IV: an inequality due to Bergstrom*, Amer. Math. Monthly **62** (1955), 172–173.
[10] K. FAN, *Some inequalities conerning positive-definite hermitian matrices*, Proc. Cambridge Phil. Soc. **51** (1955), 414–421.
[11] K. FAN, *Problem 4786*, Amer. Math. Monthly **65** (1958), 289.
[12] L. MIRSKY, *Maximum principles in matrix theory*, Proc. Glasgow Math. Assoc. **4** (1958), 34–37.
[13] M. FRADELIZI, A. GIANNOPOULOS AND M. MEYER, *Some inequalities about mixed volumes*, Israel J. Math. **135** (2003), 157–179.
[14] A. GIANNOPOULOS, M. HARTZOULAKI AND G. PAOURIS, *On a local version of the Aleksandrov-Fenchel inequality for the quermassintegrtals of a convex body*, Proc. Amer. Math. Soc. **130** (2002), 2403–2412.
[15] A. DEMBO, T. M. COVER AND J. A. THOMAS, *Information theoretic inequalities*, IEEE Trans. Infor. Theory **37** (1991), 1501–1518.

[16] X. Y. LI, G. S. LENG, *Some inequalities about dual mixed volumes of star bodies*, Acta Math. Sci. **25** (2005), 505–510.
[17] X. Y. LI, *Inequalities and extremum problems for convex bodies and star bodies*, The dissertation for the degree of doctor, 2004, 35–36.

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