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# CONCENTRATION–COMPACTNESS PRINCIPLE FOR GENERALIZED MOSER–TRUDINGER INEQUALITIES: CHARACTERIZATION OF THE NON–COMPACTNESS IN THE RADIAL CASE

ROBERT ČERNÝ

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*Abstract.* Let $B(R) \subset \mathbb{R}^n$, $n \geqslant 2$, be an open ball. By a result from [1], the Moser functional with the borderline exponent from the Moser inequality fails to be sequentially weakly continuous on the set of radial functions from the unit ball in $W_0^{1,n}(B(R))$ only in the exceptional case of sequences acting like a concentrating Moser sequence (in particular, these sequences are weakly converging to zero).

We extend this result to the case of a nontrivial weak limit and the Moser functional with the borderline exponent from the Concentration-Compactness Alternative. The same result is obtained for the Orlicz-Sobolev space $W_0L^n\log^{\alpha}L(B(R))$ with $\alpha < n-1$. We also consider the case of Orlicz-Sobolev spaces embedded into multiple exponential spaces.

## 1. Introduction

Throughout the paper, $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, $\omega_{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure of the surface of the unit sphere in $\mathbb{R}^n$ and the $n$-dimensional Lebesgue measure is denoted by $\mathscr{L}^n$. By $\nabla u$ we denote the generalized gradient of $u$ and $u^{\#}$ is the Schwarz symmetrization of $u$ (the definition is given in Section 2). The space $W_0^{1,n}(\Omega)$ or $W_0L^{\Phi}(\Omega)$ (where $\Phi$ is a Young function) stands for the closure of $C_0^{\infty}(\Omega)$ in $W^{1,n}(\Omega)$ or $WL^{\Phi}(\Omega)$, respectively. We use the standard notation $n' = \frac{n}{n-1}$.

For functions from $W_0^{1,n}(\Omega)$ the famous Moser-Trudinger inequality [21] concerning a classical embedding theorem by Trudinger [25] states that

$$\sup_{||\nabla u||_{L^n(\Omega)} \leqslant 1} \int_{\Omega} \exp(K|u(x)|^{n'})\,dx \begin{cases} \leqslant C(n,K,\mathscr{L}^n(\Omega)) & \text{when } K \leqslant n\omega_{n-1}^{\frac{1}{n-1}} \\ = \infty & \text{when } K > n\omega_{n-1}^{\frac{1}{n-1}}. \end{cases} \tag{1.1}$$

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The proof in the case of $K > n\omega_{n-1}^{\frac{1}{n-1}}$ easily follows from the properties of the Moser functions $\tilde{m}_t \in W_0^{1,n}(B(R))$, $t \in (0,1)$, defined by

$$\tilde{m}_t(x) = \begin{cases} \omega_{n-1}^{-\frac{1}{n}} \log^{\frac{1}{n'}}(\frac{1}{t}) & \text{for } |x| \in [0,tR] \\ \omega_{n-1}^{-\frac{1}{n}} \log^{-\frac{1}{n}}(\frac{1}{t}) \log(\frac{R}{|x|}) & \text{for } |x| \in [tR,R]. \end{cases} \tag{1.2}$$

From (1.1) and the Vitali Convergence Theorem (see e.g. [16, page 187]), it follows that if $p < 1$, then the functional

$$J_p(u) = \int_\Omega \exp(n\omega_{n-1}^{\frac{1}{n-1}} p|u(x)|^{n'})\,dx \tag{1.3}$$

is sequentially weakly continuous on the unit ball in $W_0^{1,n}(\Omega)$. That is,

$$u_k \rightharpoonup u \quad \text{and} \quad ||\nabla u_k||_{L^n(\Omega)} \leqslant 1 \qquad \Longrightarrow \qquad J_p(u_k) \to J_p(u).$$

If $p \geqslant 1$, then it is well-known and easy to check that the functional $J_p$ is generally not sequentially weakly continuous on the unit ball in $W_0^{1,n}(\Omega)$. Indeed, if $p > 1$ and $\Omega$ contains the origin, we fix $R > 0$ such that $B(R) \subset \Omega$ and we obtain $J_p(\tilde{m}_t) \to \infty$ as $t \to 0$, while for every sequence $t_k \subset (0,1)$, such that $t_k \to 0$, we have $\tilde{m}_{t_k} \rightharpoonup 0$ and $J_p(0) = \mathscr{L}^n(\Omega) < \infty$ (in the case of $0 \notin \Omega$, we use translated Moser functions). If $p = 1$, we fix $R > 0$, we set $\Omega = B(R)$ and it is easy to check that there are $C_0 > \mathscr{L}^n(B(R)) = J_1(0)$ and $t_0 \in (0,1)$ such that $J_1(\tilde{m}_t) \geqslant C_0$ for every $t \in (0,t_0)$.

In recent paper [1] the following characterization of the sequential weak continuity of the functional $J_p$ concerning the case of $p = 1$ and $u_k \rightharpoonup 0$, where $u_k$ are radial functions from $W_0^{1,n}(B(R))$, is given.

THEOREM 1.1. *Let $n \in \mathbb{N}$, $n \geqslant 2$ and $R > 0$. Suppose that $\{u_k\} \subset W_0^{1,n}(B(R))$ are radial functions such that $||\nabla u_k||_{L^n(B(R))} \leqslant 1$ and $u_k \rightharpoonup u$ in $W_0^{1,n}(B(R))$. If*

$$\limsup_{k\to\infty} J_1(u_k) > J_1(u),$$

*then there are $\{u_{k_m}\} \subset \{u_k\}$ and $\{t_m\} \subset (0,1)$, $t_m \to 0$, such that*

$$u_{k_m} - \tilde{m}_{t_m} \overset{m\to\infty}{\longrightarrow} 0 \qquad \text{in } W_0^{1,n}(B(R)).$$

In fact, Theorem 1.1 gives some information only in the case of $u = 0$ a.e. Otherwise (i.e. when $u$ is nontrivial), Theorem 1.2 below and the Vitali Convergence Theorem imply $\lim_{k\to\infty} J_1(u_k) = J_1(u)$.

Let us note that, in paper [1], a more difficult version of Theorem 1.1 concerning the case of non-radial functions on an open set $\Omega \subset \mathbb{R}^2$ is given. In that case, one has to consider a translated Moser sequence. It is an open problem whether some analogue of the result as Theorem 1.1 for non-radial functions in the case of $n \geqslant 2$ holds.

If $p > 1$ and $u_k \rightharpoonup u$ (we do not mind whether $u$ is trivial or not), then there are many sequences distant from $\{\tilde{m}_{t_k}\}$ such that $J_p(u_k) \to \infty$ while we always have $J_p(u) < \infty$ by the Trudinger embedding (for example, fix any $\rho \in [1,p)$ and consider $u_k = \rho^{-\frac{n-1}{n}} \tilde{m}_{t_k}$, with $t_k \to 0$, and observe that $u_k \rightharpoonup 0$ in $W_0^{1,n}(B(R))$).

A natural question to ask is what happens if the limit function $u$ in Theorem 1.1 is nontrivial (which means in this context that $u$ is a nonzero function). The aim of this paper is to answer this question in the radial case. The result is the following. If $0 \leqslant ||\nabla u||_{L^n(B(R))} < 1$, then there is $P > 1$ depending on $||\nabla u||_{L^n(B(R))}$ such that the functional $J_P$ behaves in a similar way as the one in Theorem 1.1, while for every $p < P$ we have $J_p(u_k) \to J_p(u)$ and for every $p > P$ we generally do not have that $\{J_p(u_k)\}$ is a bounded sequence. The constant $P$ is the borderline exponent from the Concentration-Compactness Alternative by Lions [19].

We are going to prove our result in higher generality. We replace the Sobolev space $W_0^{1,n}(B(R))$ by more general Orlicz-Sobolev spaces embedded into exponential and multiple exponential Orlicz spaces.

For the convenience of the reader we first focus on the case of $W_0^{1,n}(B(R))$, then we deal with the general case of $W_0 L^{\Phi}(B(R))$.

**Sobolev case**

An often used improvement of the Moser-Trudinger inequality is the following result from [5] and [19, Theorem I.6 and Remark I.18] which concerns one of the cases in the Concentration-Compactness Alternative for the Moser-Trudinger inequality.

THEOREM 1.2. *Let $n \in \mathbb{N}$, $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a sequence satisfying*

$$||\nabla u_k||_{L^n(\Omega)} \leqslant 1, \qquad u_k \rightharpoonup u \quad \text{in } W_0^{1,n}(\Omega) \qquad \text{and} \qquad u_k \to u \quad \text{a.e. in } \Omega$$

*for some $u \in W_0^{1,n}(\Omega)$. Let us set*

$$\theta = ||\nabla u||^n_{L^n(\Omega)} \in [0,1] \qquad \text{and} \qquad P = (1-\theta)^{-\frac{1}{n-1}} \tag{1.4}$$

*(where we read $P = \infty$ if $\theta = 1$). Then for every $p < P$ we have*

$$\int_\Omega \exp(n\omega_{n-1}^{\frac{1}{n-1}} p |u_k(x)|^{\frac{n}{n-1}})\, dx \leqslant C \qquad \text{where } C \text{ is independent of } k.$$

In the version of Theorem 1.1 with a nontrivial weak limit, it is natural to work with the functional $J_p$ where $p = P$. Indeed, if $p < P$, we can again use the Vitali Convergence Theorem. Furthermore, it is shown in [5], that if we take a suitable function $u \in W_0^{1,n}(B(3R))$ and if we set

$$u_k = u + (1-\theta)^{\frac{1}{n}} \tilde{m}_{\frac{1}{k}},$$

then we have $||\nabla u_k||_{L^n(B(3R))} = 1$, $u_k \rightharpoonup u$ and $J_p(u_k) \to \infty$ for every $p > P$. Hence for $p > P$, we can again construct many sequences such that $u_k \rightharpoonup u$ and $J_p(u_k) \to \infty$, while $J_p(u) < \infty$.

Now, let us state our results.

THEOREM 1.3. *Let $n \in \mathbb{N}$, $n \geqslant 2$ and $R > 0$. Let $\{u_k\} \subset W_0^{1,n}(B(R))$ be radial functions such that $||\nabla u_k||_{L^n(B(R))} \leqslant 1$ and $u_k \rightharpoonup u$ in $W_0^{1,n}(B(R))$. Let $\theta \in [0,1]$ and $P \in [1,\infty]$ be defined by (1.4). If $\theta < 1$ and*

$$\limsup_{k\to\infty} J_P(u_k) > J_P(u),$$

*then there are $\{u_{k_m}\} \subset \{u_k\}$ and $\{t_m\} \subset (0,1)$, $t_m \to 0$, such that*

$$u_{k_m} - u - (1-\theta)^{\frac{1}{n}} \tilde{m}_{t_m} \overset{m\to\infty}{\to} 0 \qquad \text{in } W_0^{1,n}(B(R)).$$

We are also able to prove the following result which is a bit stronger, since we do not suppose that $u_k$ are radial. On the other hand, we obtain the assertion only for the symmetrized functions $(u_{k_m} - u)^\#$ in the place of $u_k - u$. However, the result in terms of $(u_{k_m} - u)^\#$ instead of $u_{k_m}^\# - u^\#$ seems to be more suitable for possible future ambitions to obtain a version of Theorem 1.3 without the assumption concerning the symmetry.

THEOREM 1.4. *Let $n \in \mathbb{N}$, $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $R > 0$ be such that $\mathscr{L}^n(\Omega) = \mathscr{L}^n(B(R))$. Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be such that $||\nabla u_k||_{L^n(\Omega)} \leqslant 1$ and $u_k \rightharpoonup u$ in $W_0^{1,n}(\Omega)$. Let $\theta \in [0,1]$ and $P \in [1,\infty]$ be defined by (1.4). If $\theta < 1$ and*

$$\limsup_{k\to\infty} J_P(u_k) > J_P(u),$$

*then there are $\{u_{k_m}\} \subset \{u_k\}$ and $\{t_m\} \subset (0,1)$, $t_m \to 0$, such that*

$$(u_{k_m} - u)^\# - (1-\theta)^{\frac{1}{n}} \tilde{m}_{t_m} \overset{m\to\infty}{\to} 0 \qquad \text{in } W_0^{1,n}(B(R)).$$

Both previous theorems are contained in our general result concerning the Orlicz-Sobolev spaces embedded into exponential and multiple exponential spaces (see Theorem 1.7 below).

### Orlicz-Sobolev case

First, let us recall some well-known results concerning embeddings into exponential and multiple exponential spaces. If $\ell \in \mathbb{N}$ and $\alpha < n-1$, we set

$$\gamma = \frac{n}{n-1-\alpha} > 0, \qquad B = 1 - \frac{\alpha}{n-1} = \frac{n}{(n-1)\gamma} > 0$$
$$\text{and} \quad K_{\ell,n,\alpha} = \begin{cases} B^{\frac{1}{B}} n \omega_{n-1}^{\frac{\gamma}{n}} & \text{for } \ell = 1 \\ B^{\frac{1}{B}} \omega_{n-1}^{\frac{\gamma}{n}} & \text{for } \ell \geqslant 2. \end{cases} \tag{1.5}$$

The Sobolev-type space $W_0L^n\log^\alpha L(\Omega)$, built on the Zygmund space $L^n\log^\alpha L(\Omega)$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(t^\gamma)$ for large $t$ (see [17] and [10]). Moreover it is shown in [10] (see also [8] and [11]) that in the limiting case $\alpha = n-1$ we have the embedding into a double exponential space, i.e. the space $W_0L^n\log^{n-1}L\log^\alpha\log L(\Omega)$, $\alpha < n-1$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(\exp(t^\gamma))$ for large $t$. Further, in the limiting case $\alpha = n-1$ we have the embedding into a triple exponential space and so on. The borderline case is always $\alpha = n-1$ and for $\alpha > n-1$ we have the embedding into $L^\infty(\Omega)$. It is well-known that the Zygmund space $L^n\log^\alpha L(\Omega)$ coincides with the Orlicz space $L^\Phi(\Omega)$, where

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log^\alpha(t)} = 1,$$

the space $L^n\log^{n-1}L\log^\alpha\log L(\Omega)$ coincides with $L^\Phi(\Omega)$ where

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log^{n-1}(t)\log^\alpha(\log(t))} = 1,$$

and so on. For other results concerning these spaces and their precise definitions we refer the reader to [11], [12], [13], [14], [15] and [22].

The following notation is useful when dealing with the multiple logarithmic and multiple exponential spaces. Let us write

$$\log_{[1]}(t) = \log(t) \qquad \text{and} \qquad \log_{[j]}(t) = \log(\log_{[j-1]}(t)) \quad \text{for } j \geqslant 2,\ j\in\mathbb{N},$$

and

$$\exp_{[1]}(t) = \exp(t) \qquad \text{and} \qquad \exp_{[j]}(t) = \exp(\exp_{[j-1]}(t)) \quad \text{for } j \geqslant 2,\ j\in\mathbb{N}.$$

Let $\ell\in\mathbb{N}$ and $\alpha < n-1$. Then we have the above mentioned embedding results for any Young function $\Phi$ satisfying

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\left(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}(t)\right)\log_{[\ell]}^\alpha(t)} = 1 \tag{1.6}$$

(for $\ell = 1$ we read (1.6) as $\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log_{[1]}^\alpha(t)} = 1$). As $\Omega$ is bounded, all Young functions satisfying (1.6) give us the same Orlicz-Sobolev space.

Now, let us recall the generalized Moser-Trudinger inequality.

THEOREM 1.5. *Let $K\geqslant 0$, $\ell\in\mathbb{N}$, $n\in\mathbb{N}$, $n\geqslant 2$, $\alpha < n-1$ and let $\Omega\subset\mathbb{R}^n$ be a bounded domain. Let $\Phi$ be a Young function satisfying* (1.6).

(i) *If $u\in W_0L^\Phi(\Omega)$, then*

$$\int_\Omega \exp_{[\ell]}\left(K|u(x)|^\gamma\right)dx < \infty.$$

(ii) *If* $K < K_{\ell,n,\alpha}$, *then*

$$\sup_{u\in W_0L^\Phi(\Omega),||\Phi(\nabla u)||_{L^1(\Omega)}\leqslant 1}\int_\Omega \exp_{[\ell]}\big(K|u(x)|^\gamma\big)\,dx \leqslant C(\ell,n,\alpha,\Phi,\mathscr{L}^n(\Omega),K).$$

(iii) *If* $K > K_{\ell,n,\alpha}$, *then*

$$\sup_{u\in W_0L^\Phi(\Omega),||\Phi(\nabla u)||_{L^1(\Omega)}\leqslant 1}\int_\Omega \exp_{[\ell]}\big(K|u(x)|^\gamma\big)\,dx = \infty.$$

The first assertion follows from [10, Remarks 3.11(iv)]. The remaining two assertions follow from [18, Theorem 1.1 and Theorem 1.2] (cases $\ell = 1$ and $\ell = 2$) and [7, Theorem 1.1 and Theorem 1.2] (case $\ell \geqslant 3$). It is also shown in [18] and [7] that if $K = K_{\ell,n,\alpha}$, then the finiteness of the supremum depends on the choice of $\Phi$.

Now, let us recall the result from [4] concerning the improvement of the Moser-Trudinger inequality in the case of a nontrivial weak limit.

THEOREM 1.6. *Let* $\ell\in\mathbb{N}$, $n\in\mathbb{N}$, $n\geqslant 2$, $\alpha < n-1$ *and let* $\Omega\subset\mathbb{R}^n$ *be a bounded domain. Let* $\Phi$ *be a Young function satisfying* (1.6). *Let* $\{u_k\}\subset W_0L^\Phi(\Omega)$ *be a sequence satisfying*

$$||\Phi(|\nabla u_k|)||_{L^1(\Omega)} \leqslant 1, \qquad u_k \rightharpoonup u \quad \text{in } W_0L^\Phi(\Omega) \qquad \text{and} \qquad u_k \to u \quad \text{a.e. in } \Omega$$

*for some* $u\in W_0L^\Phi(\Omega)$. *Then for every*

$$p < P := \Big(1 - ||\Phi(|\nabla u|)||_{L^1(\Omega)}\Big)^{-\frac{\gamma}{n}}$$

*(where we define* $P = \infty$ *if* $||\Phi(|\nabla u|)||_{L^1(\Omega)} = 1$*) we have*

$$\int_\Omega \exp_{[\ell]}(K_{\ell,n,\alpha}p|u_k(x)|^\gamma)\,dx \leqslant C \qquad \text{where } C \text{ is independent of } k.$$

See [6] and [3] for the full statement of the Concentration-Compactness Principle concerning the spaces $W_0L^\Phi(\Omega)$ with $\Phi$ satisfying (1.6).

In view of Theorem 1.6 it is natural to extend the definition of the functional $J_p$ from (1.3) to

$$J_p(u) = \int_\Omega \exp_{[\ell]}(K_{\ell,n,\alpha}p|u_k(x)|^\gamma)\,dx.$$

Next, we define the functions plying the role of the Moser functions in the Orlicz-Sobolev setting. First, let us fix $L > 1$ such that

$$\log_{[\ell]}(L) \quad \text{is well defined and positive.}$$

We set for every $t\in(0,\frac{1}{L})$

$$m_t(x) = \begin{cases} B^{\frac{1-n}{n}}\omega_{n-1}^{-\frac{1}{n}}\log_{[\ell]}^{\frac{1}{\gamma}}(\frac{1}{t}) & \text{for } |x|\in[0,tR] \\ B^{\frac{1-n}{n}}\omega_{n-1}^{-\frac{1}{n}}\log_{[\ell]}^{\frac{1}{\gamma}-B}(\frac{1}{t})\log_{[\ell]}^{B}(\frac{R}{|x|}) & \text{for } |x|\in[tR,\frac{R}{L}] \\ (\frac{L}{L-1}-\frac{L}{L-1}\frac{|x|}{R})B^{\frac{1-n}{n}}\omega_{n-1}^{-\frac{1}{n}}\log_{[\ell]}^{\frac{1}{\gamma}-B}(\frac{1}{t})\log_{[\ell]}^{B}(L) & \text{for } |x|\in[\frac{R}{L},R]. \end{cases} \tag{1.7}$$

Now, we introduce two conditions on the Young function $\Phi$ (satisfying (1.6)) that guarantee the uniform convexity of the space $W_0L^\Phi(\Omega)$ equipped with the Luxemburg norm corresponding to $\Phi$. These conditions are

$$\text{for every } \varepsilon > 0 \text{ there are } k_\varepsilon > 1 \text{ and } t_\varepsilon > 0 \text{ such that} \\ \frac{\Phi'((1+\varepsilon)t)}{\Phi'(t)} \geqslant k_\varepsilon \qquad \text{for every } t \geqslant t_\varepsilon \tag{1.8}$$

and

$$\Phi \text{ is strictly convex.} \tag{1.9}$$

By the criterion from [23, Theorem 10 in section 7.2], the uniform convexity follows from (1.8), (1.9) and the $\Delta_2$-condition. It can be easily checked that the $\Delta_2$-condition follows from (1.6). Notice that the above criterion can be also applied to the uniform convexity of the Luxemburg norm corresponding to the Young function $\beta\Phi$, with arbitrary $\beta > 0$.

Now, we can state our main result.

THEOREM 1.7. *Let $\ell \in \mathbb{N}$, $n \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $R > 0$ be such that $\mathscr{L}^n(\Omega) = \mathscr{L}^n(B(R))$. Let $\Phi$ be a Young function satisfying (1.6), (1.8) and (1.9). Let $\{u_k\} \subset W_0L^\Phi(\Omega)$ be a sequence satisfying*

$$\int_\Omega \Phi(|\nabla u_k(x)|)\,dx \leqslant 1 \qquad \text{and} \qquad u_k \rightharpoonup u \quad \text{in } W_0L^\Phi(\Omega)$$

*for some $u \in W_0L^\Phi(\Omega)$. Let us set*

$$\theta := \int_\Omega \Phi(|\nabla u(x)|)\,dx \in [0,1], \qquad \xi := \int_{B(R)} \Phi(|\nabla u^\#(x)|)\,dx \in [0,1],$$

$$P_\theta = (1-\theta)^{-\frac{\gamma}{n}} \in [1,\infty] \qquad \text{and} \qquad P_\xi = (1-\xi)^{-\frac{\gamma}{n}} \in [1,\infty].$$

(i) *If $\theta < 1$ and $\limsup_{k\to\infty} J_{P_\theta}(u_k) > J_{P_\theta}(u)$, then there are $\{u_{k_m}\} \subset \{u_k\}$ and $\{t_m\} \subset (0,1)$, $t_m \to 0$, such that*

$$(u_{k_m} - u)^\# - (1-\theta)^{\frac{1}{n}} m_{t_m} \overset{m\to\infty}{\longrightarrow} 0 \qquad \text{in } W_0L^\Phi(B(R)). \tag{1.10}$$

(ii) *If $\xi < 1$ and $\limsup_{k\to\infty} J_{P_\xi}(u_k) > J_{P_\xi}(u)$, then there are $\{u_{k_m}\} \subset \{u_k\}$ and $\{t_m\} \subset (0,1)$, $t_m \to 0$, such that*

$$u^\#_{k_m} - u^\# - (1-\xi)^{\frac{1}{n}} m_{t_m} \overset{m\to\infty}{\longrightarrow} 0 \qquad \text{in } W_0L^\Phi(B(R)). \tag{1.11}$$

(iii) *In the special case of $u = 0$ a.e. (hence $\theta = \xi = 0$), we have $P_\theta = P_\xi = 1$, (1.10) and (1.11) read*

$$u^\#_{k_m} - m_{t_m} \overset{m\to\infty}{\longrightarrow} 0 \qquad \text{in } W_0L^\Phi(B(R))$$

*and we do not need to assume (1.8) and (1.9).*

Notice that since $u$ and $u^{\#}$ are equidistributed (and similarly for $u_k$ and $u_k^{\#}$), we can read the condition $J_{P_\xi}(u_k) \to J_{P_\xi}(u)$ also as $J_{P_\xi}(u_k^{\#}) \to J_{P_\xi}(u^{\#})$.

Theorem 1.7 implies Theorems 1.3 and 1.4. Indeed, in the case of radial functions, we have $\theta = \xi$. Next, for $\alpha = 0$, we have $B = 1$. Thus, we observe that $m_t = \tilde{m}_t$ in $B(\frac{R}{L})$.

Furthermore, we have $\log^{-\frac{1}{n}}(\frac{1}{t}) \to 0$ as $t \to 0$ and thus it is easy to check that

$$||\nabla m_t||_{L^n(B(R)\setminus B(\frac{R}{L}))} \overset{t\to 0}{\to} 0 \qquad \text{and} \qquad ||\nabla \tilde{m}_t||_{L^n(B(R)\setminus B(\frac{R}{L}))} \overset{t\to 0}{\to} 0.$$

Hence we obtain

$$m_{t_m} - \tilde{m}_{t_m} \overset{m\to\infty}{\to} 0 \qquad \text{in } W_0^{1,n}(B(R)) \tag{1.12}$$

and thus Theorems 1.3 and 1.4 follow from Theorem 1.7.

REMARK 1.1. For every $\ell \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$, there exists a Young function satisfying conditions (1.6), (1.8) and (1.9). For example, we can consider a Young function $\Phi_0$ satisfying

$$\Phi_0 \text{ is strictly convex,}$$
$$\Phi_0(t) = \begin{cases} t^n & \text{for } t \in (0,t_0) \\ t^n\left(\prod_{j=1}^{\ell-1} \log_{[j]}^{n-1}(t)\right) \log_{[\ell]}^{\alpha}(t) & \text{for } t \in (t_1,\infty) \end{cases} \tag{1.13}$$

for suitably chosen $0 < t_0 < t_1$. The function $\Phi_0$ obviously satisfies (1.6) and (1.9). The proof that $\Phi_0$ satisfies (1.8) is given in Section 3.

The paper is organized as follows. In Preliminaries we recall several basic facts concerning Orlicz-Sobolev spaces. In Section 3 we give some notes concerning the uniform convexity of the spaces $W_0^{1,n}(\Omega)$ and $W_0L^{\Phi}(\Omega)$.

For the convenience of the reader interested in the Sobolev case only, we give simple proofs of Theorems 1.3 and 1.4 in Section 4. Let us also note that in Section 4 we do not use any results from Preliminaries.

The rest of the paper (Sections 5 and 6) is devoted to the more general Orlicz-Sobolev case. In Section 5 we study properties of the Moser-type functions defined in (1.7). The proof of Theorem 1.7 is given in the sixth section.

Since Theorems 1.3 and 1.4 follow from Theorem 1.7 (see (1.12)), the reader interested in the Orlicz-Sobolev case may skip Section 4. In fact, in the Orlicz-Sobolev case we use the same strategy as in the Sobolev case, but we have to overcome several technical difficulties. These difficulties are due to the fact that several phenomenons and constants are related to the Luxemburg norm, while others are related to the modular. In the Sobolev case, the modular is just the $n$-th power of the norm and thus passing from one to another is easy. In the Orlicz-Sobolev case, the relation between the Luxemburg norm and the modular is much more complicated. However, some careful estimates can be achieved using the observation that the Luxemburg norm is very close to 1 if and only if the modular is very close to 1 (see (2.4)). Furthermore, if we deal with functions possessing very large gradients, then their modulars (with respect to a Young function satisfying (1.6)) are very close to the $n$-power of the norm, as, for example,

$$\frac{(\lambda t)^n \log^\alpha(\lambda t)}{t^n \log^\alpha(t)} \approx \lambda^n \qquad \text{for } \lambda > 0 \text{ and } t \text{ very large.}$$

Sometimes it is possible to pass from the Luxemburg norm with respect to $\Phi$ to the (equivalent) Luxemburg norm with respect to $\beta\Phi$, where $\beta > 0$ is chosen so that we can use (2.4) for the second norm. In fact, the proof of (1.10) rests upon this strategy and thus all the preliminary computations and estimates have to be done for the Luxemburg norm with respect to $\beta\Phi$, with $\beta > 0$ being a general number.

## 2. Preliminaries

By $B(R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered at the origin with the radius $R > 0$. By $C$ we denote a generic positive constant which may depend on $\ell$, $n$, $\alpha$, $\Phi$ and $\mathscr{L}^n(\Omega)$. This constant may vary from expression to expression as usual. When integrating with respect to the $n$-dimensional Lebesgue measure we simply write $\int_\Omega \Phi(|\nabla u|)$ instead of $\int_\Omega \Phi(|\nabla u(x)|)\,dx$, etc.

### Young functions and Orlicz spaces

A function $\Phi : [0,\infty) \to [0,\infty)$ is a Young function if $\Phi$ is increasing, convex, $\Phi(0) = 0$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$.

We denote by $L^\Phi(\Omega)$ the Orlicz space corresponding to a Young function $\Phi$ on a set $\Omega$ with the Lebesgue measure. The space $L^\Phi(\Omega)$ is equipped with the Luxemburg norm

$$||u||_{L^\Phi(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Phi\Big(\frac{|u|}{\lambda}\Big) \leqslant 1\Big\}. \tag{2.1}$$

### $\Delta_2$-condition

We say that a Young function $\Phi$ satisfies the $\Delta_2$-condition, if there are $t_\Delta \geqslant 0$ and $C_\Delta > 1$ such that

$$\Phi(2t) \leqslant C_\Delta \Phi(t) \qquad \text{whenever } t \geqslant t_\Delta.$$

It is easy to see that if $\Phi$ satisfies the $\Delta_2$-condition for one fixed $t_\Delta > 0$ then it satisfies this condition with arbitrary $\tilde{t}_\Delta > 0$ with a different constant $\tilde{C}_\Delta > 1$. But we cannot take $\tilde{t}_\Delta = 0$ provided $t_\Delta > 0$ in general. From the $\Delta_2$-condition one easily proves that

$$\int_\Omega \Phi\Big(\frac{|u|}{||u||_{L^\Phi(\Omega)}}\Big) = 1 \qquad \text{whenever } ||u||_{L^\Phi(\Omega)} > 0, \tag{2.2}$$

the convergence in the norm is equivalent to the convergence in the modular (2.3)

$$||u_k||_{L^\Phi(\Omega)} \overset{k\to\infty}{\to} 1 \quad \Longleftrightarrow \quad \int_\Omega \Phi(|u_k|) \overset{k\to\infty}{\to} 1, \tag{2.4}$$

$$\text{norm is bounded away from 1 from below} \iff \text{modular is bounded away from 1 from below,}$$

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It is not difficult to check the $\Delta_2$-condition for our Young functions satisfying (1.6).

**Orlicz-Sobolev spaces**

Let $\Phi$ be a Young function satisfying (1.6). We define the Orlicz-Sobolev space $WL^{\Phi}(\Omega)$ as the set

$$WL^{\Phi}(\Omega) = \{u\colon u, |\nabla u| \in L^{\Phi}(\Omega)\}$$

equipped with the norm

$$\|u\|_{WL^{\Phi}(\Omega)} = \|u\|_{L^{\Phi}(\Omega)} + \|\nabla u\|_{L^{\Phi}(\Omega)},$$

where $|\nabla u|$ is the Euclidean norm in $\mathbb{R}^n$ of the generalized gradient $\nabla u$ of $u$.

We put $W_0L^{\Phi}(\Omega)$ for the closure of $C_0^{\infty}(\Omega)$ in $WL^{\Phi}(\Omega)$. The space $W_0L^{\Phi}(\Omega)$ is a reflexive Banach space and it is compactly embedded into $L^{\Phi}(\Omega)$. As $\Omega$ is bounded, on $W_0L^{\Phi}(\Omega)$ we can also use the Dirichlet norm

$$\|u\|_{W_0L^{\Phi}(\Omega)} := \|\nabla u\|_{L^{\Phi}(\Omega)},$$

which is equivalent to the standard Sobolev-type norm given above.

We write that $u_k \rightharpoonup u$ in $W_0L^{\Phi}(\Omega)$, if

$$\int_{\Omega} \frac{\partial u_k}{\partial x_i} v\,dx \to \int_{\Omega} \frac{\partial u}{\partial x_i} v\,dx \qquad \text{for every } v \in L^{\Psi}(\Omega) \text{ and } i \in \{1,\dots,n\}$$

where $\Psi$ is the associated Young function to $\Phi$.

Finally, let us recall that the norm in the space $W_0L^{\Phi}(\Omega)$ is weakly lower semicontinuous and so is the modular of the gradient.

**Non-increasing radially symmetric rearrangement**

The non-increasing rearrangement $u^*$ of a measurable function $u$ on $\Omega$ is

$$u^*(y) = \inf\Big\{s > 0 : \mathscr{L}^n(\{x \in \Omega : |u(x)| > s\}) \leqslant y\Big\}, \quad y > 0.$$

We also define the non-increasing radially symmetric rearrangement $u^{\#}$ by

$$u^{\#}(x) = u^*\Big(\frac{\omega_{n-1}}{n}|x|^n\Big) \qquad \text{for } x \in B(R), \quad \mathscr{L}^n(B(R)) = \mathscr{L}^n(\Omega).$$

For an introduction to these rearrangements see e.g. [24]. When dealing with a radial function $u^\#$ on $B(R)$, it is often convenient for us to work with its one-dimensional representative $h:[0,R]\mapsto[0,\infty)$ defined by

$$h(|x|) := u^\#(x). \tag{2.5}$$

REMARK 2.1. For every $u\in W_0^{1,1}(\Omega)$ its one-dimensional representative $h$ defined in (2.5) is locally absolutely continuous on $(0,R]$ (and thus differentiable almost everywhere).

*Proof.* Fix $\delta\in(0,R)$. By [20, Section 1.1.3], every function from $W^{1,1}(\Omega)$ satisfies ACL, i.e. it is absolutely continuous on almost all lines parallel to coordinate axes. Hence the function

$$t\mapsto u(t,x_2,\dots,x_n) = h(\sqrt{t^2+x_2^2+\ldots+x_n^2})$$

is absolutely continuous for almost every $[x_2,\dots,x_n]\in\mathbb{R}^{n-1}$. In particular, we can find $[x_2,\dots,x_n]\in\mathbb{R}^{n-1}$ such that $c:=x_2^2+\ldots+x_n^2\leqslant\frac{\delta}{2}$ and the above mentioned function is absolutely continuous. Hence $t\mapsto h(\sqrt{t^2+c^2})$ is absolutely continuous and it is easy to see that $y\mapsto h(y)$ is absolutely continuous for $y\geqslant\sqrt{c^2+c^2}$, while $\sqrt{c^2+c^2}<2c\leqslant\delta$. □

We often use the Pólya-Szegö principle (see for example [2], [9], [24]).

THEOREM 2.1. *Let $\Phi$ be a Young function and let $u\in W^{1,1}(\mathbb{R}^n)$ satisfy $\mathscr{L}^n(\{x\in\mathbb{R}^n:|u(x)|>t\})<\infty$ for all $t>0$. Then*

$$\int_{\mathbb{R}^n}\Phi(|\nabla u|)\geqslant\int_{\mathbb{R}^n}\Phi(|\nabla u^\#|).$$

It is obvious that in the situation from Theorem 2.1 one also has $||\nabla u||_{L^\Phi(\mathbb{R}^n)}\geqslant||\nabla u^\#||_{L^\Phi(\mathbb{R}^n)}$. Let us also note that in the literature, there is often assumed that $u$ is non-negative in Theorem 2.1. This assumption simplifies the discussion of the equality cases, but it is irrelevant as far as the inequality is concerned, since $|\nabla|u||=|\nabla u|$ a.e.

**Preliminary results**

LEMMA 2.1. *Let $0<C_1<C_2$, $\Omega\subset\mathbb{R}^n$ be an open set and let $\Phi$ be a Young function satisfying (1.6). Then for every $\varepsilon>0$ there are $G>0$ and $\delta>0$ such that for every $u\in W_0L^\Phi(\Omega)$ satisfying $C_1\leqslant||\nabla u||_{L^\Phi(\Omega)}\leqslant C_2$ and $\int_{\{|\nabla u|<G\}}\Phi(|\nabla u|)<\delta$ we have*

$$(1-\varepsilon)||\nabla u||^n_{L^\Phi(\Omega)}\leqslant\int_\Omega\Phi(|\nabla u|)\leqslant(1+\varepsilon)||\nabla u||^n_{L^\Phi(\Omega)}.$$

*Proof.* Let us write $\lambda = ||\nabla u||_{L^\Phi(\Omega)}$ to simplify our notation. We can suppose that $\lambda > 0$, otherwise the proof trivially follows from (2.1). We are going to show that we can make $\frac{1}{\lambda^n}\int_\Omega \Phi(|\nabla u|)$ as close to 1 as we wish via a suitable choice of $G$ and $\delta$. Let $\Phi_0$ be a fixed Young function from (1.13). By (2.2) we have

$$\begin{aligned}
&\frac{1}{\lambda^n}\int_\Omega \Phi(|\nabla u|) - 1\\
&\quad = \frac{1}{\lambda^n}\int_\Omega \Phi(|\nabla u|) - \int_\Omega \Phi\Big(\frac{|\nabla u|}{\lambda}\Big)\\
&\quad = \frac{1}{\lambda^n}\int_{\{|\nabla u|<G\}} \Phi(|\nabla u|) + \frac{1}{\lambda^n}\int_{\{|\nabla u|\geqslant G\}} \Phi_0(|\nabla u|)\\
&\qquad + \frac{1}{\lambda^n}\int_{\{|\nabla u|\geqslant G\}} \Big(\Phi(|\nabla u|) - \Phi_0(|\nabla u|)\Big) - \int_{\{|\nabla u|<G\}} \Phi\Big(\frac{|\nabla u|}{\lambda}\Big)\\
&\qquad - \int_{\{|\nabla u|\geqslant G\}} \Phi_0\Big(\frac{|\nabla u|}{\lambda}\Big) - \int_{\{|\nabla u|\geqslant G\}} \Big(\Phi\Big(\frac{|\nabla u|}{\lambda}\Big) - \Phi_0\Big(\frac{|\nabla u|}{\lambda}\Big)\Big)\\
&\quad = I_1 + I_2 + I_3 - I_4 - I_5 - I_6.
\end{aligned}$$

Next, we claim that we can make $I_3$, $I_6$ and $I_2 - I_5$ as small as we wish choosing $G$ sufficiently large. This is obvious for $I_3$ and $I_6$, since we have (1.6) and $\Omega$ is bounded. If $G$ is sufficiently large, for $I_5$ we have

$$I_5 = \int_{\{|\nabla u|\geqslant G\}} \Big(\frac{|\nabla u|}{\lambda}\Big)^n \Big(\prod_{j=1}^{\ell-1} \log_{[j]}^{n-1}\Big(\frac{|\nabla u|}{\lambda}\Big)\Big) \log_{[\ell]}^{\alpha}\Big(\frac{|\nabla u|}{\lambda}\Big).$$

We can write

$$\left|\frac{\log(\frac{|\nabla u|}{\lambda})}{\log(|\nabla u|)} - 1\right| = \left|\frac{\log(\frac{1}{\lambda})}{\log(|\nabla u|)}\right| \leqslant \frac{\max\{|\log(C_1)|, |\log(C_2)|\}}{\log(G)}$$

and similarly for the remaining iterated logarithms. Hence choosing $G$ very large, we can make each $\log_{[j]}(\frac{|\nabla u|}{\lambda})$ as close to $\log_{[j]}(|\nabla u|)$ as we wish and thus $I_5$ can be made as close to $I_2$ as we wish.

Finally, for $G$ fixed, we choose $\delta > 0$ so small that $I_1$ and $I_4$ are as small as we wish (let us recall that $\frac{1}{\lambda}$ is bounded and $\Phi$ satisfies the $\Delta_2$-condition). Thus, we are done. □

LEMMA 2.2. *Let $R > 0$ and let $\Phi$ be a Young function satisfying* (1.6). *Then there is $G > 0$ with the following property:*

*For every $\beta > 0$ and $\varepsilon > 0$, there is $y_0 > 0$ such that if $u \in W_0L^\Phi(B(R))$ is a radial function, then it satisfies*

$$y \in (0, y_0) \quad \Longrightarrow \quad |h(y)| \leqslant GR + (1+\varepsilon)\beta^{-\frac{1}{n}} B^{\frac{1-n}{n}} \omega_{n-1}^{-\frac{1}{n}} \log_{[\ell]}^{\frac{1}{\gamma}}\Big(\frac{1}{y}\Big) ||\nabla u||_{L^{\beta\Phi}(\{|\nabla u|>G\})}, \tag{2.6}$$

*where* $||\nabla u||_{L^{\beta\Phi}(\{|\nabla u|>G\})}$ *denotes the Luxemburg norm with respect to the Young function* $\beta\Phi$ *on the set* $\{|\nabla u|>G\}$.

*In particular, setting* $\varepsilon = 1$ *we have* $y_0 > 0$ *such that*

$$y \in (0, y_0) \quad \Longrightarrow \quad |h(y)| \leqslant C + C\log_{[\ell]}^{\frac{1}{\gamma}}\left(\frac{1}{y}\right)||\nabla u||_{L^{\beta\Phi}(B(R))}. \tag{2.7}$$

*Proof.* The proof for the case $\beta = 1$ and $\ell = 1$ can be found in [18] (see the proof of Theorem 1.1), the case $\beta = 1$ and $\ell \geqslant 2$ is treated in [7]. The case $\beta \neq 1$ can be obtained by a minor modifications of these proofs. We omit the details. □

## 3. Uniform convexity of $W_0^{1,n}(\Omega)$ and $W_0L^{\Phi}(\Omega)$

A Banach space is uniformly convex if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$||u|| = ||v|| = 1,\ ||u-v|| > \varepsilon \qquad \Longrightarrow \qquad \left|\left|\frac{u+v}{2}\right|\right| < 1-\delta.$$

We already know that the space $W_0L^{\Phi}(\Omega)$ equipped with the Luxemburg norm corresponding to a Young function $\Phi$ satisfying (1.6), (1.8) and (1.9) is uniformly convex. This is also true for the space $W_0^{1,n}(\Omega)$ which can be considered as a space $W_0L^{\Phi}(\Omega)$ with $\Phi(t) = t^n$.

It is a well-known fact that if a sequence converges weakly in a uniformly convex Banach space, that is $u_k \rightharpoonup u$, and $||u_k|| \to ||u||$ (where $\|\cdot\|$ is a norm in this space), then then $u_k \to u$ (strong convergence in norm). We shall need a slight modification of this property.

LEMMA 3.1. *In every uniformly convex Banach space the following assertion holds. For every* $\varepsilon > 0$ *there is* $\delta \in (0,1)$ *such that*

$$u_k \rightharpoonup u,\ ||u|| = 1,\ ||u_k|| \leqslant 1+\delta \text{ for every } k \qquad \Longrightarrow \qquad ||u_k - u|| < \varepsilon \text{ for every } k \text{ sufficiently large.}$$

*Proof.* The proof is standard. □

REMARK 3.1. The homogeneity of the norm implies, that Lemma 3.1 holds with general $||u|| > 0$ and $||u_k|| \leqslant (1+\delta)||u||$. In the case of our Orlicz-Sobolev spaces, these assumptions can be also replaced by

$$\int_\Omega \Phi(|\nabla u|) > 0 \qquad \text{and} \qquad \int_\Omega \Phi(|\nabla u_k|) \leqslant (1+\delta)\int_\Omega \Phi(|\nabla u|).$$

Indeed, we can apply Lemma 3.1 with respect to the norm given by the Young function $\beta\Phi$, where the constant $\beta$ is chosen so that

$$\int_\Omega \beta\Phi(|\nabla u|) = 1.$$

Since $\beta\Phi$ satisfies the $\Delta_2$-condition, we have that $||\nabla u||_{L^{\beta\Phi}(\Omega)} = 1$ and $||\nabla u_k||_{L^{\beta\Phi}(\Omega)}$ are close to $1$. Therefore, by Lemma 3.1, we have that $u_k - u$ are small in the norm corresponding to $\beta\Phi$ and thus they are also small in the equivalent norm corresponding to $\Phi$.

In the rest of this section we prove Remark 1.1.

*Proof of Remark* 1.1. It is enough to check condition (1.8), the remaining properties are obviously satisfied. By (1.13) we have for $t > t_1$

$$\frac{\Phi_0'((1+\varepsilon)t)}{\Phi_0'(t)} = \frac{n(1+\varepsilon)^{n-1}t^{n-1}(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}((1+\varepsilon)t))\log_{[\ell]}^{\alpha}((1+\varepsilon)t))\Upsilon((1+\varepsilon)t)}{nt^{n-1}(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}(t))\log_{[\ell]}^{\alpha}(t)\Upsilon(t)}, \tag{3.1}$$

where

$$\Upsilon(t) = 1 + \sum_{j=1}^{\ell-1}\frac{n-1}{n}\Big(\prod_{i=1}^{j}\log_{[i]}^{-1}(t)\Big) + \frac{\alpha}{n}\Big(\prod_{i=1}^{\ell}\log_{[i]}^{-1}(t)\Big).$$

Now, we have

$$\frac{n(1+\varepsilon)^{n-1}t^{n-1}}{nt^{n-1}} = (1+\varepsilon)^{n-1} \geqslant (1+\varepsilon), \tag{3.2}$$

and

$$\frac{\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}((1+\varepsilon)t)}{\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}(t)} \geqslant 1. \tag{3.3}$$

Furthermore, we see that taking $t_\varepsilon$ sufficiently large we can make both $\Upsilon((1+\varepsilon)t)$ and $\Upsilon(t)$ as close to $1$ as we wish and thus

$$\frac{\Upsilon((1+\varepsilon)t)}{\Upsilon(t)} \geqslant (1+\varepsilon)^{-\frac{1}{3}}. \tag{3.4}$$

Finally, we have

$$\frac{\log((1+\varepsilon)t)}{\log(t)} = 1 + \frac{\log(1+\varepsilon)}{\log(t)} \leqslant 1 + \frac{\varepsilon}{\log(t)}$$

and thus for $t$ large enough we obtain

$$\frac{\log^{\alpha}((1+\varepsilon)t)}{\log^{\alpha}(t)} \geqslant \Big(\frac{\log((1+\varepsilon)t)}{\log(t)}\Big)^{-|\alpha|} \geqslant \Big(1+\frac{\varepsilon}{\log(t)}\Big)^{-|\alpha|} \geqslant (1+\varepsilon)^{-\frac{1}{3}}.$$

For the iterated logarithm it is also easy to prove (see for example [7, Lemma 2.2]) that

$$\frac{\log_{[\ell]}^{\alpha}((1+\varepsilon)t)}{\log_{[\ell]}^{\alpha}(t)} \geqslant (1+\varepsilon)^{-\frac{1}{3}} \tag{3.5}$$

for $t$ large enough. Thus, by (3.1), (3.2), (3.3), (3.4) and (3.5) we can set $k_\varepsilon = (1+\varepsilon)^{\frac{1}{3}}$ and we are done. $\square$

## 4. Proofs of Theorem 1.3 and Theorem 1.4

For every $t \in (0,1)$, we define the following functional acting on functions from $W_0^{1,n}(B(R))$

$$\langle \tilde{m}_t^*, u\rangle := \int_{B(R)} |\nabla \tilde{m}_t(x)|^{n-2} \nabla \tilde{m}_t(x) \cdot \nabla u(x)\, dx.$$

In the sequel, we are interested in radial functions only. Suppose that $h$ is the one-dimensional representative of a radial function $u$ (see (2.5)) and let $g_t$ represent $\tilde{m}_t$.

PROPOSITION 4.1. *Let $R > 0$. The Moser functions and the functional defined above have the following properties:*

$$||\nabla \tilde{m}_t||_{L^n(B(R))} = 1 \qquad \text{for every } t \in (0,1), \tag{4.1}$$

$$\langle \tilde{m}_t^*, u\rangle = \omega_{n-1}^{\frac{1}{n}} \log^{-\frac{1}{n'}}\left(\frac{1}{t}\right) h(Rt) = \frac{h(Rt)}{g_t(Rt)} \qquad \text{for every } t \in (0,1), \tag{4.2}$$

$$|\langle \tilde{m}_t^*, u\rangle| \leqslant ||\nabla u||_{L^n(B(R))} \qquad \text{for every } t \in (0,1), \tag{4.3}$$

$$\langle \tilde{m}_t^*, u\rangle \overset{t\to 0}{\to} 0 \quad \text{for every fixed radial function } u \in W_0^{1,n}(B(R)) \tag{4.4}$$

*and*

$$J_p(u) = \omega_{n-1} R^n \int_0^1 y^{n(1-p|\langle \tilde{m}_y^*, u\rangle|^{n'})} \frac{dy}{y}. \tag{4.5}$$

*Proof.* Property (4.1) is well-known and easy to compute from

$$|\nabla \tilde{m}_t(x)| = \left(\sum_{i=1}^n \left(\frac{\partial}{\partial x_i} \tilde{m}_t(x)\right)^2\right)^{\frac{1}{2}} = |g_t'(|x|)|$$

and (see (1.2))

$$g_t'(|x|) = \begin{cases} 0 & \text{for } |x| \in [0, tR] \\ -\omega_{n-1}^{-\frac{1}{n}} \log^{-\frac{1}{n}}\left(\frac{1}{t}\right) \frac{1}{|x|} & \text{for } |x| \in [tR, R]. \end{cases} \tag{4.6}$$

Furthermore, property (4.3) is easily obtained using Hölder's inequality.

Let us prove (4.2). By (4.6) and (1.2) we have

$$\begin{aligned}\langle m_t^*, u\rangle &= \int_0^R |g_t'(y)|^{n-2} g_t'(y) h'(y) \omega_{n-1} y^{n-1}\, dy = -\omega_{n-1}^{-\frac{n-1}{n}+1} \log^{-\frac{n-1}{n}}\left(\frac{1}{t}\right) \int_{tR}^R h'(y)\, dy \\ &= \omega_{n-1}^{\frac{1}{n}} \log^{-\frac{1}{n'}}\left(\frac{1}{t}\right) h(Rt) = \frac{h(Rt)}{g_t(Rt)}.\end{aligned}$$

We proceed to the proof of (4.4). Fix $\varepsilon > 0$. From the absolute continuity of the Lebesgue integral there is $\tau \in (0,R)$ such that $||\nabla u||_{L^n(B(\tau))} < \varepsilon$. Furthermore, for

$0 < y_1 < y_2 \leqslant R$ one has by Hölder's inequality

$$\begin{aligned}
|h(y_1) - h(y_2)| &\leqslant \int_{y_1}^{y_2} |h'(y)|\,dy \\
&= \int_{y_1}^{y_2} |h'(y)|\omega_{n-1}^{\frac{1}{n}} y^{\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}} y^{-\frac{n-1}{n}}\,dy \\
&\leqslant \omega_{n-1}^{-\frac{1}{n}} \Big(\int_{y_1}^{y_2} |h'(y)|^n \omega_{n-1} y^{n-1}\,dy\Big)^{\frac{1}{n}} \Big(\int_{y_1}^{y_2} \frac{dy}{y}\Big)^{\frac{1}{n'}} \\
&= \omega_{n-1}^{-\frac{1}{n}} ||\nabla u||_{L^n(B(y_2)\setminus B(y_1))} \log^{\frac{1}{n'}}\Big(\frac{y_2}{y_1}\Big).
\end{aligned}$$

Therefore we have from (4.2) and $h(R) = 0$ for $t$ small enough

$$\begin{aligned}
|\langle m_t^*, u\rangle| &\leqslant \omega_{n-1}^{\frac{1}{n}} \log^{-\frac{1}{n'}}\Big(\frac{1}{t}\Big)\Big(|h(\tau) - h(R)| + |h(Rt) - h(\tau)|\Big) \\
&\leqslant \log^{-\frac{1}{n'}}\Big(\frac{1}{t}\Big)\Big(||\nabla u||_{L^n(B(R))} \log^{\frac{1}{n'}}\Big(\frac{R}{\tau}\Big) + ||\nabla u||_{L^n(B(\tau))} \log^{\frac{1}{n'}}\Big(\frac{\tau}{Rt}\Big)\Big) \\
&\leqslant \log^{-\frac{1}{n'}}\Big(\frac{1}{t}\Big)\Big(C \log^{\frac{1}{n'}}\Big(\frac{R}{\tau}\Big) + \varepsilon \log^{\frac{1}{n'}}\Big(\frac{\tau}{Rt}\Big)\Big) \\
&\leqslant \log^{-\frac{1}{n'}}\Big(\frac{1}{t}\Big)\Big(2\varepsilon \log^{\frac{1}{n'}}\Big(\frac{1}{t}\Big)\Big) = 2\varepsilon
\end{aligned}$$

and (4.4) follows.

Now, we proceed to the proof of the last property. We have from (1.3) and (4.2)

$$\begin{aligned}
J_p(u) &= \int_0^R \exp\Big(n\omega_{n-1}^{\frac{1}{n-1}} p|h(y)|^{n'}\Big)\omega_{n-1} y^{n-1}\,dy \\
&= \omega_{n-1} R^n \int_0^1 \exp\Big(n\omega_{n-1}^{\frac{1}{n-1}} p|h(Rt)|^{n'}\Big) t^{n-1}\,dt \\
&= \omega_{n-1} R^n \int_0^1 \exp\Big(np \log\Big(\frac{1}{t}\Big)|\langle \tilde{m}_t^*, u\rangle|^{n'}\Big) t^{n-1}\,dt \\
&= \omega_{n-1} R^n \int_0^1 t^{n(1-p|\langle m_t^*, u\rangle|^{n'})} \frac{dt}{t}.
\end{aligned}$$

Thus, we are done. □

The following Lemmata 4.1 and 4.2 are Sobolev versions of more general Lemmata 5.1 and 5.3, respectively. Since there would be only a minor simplification of the proofs in the Sobolev setting, instead of the proofs we just give the reference to the proofs in the Orlicz-Sobolev setting.

LEMMA 4.1. *Let $\{t_k\} \subset (0,1)$, $t_k \to 0$ and let the sequence $\{u_k\} \subset W_0^{1,n}(\Omega)$ satisfy $||\nabla u_k||_{L^n(\Omega)} \leqslant C$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that the following assertion holds:*

*If $u \in W_0^{1,n}(\Omega)$ satisfies $||\nabla u||_{L^n(\Omega)} < \delta$ and there is $k_0 \in \mathbb{N}$ such that*

$$\langle \tilde{m}_{t_k}^*, u_k^\# \rangle \geqslant 1 - \varepsilon \qquad \textit{for } k \geqslant k_0,$$

*then there is* $k_1 \in \mathbb{N}$ *such that*

$$\langle \tilde{m}_{t_k}^*, (u_k - u)^\# \rangle \geqslant 1 - 5\varepsilon \qquad \text{for } k \geqslant k_1.$$

*Proof.* We can use the proof of Lemma 5.1. Since we have $\Phi(t) = t^n$, we are interested only in the first of the two cases considered when proving (5.16). □

LEMMA 4.2. *Let* $\{t_k\} \subset (0,1)$, $t_k \to 0$ *and let* $\{u_k\} \subset W_0^{1,n}(B(R))$ *be radial functions. If* $||\nabla u_k||_{L^n(B(R))} \leqslant 1 + \mathrm{o}(1)$ *and* $\langle \tilde{m}_{t_k}^*, u_k \rangle \to 1$, *then*

$$u_k - \tilde{m}_{t_k} \to 0 \qquad \text{in } W_0^{1,n}(B(R)).$$

*Proof.* We can use the proof of Lemma 5.3. It is enough to set $\beta = 1$ and replace the collection of Moser-type functions $\{m_{t_k}\}$ by our Moser functions $\{\tilde{m}_{t_k}\}$. We also use the estimates from Lemma 4.1 instead of the estimates from Lemma 5.1. □

*Proof of Theorem* 1.4. Assume that $\theta \in (0,1)$ (for $\theta = 0$, the proof follows from Theorem 1.1) and $\limsup_{k\to\infty} J_P(u_k) > J_P(u)$. Passing to a subsequence we can suppose that the limit exists and $\lim_{k\to\infty} J_P(u_k) > J_P(u)$. Passing to a subsequence again we can also suppose that $u_k \to u$ in $L^n(\Omega)$ and $u_k \to u$ a.e. in $\Omega$. Since the symmetric rearrangement preserves the convergence in Lebesgue spaces (see [24, Theorem 1.D]), we can also suppose that $u_k^\# \to u^\#$ in $L^n(B(R))$ and $u_k^\# \to u^\#$ a.e. in $B(R)$.

*Step* 1. We find a sequence $\{t_k\} \subset (0,1)$, $t_k \to 0$, such that

$$\liminf_{k\to\infty} \langle \tilde{m}_{t_k}^*, u_k^\# \rangle \geqslant (1-\theta)^{\frac{1}{n}} \tag{4.7}$$

and

$$\liminf_{k\to\infty} \langle \tilde{m}_{t_k}^*, (u_k - u)^\# \rangle \geqslant (1-\theta)^{\frac{1}{n}} \tag{4.8}$$

(passing to a subsequence of $\{u_k\}$ if necessary). To prove (4.7) assume that there are $\delta > 0$, $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that

$$\langle \tilde{m}_t^*, u_k^\# \rangle \leqslant (1-\varepsilon)(1-\theta)^{\frac{1}{n}} \qquad \text{for every } t \in (0,\delta) \text{ and every } k \geqslant k_0.$$

Hence using (4.5), $P = (1-\theta)^{-\frac{1}{n-1}}$ and (4.3) we obtain for $k \geqslant k_0$

$$\begin{aligned} J_P(u_k) &= \omega_{n-1} R^n \int_0^1 y^{n(1-P|\langle \tilde{m}_y^*, u_k \rangle|^{n'})} \frac{dy}{y} \\ &\leqslant \omega_{n-1} R^n \int_0^\delta y^{n-n(1-\varepsilon)^{n'}-1}\, dy + \omega_{n-1} R^n \int_\delta^1 y^{n-nP-1}\, dy < \infty. \end{aligned}$$

By the Lebesgue Dominated Convergence Theorem $J_P(u_k) \to J_P(u)$ which is a contradiction. Thus we can select $\{t_k\} \subset (0,1)$, $t_k \to 0$, and a suitable subsequence of $\{u_k\}$ such that (4.7) holds.

We proceed to the proof of (4.8). First, let us introduce the following notation. Given $L > 0$, we define

$$u_L(x) = \min\{|u(x)|, L\}\,\mathrm{sgn}(u(x)) \qquad \text{and} \qquad u^L(x) = u(x) - u_L(x).$$

Similarly we define $u_k^L$ and $(u_k)_L$, $k \in \mathbb{N}$. It can be easily seen that

$$\int_\Omega |\nabla u_k|^n = \int_\Omega |\nabla u_k^L|^n + \int_\Omega |\nabla (u_k)_L|^n, \quad u_k^L \to u^L \text{ a.e. in } \Omega \text{ and } (u_k)_L \to u_L \text{ a.e. in } \Omega.$$

Moreover $u_k^L$ form a bounded sequence in $W_0^{1,n}(\Omega)$ and thus there is a weakly convergent subsequence. Since $u_k^L$ converge almost everywhere to $u^L$, it is easy to see that

$$u_k^L \rightharpoonup u^L \text{ in } W_0^{1,n}(\Omega) \qquad \text{and} \qquad (u_k)_L \rightharpoonup u_L \text{ in } W_0^{1,n}(\Omega).$$

The proof of (4.8) is obtained establishing the following chain of inequalities

$$\begin{aligned}(1-\theta)^{\frac{1}{n}} &\leqslant \liminf_{k\to\infty} \langle \tilde{m}^*_{t_k}, u_k^\# \rangle \leqslant \liminf_{k\to\infty} \langle \tilde{m}^*_{t_k}, (u_k^L)^\# \rangle \\ &\leqslant \liminf_{k\to\infty} \langle \tilde{m}^*_{t_k}, (u_k^L - u^L)^\# \rangle + \varepsilon \leqslant \liminf_{k\to\infty} \langle \tilde{m}^*_{t_k}, (u_k - u)^\# \rangle + \varepsilon,\end{aligned} \tag{4.9}$$

with $\varepsilon > 0$ being an arbitrarily small number and $L$ depending on $\varepsilon$ is specified below.

The first inequality in (4.9) is just (4.7). The second inequality easily follows from $g_{t_k}(Rt_k) \to \infty$ (see (1.2)), (4.2) and

$$u_k^\# - L \leqslant (u_k^L)^\# \leqslant u_k^\#.$$

The third inequality follows from Lemma 4.1 (up to a normalization), since we can make $\int_\Omega |\nabla u^L|^n$ as small as we wish via a choice of a sufficiently large $L$. The last inequality follows from (4.2), since $g_{t_k}(Rt_k) \to \infty$ and

$$|u_k^L - u^L| = |u_k - u + (u_L - (u_k)_L)| \leqslant |u_k - u| + |u_L| + |(u_k)_L| \leqslant |u_k - u| + 2L.$$

This completes the proof of (4.8).

*Step* 2. In this step we prove

$$\limsup_{k\to\infty} ||\nabla (u_k - u)^\#||_{L^n(B(R))} \leqslant (1-\theta)^{\frac{1}{n}}. \tag{4.10}$$

Fix $\varepsilon > 0$. First we fix $L > 0$ so large that

$$\int_\Omega |\nabla u^L|^n = \tau, \tag{4.11}$$

where $\tau \in (0, \frac{1}{2}\min\{\theta, 1-\theta\})$ is a small number specified below.

By the Pólya-Szegö inequality (Theorem 2.1) we have

$$\begin{aligned}||\nabla (u_k - u)^\#||_{L^n(B(R))} &\leqslant ||\nabla (u_k - u)||_{L^n(\Omega)} \\ &\leqslant ||\nabla ((u_k)_L - u_L)||_{L^n(\Omega)} + ||\nabla u_k^L||_{L^n(\Omega)} + ||\nabla u^L||_{L^n(\Omega)} \\ &= I_1 + I_2 + I_3.\end{aligned} \tag{4.12}$$

If $\tau$ is small enough, then (4.11) implies that $I_3 < \varepsilon$.

Next, since $(u_k)_L \rightharpoonup u_L$, by the weak lower semicontinuity of the norm we have for $k$ large enough

$$\int_\Omega |\nabla (u_k)_L|^n \geqslant \int_\Omega |\nabla u_L|^n - \tau = \int_\Omega |\nabla u|^n - \int_\Omega |\nabla u^L|^n - \tau = \theta - 2\tau$$

and thus

$$\int_\Omega |\nabla u_k^L|^n = \int_\Omega |\nabla u_k|^n - \int_\Omega |\nabla (u_k)_L|^n \leqslant 1 - \theta + 2\tau.$$

Hence, if $\tau$ is small enough, we obtain $I_2 < (1-\theta)^{\frac{1}{n}} + \varepsilon$.

Let us proceed to the proof that $I_1 < \varepsilon$. We obtain from (4.9) for $k$ large enough

$$\langle \tilde{m}_{t_k}^*, (u_k^L)^\# \rangle \geqslant (1-\theta-\tau)^{\frac{1}{n}}$$

and thus, by (4.3) we have for $k$ large enough

$$||\nabla (u_k^L)^\#||_{L^n(B(R))} \geqslant (1-\theta-\tau)^{\frac{1}{n}}.$$

This implies by the Pólya-Szegö inequality (Theorem 2.1)

$$||\nabla u_k^L||_{L^n(\Omega)} \geqslant (1-\theta-\tau)^{\frac{1}{n}}$$

and thus we obtain

$$||\nabla (u_k)_L||_{L^n(\Omega)} = \Big(||\nabla u_k||^n_{L^n(\Omega)} - ||\nabla u_k^L||^n_{L^n(\Omega)}\Big)^{\frac{1}{n}} \leqslant (\theta+\tau)^{\frac{1}{n}}. \tag{4.13}$$

Furthermore, we have by (4.11)

$$||\nabla u_L||_{L^n(\Omega)} = \Big(||\nabla u||^n_{L^n(\Omega)} - ||\nabla u^L||^n_{L^n(\Omega)}\Big)^{\frac{1}{n}} = (\theta-\tau)^{\frac{1}{n}}. \tag{4.14}$$

Now, if $\tau > 0$ is sufficiently small, we can use Lemma 3.1 (recall that we have (4.13), (4.14) and $(u_k)_L \rightharpoonup u_L$) and the homogeneity of the norm to obtain

$$I_1 < \varepsilon \qquad \text{for } k \text{ large enough.}$$

Hence we have $I_1 + I_2 + I_3 \leqslant \varepsilon + (1-\theta)^{-\frac{1}{n}} + \varepsilon + \varepsilon$. This concludes the proof of (4.10).

*Step* 3. Our aim is to prove

$$(1-\theta)^{-\frac{1}{n}}(u_k-u)^\# - \tilde{m}_{t_k} \overset{k\to\infty}{\to} 0 \qquad \text{in } W_0^{1,n}(B(R)). \tag{4.15}$$

Combining (4.8) and (4.10) with (4.3) we obtain

$$\langle \tilde{m}_{t_k}^*, (1-\theta)^{-\frac{1}{n}}(u_k-u)^\# \rangle \overset{k\to\infty}{\to} 1 \qquad \text{and} \qquad ||(1-\theta)^{-\frac{1}{n}}\nabla (u_k-u)^\#||_{L^n(B(R))} \overset{k\to\infty}{\to} 1.$$

Now, we complete the proof of (4.15) using Lemma 4.2. Thus, we are done. □

*Proof of Theorem 1.3.* Let us suppose that $\theta \in (0,1)$ (for $\theta = 0$, the proof follows from Theorem 1.1) and $\limsup_{k\to\infty} J_P(u_k) > J_P(u)$. We can suppose that $\lim_{k\to\infty} J_P(u_k)$ exists, it satisfies $\lim_{k\to\infty} J_P(u_k) > J_P(u)$, $u_k \to u$ in $L^n(\Omega)$, $u_k \to u$ a.e. in $\Omega$. Recall that we suppose that $u_k$ and $u$ are radial functions and $\Omega = B(R)$ now.

*Step* 1. The aim of this step is to show that passing to a subsequence we can find $\{t_k\} \subset (0,1)$, $t_k \to 0$, such that

$$\liminf_{k\to\infty} \langle \tilde{m}^*_{t_k}, u_k \rangle \geqslant (1-\theta)^{\frac{1}{n}} \tag{4.16}$$

and

$$\liminf_{k\to\infty} \langle \tilde{m}^*_{t_k}, u_k - u \rangle \geqslant (1-\theta)^{\frac{1}{n}} \tag{4.17}$$

Inequality (4.16) is proved in the same way as (4.7). Next, (4.17) easily follows from (4.4) and (4.16).

*Step* 2. In this step we prove

$$\limsup_{k\to\infty} ||\nabla(u_k - u)||_{L^n(B(R))} \leqslant (1-\theta)^{\frac{1}{n}}. \tag{4.18}$$

The proof of (4.10) is still valid for our radial functions. From (4.12) we can see that the quantity $||\nabla(u_k - u)||_{L^n(B(R))}$ is again estimated by $I_1 + I_2 + I_3$.

*Step* 3. Our aim is to prove

$$(1-\theta)^{-\frac{1}{n}}(u_k - u) - \tilde{m}_{t_k} \overset{k\to\infty}{\to} 0 \qquad \text{in } W^{1,n}_0(B(R)). \tag{4.19}$$

Combining (4.17) and (4.18) with (4.3) we obtain

$$\langle \tilde{m}^*_{t_k}, (1-\theta)^{-\frac{1}{n}}(u_k - u) \rangle \overset{k\to\infty}{\to} 1 \qquad \text{and} \qquad ||(1-\theta)^{-\frac{1}{n}} \nabla(u_k - u)||_{L^n(B(R))} \overset{k\to\infty}{\to} 1.$$

Now, we complete the proof of (4.19) using Lemma 4.2. □

## 5. Properties of the Moser-type functions

In this section we study properties of the functions $m_t$, $t \in (0, \frac{1}{L})$, defined in (1.7).

For every $t \in (0, \frac{1}{L})$, we define the following functional acting on functions from $W_0 L^\Phi(B(R))$

$$\langle m^*_t, u \rangle := \int_{B(R)} \frac{\Phi_0(|\nabla m_t(x)|)}{|\nabla m_t(x)|^2} \nabla m_t(x) \cdot \nabla u(x)\, dx,$$

with the convention that the integrand reads zero in the points where $|\nabla m_t(x)| = 0$. The function $\Phi_0$ is a fixed Young function coming from (1.13).

In the sequel, we are interested in radial functions only. Suppose that $h$ is the one-dimensional representative of a radial function $u$ (see (2.5)) and let $g_t$ represent $m_t$.

PROPOSITION 5.1. *Let $R>0$, $\beta>0$ and let $\Phi$ be a Young function satisfying condition* (1.6). *The Moser-type functions and the functional defined above have the following properties:*

$$||\nabla m_t||_{L^{\beta\Phi}(B(R))} = (1+\mathrm{o}(1))\beta^{\frac{1}{n}} \quad as \quad t\to 0, \tag{5.1}$$

$$\left|\langle m_t^*,u\rangle - \frac{h(Rt)}{g_t(Rt)}\right| \leqslant \psi(t)||\nabla u||_{L^{\beta\Phi}(B(R))} \qquad where\ \psi(t)\overset{t\to 0}{\to} 0, \tag{5.2}$$

$$|\langle m_t^*,u\rangle| \leqslant (1+\tilde{\psi}(t))\beta^{-\frac{1}{n}}||\nabla u||_{L^{\beta\Phi}(B(R))} \qquad where\ \tilde{\psi}(t)\overset{t\to 0}{\to} 0 \tag{5.3}$$

*and*

$$\langle m_t^*,u\rangle \overset{t\to 0}{\to} 0 \quad \text{for every fixed radial function } u\in W_0L^{\Phi}(B(R)). \tag{5.4}$$

*Proof of property* (5.1) *from Proposition* 5.1. The proof can be done by an easy modification of the proof of Theorem 1.2 of [18] (see also [7, proof of Theorem 1.2]). The details are left to the reader. □

*Proof of property* (5.2) *from Proposition* 5.1. Since we have the equivalence of the norms $||\cdot||_{L^{\Phi}(B(R))}$ and $||\cdot||_{L^{\beta\Phi}(B(R))}$ and the homogeneity of both sides of (5.2), it is enough to consider the case of $\beta=1$ and $||\nabla u||_{L^{\Phi}(B(R))}=1$.

From (1.7) we obtain

$$-g_t'(y) = \begin{cases} 0 & \text{for } y\in(0,tR) \\ B^{\frac{1}{n}}\omega_{n-1}^{-\frac{1}{n}}\log_{[\ell]}^{\frac{1}{\gamma}-B}(\frac{1}{t})\log_{[\ell]}^{B-1}(\frac{R}{y})(\prod_{j=1}^{\ell-1}\log_{[j]}^{-1}(\frac{R}{y}))\frac{1}{y} & \text{for } y\in(tR,\frac{R}{L}) \\ \frac{L}{L-1}\frac{1}{R}B^{\frac{1-n}{n}}\omega_{n-1}^{-\frac{1}{n}}\log_{[\ell]}^{\frac{1}{\gamma}-B}(\frac{1}{t})\log_{[\ell]}^{B}(L) & \text{for } y\in(\frac{R}{L},R). \end{cases} \tag{5.5}$$

Let $y_0<\frac{R}{L}$ be the constant from (2.7). If $t$ is so small that $tR<y_0$, we can write

$$\begin{aligned}\langle m_t^*,u\rangle &= \int_{B(R)}\frac{\Phi_0(|\nabla m_t|)}{|\nabla m_t|^2}\nabla m_t\cdot\nabla u = \int_0^R\frac{\Phi_0(-g_t'(y))}{-g_t'(y)}(-h'(y))\omega_{n-1}y^{n-1}\,dy \\ &= \int_0^{tR}+\int_{tR}^{y_0}+\int_{y_0}^{R} = I_1+I_2+I_3.\end{aligned}$$

From (5.5) we obtain $I_1=0$. Furthermore, by (1.5) we have $\frac{1}{\gamma}-B<0$. Hence $\log_{[\ell]}^{\frac{1}{\gamma}-B}(\frac{1}{t})\to 0$ as $t\to 0$ and thus (1.13) and (5.5) yield for $t$ small enough

$$\sup_{y\in(y_0,R)}\frac{\Phi_0(-g_t'(y))}{-g_t'(y)} \leqslant C\log_{[\ell]}^{(n-1)(\frac{1}{\gamma}-B)}\Big(\frac{1}{t}\Big).$$

Hence we can use $h(R)=0$ (as $u\in W_0L^{\Phi}(B(R))$) and $|h(y_0)|\leqslant C$ (by (2.7)) to obtain

$$\begin{aligned}I_3 &\leqslant \int_{y_0}^R C\log_{[\ell]}^{(n-1)(\frac{1}{\gamma}-B)}\Big(\frac{1}{t}\Big)(-h'(y))\,dy = C\log_{[\ell]}^{(n-1)(\frac{1}{\gamma}-B)}\Big(\frac{1}{t}\Big)h(y_0) \\ &\leqslant C\log_{[\ell]}^{(n-1)(\frac{1}{\gamma}-B)}\Big(\frac{1}{t}\Big)\overset{t\to 0}{\to} 0.\end{aligned}$$

It remains to prove that

$$\Big|I_2-\frac{h(Rt)}{g_t(Rt)}\Big|\leqslant\psi_1(t)\qquad\text{where }\psi_1(t)\overset{t\to0}{\to}0.\tag{5.6}$$

Fix $\varepsilon>0$. We observe that for $\theta,\beta\in\mathbb{R}$ and $s>0$ large enough we have

$$\log(\theta s^\beta)\approx\beta\log(s)\qquad\text{and}\qquad\log_{[j]}(\theta s^\beta)\approx\log_{[j]}(s)\quad\text{for }j\geqslant2.\tag{5.7}$$

Hence we can find $M>1$ so large that for every $t>0$ small enough we have by the second line of (5.5)

$$y\in\Big(tR,\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}\Big(\frac1t\Big)\Big)\qquad\Longrightarrow\qquad|g_t'(y)|\in\Big(\frac{1}{y^{1-\varepsilon}},\frac{1}{y^{1+\varepsilon}}\Big)\tag{5.8}$$

From (1.13), (5.7) and (5.8) we obtain for every $t>0$ small enough

$$\begin{aligned}&y\in\Big(tR,\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}\Big(\frac1t\Big)\Big)\\&\qquad\Longrightarrow\qquad\frac{\frac{\Phi_0(-g_t'(y))}{-g_t'(y)}}{(-g_t'(y))^{n-1}(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}(\frac{R}{y}))\log_{[\ell]}^{\alpha}(\frac{R}{y})}\in(1-C\varepsilon,1+C\varepsilon).\end{aligned}\tag{5.9}$$

Furthermore, we have from (5.5), $(n-1)(\frac{1}{\gamma}-B)=-\frac{1}{\gamma}$ and $(n-1)(B-1)=-\alpha$ (see (1.5))

$$\begin{aligned}J:=&\int_{tR}^{\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}(\frac1t)}(-g_t'(y))^{n-1}\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}\Big(\frac{R}{y}\Big)\Big)\log_{[\ell]}^{\alpha}\Big(\frac{R}{y}\Big)(-h'(y))\omega_{n-1}y^{n-1}\,dy\\=&\int_{tR}^{\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}(\frac1t)}B^{\frac{n-1}{n}}\omega_{n-1}^{-\frac{n-1}{n}}\log_{[\ell]}^{(n-1)(\frac{1}{\gamma}-B)}\Big(\frac1t\Big)\log_{[\ell]}^{(n-1)(B-1)}\Big(\frac{R}{y}\Big)\\&\qquad\times\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{1-n}\Big(\frac{R}{y}\Big)\Big)\frac{1}{y^{n-1}}\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}\Big(\frac{R}{y}\Big)\Big)\log_{[\ell]}^{\alpha}\Big(\frac{R}{y}\Big)(-h'(y))\omega_{n-1}y^{n-1}\,dy\\=&\omega_{n-1}^{\frac1n}B^{\frac{n-1}{n}}\log_{[\ell]}^{-\frac{1}{\gamma}}\Big(\frac1t\Big)\int_{tR}^{\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}(\frac1t)}(-h'(y))\,dy\\=&\omega_{n-1}^{\frac1n}B^{\frac{n-1}{n}}\log_{[\ell]}^{-\frac{1}{\gamma}}\Big(\frac1t\Big)\Big(h(Rt)-h\Big(\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}\Big(\frac1t\Big)\Big)\Big).\end{aligned}\tag{5.10}$$

Therefore we have from (1.7) and (2.7)

$$\begin{aligned}\Big|\frac{h(Rt)}{g_t(Rt)}-J\Big|&=\omega_{n-1}^{\frac1n}B^{\frac{n-1}{n}}\log_{[\ell]}^{-\frac{1}{\gamma}}\Big(\frac1t\Big)\Big|h\Big(\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}\Big(\frac1t\Big)\Big)\Big|\\&\leqslant C\log_{[\ell]}^{-\frac{1}{\gamma}}\Big(\frac1t\Big)\Big(C+C\log_{[\ell]}^{\frac{1}{\gamma}}\Big(\log_{[\ell]}^{-M(\frac{1}{\gamma}-B)}\Big(\frac1t\Big)\Big)\Big)\overset{t\to0}{\to}0.\end{aligned}\tag{5.11}$$

Notice that (2.7) also yields for $t$ small enough

$$\log_{[\ell]}^{-\frac{1}{\gamma}}\Big(\frac{1}{t}\Big)h(Rt)\leqslant \log_{[\ell]}^{-\frac{1}{\gamma}}\Big(\frac{1}{t}\Big)\Big(C+C\log_{[\ell]}^{\frac{1}{\gamma}}\Big(\frac{1}{Rt}\Big)\Big)\leqslant C. \tag{5.12}$$

Next, we decompose $I_2$ into three integrals defined by

$$\begin{aligned}
I_2 &= \int_{tR}^{y_0}\frac{\Phi_0(-g_t'(y))}{-g_t'(y)}(-h'(y))\omega_{n-1}y^{n-1}\,dy\\
&= \int_{tR}^{\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}(\frac{1}{t})} + \int_{(\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}(\frac{1}{t}),y_0)\cap\{-g_t'>t_1\}} + \int_{(\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}(\frac{1}{t}),y_0)\cap\{-g_t'\leqslant t_1\}}\\
&= J_1+J_2+J_3,
\end{aligned}$$

where $t_1>0$ comes from (1.13). From (5.9), (5.10), (5.11) and (5.12) we can see that for $t$ sufficiently small, we can make $J_1$ as close to $\frac{h(Rt)}{g_t(Rt)}$ as we wish. It remains to estimate $J_2$ and $J_3$ (by an expression approaching zero).

For $t$ small enough and $y\in(\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}(\frac{1}{t}),y_0)\cap\{-g_t'>t_1\}$ we use (1.13), (5.5) and (5.7) to obtain

$$\begin{aligned}
&\frac{\Phi_0(-g_t'(y))}{-g_t'(y)}\\
&= (-g_t'(y))^{n-1}\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}(-g_t'(y))\Big)\log_{[\ell]}^{\alpha}(-g_t'(y))\\
&\leqslant C(-g_t'(y))^{n-1}\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}\Big(\log_{[\ell]}\Big(\frac{1}{t}\Big)\Big)\Big)\log_{[\ell]}^{|\alpha|}\Big(\log_{[\ell]}\Big(\frac{1}{t}\Big)\Big)\\
&\leqslant C\log_{[\ell]}^{(n-1)(\frac{1}{\gamma}-B)}\Big(\frac{1}{t}\Big)\log_{[\ell]}^{(n-1)|B-1|}\Big(\log_{[\ell]}\Big(\frac{1}{t}\Big)\Big)\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}\Big(\log_{[\ell]}\Big(\frac{1}{t}\Big)\Big)\Big)\frac{1}{y^{n-1}}\\
&\qquad\times\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}\Big(\log_{[\ell]}\Big(\frac{1}{t}\Big)\Big)\Big)\log_{[\ell]}^{|\alpha|}\Big(\log_{[\ell]}\Big(\frac{1}{t}\Big)\Big)\\
&\leqslant C\log_{[\ell]}^{\frac{n-1}{2}(\frac{1}{\gamma}-B)}\Big(\frac{1}{t}\Big)\frac{1}{y^{n-1}}.
\end{aligned}$$

This estimate, (2.7) and $(n-1)(\frac{1}{\gamma}-B)=-\frac{1}{\gamma}$ imply for $t$ small enough

$$\begin{aligned}
J_2 &\leqslant \int_{\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}(\frac{1}{t})}^{y_0} C\log_{[\ell]}^{\frac{n-1}{2}(\frac{1}{\gamma}-B)}\Big(\frac{1}{t}\Big)(-h'(y))dy\\
&= C\log_{[\ell]}^{-\frac{1}{2\gamma}}\Big(\frac{1}{t}\Big)\Big(h\Big(\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}\Big(\frac{1}{t}\Big)\Big)-h(y_0)\Big)\\
&\leqslant C\log_{[\ell]}^{-\frac{1}{2\gamma}}\Big(\frac{1}{t}\Big)\Big(C+C\log_{[\ell]}^{\frac{1}{\gamma}}\Big(\log_{[\ell]}^{-M(\frac{1}{\gamma}-B)}\Big(\frac{1}{t}\Big)\Big)\Big)\overset{t\to0}{\to}0.
\end{aligned} \tag{5.13}$$

It remains to estimate $J_3$. By (1.13) we have $\Phi_0(t) \leqslant Ct^n$ on $(0,t_1)$ and thus for $t$ small enough and $y \in (\log_{[\ell]}^{M(\frac{1}{\gamma}-B)}(\frac{1}{t}), y_0) \cap \{-g_t' \leqslant t_1\}$ we obtain

$$\begin{aligned}&\frac{\Phi_0(-g_t'(y))}{-g_t'(y)}\\&\leqslant C(-g_t'(y))^{n-1}\\&\leqslant C\log_{[\ell]}^{(n-1)(\frac{1}{\gamma}-B)}\left(\frac{1}{t}\right)\log_{[\ell]}^{(n-1)|B-1|}\left(\log_{[\ell]}\left(\frac{1}{t}\right)\right)\left(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}\left(\log_{[\ell]}\left(\frac{1}{t}\right)\right)\right)\frac{1}{y^{n-1}}\\&\leqslant C\log_{[\ell]}^{\frac{n-1}{2}(\frac{1}{\gamma}-B)}\left(\frac{1}{t}\right)\frac{1}{y^{n-1}}.\end{aligned}$$

Now, we can estimate $J_3$ in the same way as we have estimated $J_2$ in (5.13). This concludes the proof of (5.6) and we are done. □

*Proof of property* (5.3) *from Proposition* 5.1. According to the homogeneity of both sides of the inequality, it is enough to consider the case of $||\nabla u||_{L^{\beta\Phi}(B(R))} = 1$. By (1.7) and (2.6), for every $\varepsilon > 0$ we can find $t_1 > 0$ so small that for every $t \in (0,t_1)$ we have

$$\begin{aligned}|h(Rt)| &\leqslant (1+2\varepsilon)\beta^{-\frac{1}{n}}B^{\frac{1-n}{n}}\omega_{n-1}^{-\frac{1}{n}}\log_{[\ell]}^{\frac{1}{\gamma}}\left(\frac{1}{Rt}\right) \leqslant (1+3\varepsilon)\beta^{-\frac{1}{n}}B^{\frac{1-n}{n}}\omega_{n-1}^{-\frac{1}{n}}\log_{[\ell]}^{\frac{1}{\gamma}}\left(\frac{1}{t}\right)\\&= (1+3\varepsilon)\beta^{-\frac{1}{n}}g_t(Rt).\end{aligned}$$

From this estimate and from (5.2) we infer

$$|\langle m_t^*, u\rangle| \leqslant \frac{|h(Rt)|}{g_t(Rt)} + \psi(t) \leqslant (1+3\varepsilon)\beta^{-\frac{1}{n}} + \psi(t)$$

and thus (5.3) follows. □

*Proof of property* (5.4) *from Proposition* 5.1. Fix a radial function $u \in W_0L^\Phi(B(R))$ and $\varepsilon > 0$. First, by the absolute continuity of the Lebesgue integral we observe that choosing $\rho > 0$ sufficiently small we can make the integral $\int_{B(\rho)}\Phi(|\nabla u|)$ as small as we wish. This observation together with the fact that $\Phi$ satisfies the $\Delta_2$-condition imply that $\rho$ can be chosen so small that

$$||\nabla u||_{L^{\beta\Phi}(B(\rho))} < \varepsilon.$$

Now, let us write $u$ as a sum of two functions $u = u_1 + u_2$, i.e. $h = h_1 + h_2$, defined by

$$h_1(t) = \begin{cases} h(t) & \text{for } t \in [\rho, R] \\ h(\rho) & \text{for } t \in [0,\rho] \end{cases} \qquad \text{and} \qquad h_2(t) = \begin{cases} 0 & \text{for } t \in [\rho, R] \\ h(t) - h(\rho) & \text{for } t \in [0,\rho]. \end{cases}$$

We plainly have

$$u_1, u_2 \in W_0L^\Phi(B(R)), \qquad ||\nabla u_1||_{L^{\beta\Phi}(B(R))} \leqslant C \qquad \text{and} \qquad ||\nabla u_2||_{L^{\beta\Phi}(B(R))} < \varepsilon.$$

Therefore, (1.7) and (2.7) applied to $h_1$ and $h_2$ separately yield for every sufficiently small $t \in (0, \frac{\rho}{R})$

$$\begin{aligned}|h(Rt)| &= |h_1(Rt) + h_2(Rt)| \leqslant |h_1(Rt)| + |h_2(Rt)| = |h_1(\rho)| + |h_2(Rt)| \\ &\leqslant C + C\log_{[\ell]}^{\frac{1}{\gamma}}\left(\frac{1}{\rho}\right) + C + C\log_{[\ell]}^{\frac{1}{\gamma}}\left(\frac{1}{Rt}\right)\varepsilon \leqslant C\log_{[\ell]}^{\frac{1}{\gamma}}\left(\frac{1}{Rt}\right)\varepsilon = C\varepsilon g_t(Rt).\end{aligned}$$

Finally, from the last estimate and from (5.2) we infer for $t$ sufficiently small

$$|\langle m_t^*, u\rangle| \leqslant \left|\frac{h(Rt)}{g_t(Rt)}\right| + \psi(t)||\nabla u||_{L^{\beta\Phi}(B(R))} \leqslant C\varepsilon + \psi(t)||\nabla u||_{L^{\beta\Phi}(B(R))} \leqslant C\varepsilon$$

and (5.4) follows. □

LEMMA 5.1. *Let $\Phi$ be a Young function satisfying (1.6), $\{t_k\} \subset (0,1)$, $t_k \to 0$ and let $\{u_k\} \subset W_0L^\Phi(\Omega)$ satisfy $\int_\Omega \Phi(|\nabla u_k|) \leqslant C$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that the following assertion holds:*

*If $u \in W_0L^\Phi(\Omega)$ satisfies $\int_\Omega \Phi(|\nabla u|) < \delta$ and there is $k_0 \in \mathbb{N}$ such that*

$$\langle m_{t_k}^*, u_k^\#\rangle \geqslant 1 - \varepsilon \qquad \text{for } k \geqslant k_0,$$

*then there is $k_1 \in \mathbb{N}$ such that*

$$\langle m_{t_k}^*, (u_k - u)^\#\rangle \geqslant 1 - 5\varepsilon \qquad \text{for } k \geqslant k_1.$$

*Proof.* Let $h$ be such that $u^\#(x) = h(|x|)$ and let $h_k$ be such that $u_k^\#(x) = h_k(|x|)$, $k \in \mathbb{N}$. The proof is based on a comparison of the measure of the level sets of functions $h$ and $h_k$.

First, in view (2.3) we can choose $\delta > 0$ so small that (5.2) and (5.3) ensure for $k$ large enough

$$h(Rt_k) < \varepsilon g_{t_k}(Rt_k). \tag{5.14}$$

Next, let us show that for $k$ large enough we have

$$h_k(2Rt_k) \geqslant (1 - 3\varepsilon)g_{t_k}(Rt_k). \tag{5.15}$$

For $k \geqslant k_0$ sufficiently large, we obtain from (5.2) and $\langle m_{t_k}^*, u_k^\#\rangle \geqslant 1 - \varepsilon$

$$h_k(Rt_k) \geqslant (1 - 2\varepsilon)g_{t_k}(Rt_k).$$

Since

$$h_k(2Rt_k) = h_k(Rt_k) - (h_k(Rt_k) - h_k(2Rt_k)),$$

it remains to show that for $k$ large enough we have

$$h_k(Rt_k) - h_k(2Rt_k) \leqslant \varepsilon g_{t_k}(Rt_k). \tag{5.16}$$

Let us prove (5.16). From $\int_\Omega \Phi(|\nabla u_k|) \leqslant C$ and the Pólya-Szegö inequality (Theorem 2.1) we have

$$\begin{aligned} C \geqslant \int_\Omega \Phi(|\nabla u_k|) &\geqslant \int_{B(R)} \Phi(|\nabla u_k^\#|) = \int_0^R \Phi(-h_k'(y))\omega_{n-1}y^{n-1}dy \\ &\geqslant \int_{Rt_k}^{2Rt_k} \Phi(-h_k'(y))\omega_{n-1}y^{n-1}\,dy \geqslant \omega_{n-1}(Rt_k)^{n-1}\int_{Rt_k}^{2Rt_k} \Phi(-h_k'(y))\,dy. \end{aligned}$$

Hence

$$Ct_k^{-n} \geqslant \frac{1}{Rt_k}\int_{Rt_k}^{2Rt_k} \Phi(-h_k'(y))\,dy$$

and Jensen's inequality yields

$$Ct_k^{-n} \geqslant \Phi\Big(\frac{1}{Rt_k}\int_{Rt_k}^{2Rt_k} -h_k'(y)\,dy\Big). \tag{5.17}$$

Now, if $\Phi(t) \geqslant Ct^n$ for large arguments (i.e. we have $\ell \geqslant 2$ or $\alpha \geqslant 0$), then for $k$ large enough we infer from (1.7) and (5.17)

$$h_k(Rt_k) - h_k(2Rt_k) = \int_{Rt_k}^{2Rt_k} -h_k'(y)\,dy \leqslant C \leqslant \varepsilon g_{t_k}(Rt_k).$$

Thus, (5.16) is proved in this case.

On the other hand, if $\ell = 1$ and $\alpha < 0$, then it can be easily seen that for large arguments we have $\Phi^{-1}(t) \leqslant 2t^{\frac{1}{n}}\log^{-\frac{\alpha}{n}}(t)$. Hence we obtain from (5.17)

$$h_k(Rt_k) - h_k(2Rt_k) = \int_{Rt_k}^{2Rt_k} -h_k'(y)\,dy \leqslant Ct_k\Phi^{-1}(Ct_k^{-n}) \leqslant Ct_kCt_k^{-1}\log^{-\frac{\alpha}{n}}\Big(\frac{1}{t_k}\Big).$$

Next, as $g_{t_k}(Rt_k) = C\log^{\frac{1}{\gamma}}(\frac{1}{t_k})$ and $\frac{1}{\gamma} = \frac{n-1-\alpha}{n} > -\frac{\alpha}{n}$, we obtain (5.16) again. Having proved (5.16) in both cases, we also have (5.15).

Now, from (5.14) and (5.15) we can see that (recall that the functions $h$ and $h_k$ are non-increasing)

$$\{u_k^\# \geqslant (1-3\varepsilon)g_{t_k}(Rt_k)\} \supset B(2Rt_k) \qquad \text{while} \qquad \{u^\# \geqslant \varepsilon g_{t_k}(Rt_k)\} \subset B(Rt_k).$$

Hence

$$\mathscr{L}^n(\{(u_k-u)^\# \geqslant (1-4\varepsilon)g_{t_k}(Rt_k)\}) \geqslant \mathscr{L}^n(B(2Rt_k)) - \mathscr{L}^n(B(Rt_k)) > \mathscr{L}^n(B(Rt_k)).$$

This implies that on the sphere $\{|x| = Rt_k\}$, the value of $(u_k-u)^\#$ is estimated from below by $(1-4\varepsilon)g_{t_k}(Rt_k)$. Finally, (5.2) completes the proof. □

LEMMA 5.2. *Let* $\{t_k\} \subset (0,1)$, $t_k \to 0$, $\beta > 0$, $G > 0$, $\delta > 0$ *and let* $\Phi$ *be a Young function satisfying* (1.6)*. Then there is* $\sigma > 0$ *with the following property. If* $\{u_k\} \subset W_0L^\Phi(\Omega)$ *are radial functions such that* $\int_\Omega \Phi(|\nabla u_k|) \leqslant (1+\sigma)\beta$ *and* $\langle m^*_{t_k}, u_k\rangle \geqslant ((1-\sigma)\beta)^{\frac{1}{n}}$*, then*

$$\int_{\{|\nabla u_k| \leqslant G\}} \Phi(|\nabla u_k|) < \delta \qquad \textit{for } k \textit{ large enough.}$$

*Proof.* We can plainly find $\sigma_0 > 0$ and $C_1 = C_1(n) > 0$ such that for every $\sigma \in (0,\sigma_0)$ we have

$$(1-\sigma)\Big(\frac{(1-\sigma)^{\frac{1}{n}} - 2\sigma}{1+2\sigma}\Big)^n \geqslant (1-C_1\sigma). \tag{5.18}$$

Let us fix $\sigma \in (0,\sigma_0)$ so small that

$$(C_1+1)\sigma\beta \leqslant \delta. \tag{5.19}$$

Fix $\tilde{G} \geqslant G$. Using (1.7), (2.6) and (5.2) we obtain for $k$ large enough

$$\begin{aligned}
&(\beta - \sigma\beta)^{\frac{1}{n}} \leqslant \langle m^*_{t_k}, u_k\rangle \leqslant \frac{h_k(Rt_k)}{g_{t_k}(Rt_k)} + \sigma\beta^{\frac{1}{n}} \\
&\leqslant \frac{1}{g_{t_k}(Rt_k)}\Big(\tilde{G}R + (1+\sigma)||\nabla u_k||_{L^\Phi(\{|\nabla u_k|>\tilde{G}\})} B^{\frac{1-n}{n}} \omega_{n-1}^{-\frac{1}{n}} \log_{[\ell]}^{\frac{1}{\gamma}}\Big(\frac{1}{Rt_k}\Big)\Big) + \sigma\beta^{\frac{1}{n}} \\
&\leqslant \frac{1}{g_{t_k}(Rt_k)}\Big(\tilde{G}R + (1+2\sigma)||\nabla u_k||_{L^\Phi(\{|\nabla u_k|>\tilde{G}\})} B^{\frac{1-n}{n}} \omega_{n-1}^{-\frac{1}{n}} \log_{[\ell]}^{\frac{1}{\gamma}}\Big(\frac{1}{t_k}\Big)\Big) + \sigma\beta^{\frac{1}{n}} \\
&= \frac{1}{g_{t_k}(Rt_k)}\Big(\tilde{G}R + (1+2\sigma)||\nabla u_k||_{L^\Phi(\{|\nabla u_k|>\tilde{G}\})} g_t(Rt_k)\Big) + \sigma\beta^{\frac{1}{n}} \\
&\leqslant (1+2\sigma)||\nabla u_k||_{L^\Phi(\{|\nabla u_k|>\tilde{G}\})} + 2\sigma\beta^{\frac{1}{n}}.
\end{aligned}$$

Thus, if $\tilde{G}$ is large enough, acting in the same way as in the proof of Lemma 2.1 and using (5.18) we obtain

$$\begin{aligned}
\int_{\{|\nabla u_k|>\tilde{G}\}} \Phi(|\nabla u_k|) &\geqslant (1-\sigma)||\nabla u_k||^n_{L^\Phi(\{|\nabla u_k|>\tilde{G}\})} \\
&\geqslant (1-\sigma)\Big(\frac{(\beta-\sigma\beta)^{\frac{1}{n}} - 2\sigma\beta^{\frac{1}{n}}}{1+2\sigma}\Big)^n \geqslant (1-C_1\sigma)\beta.
\end{aligned}$$

Hence (5.19) gives

$$\begin{aligned}
\int_{\{|\nabla u_k| \leqslant G\}} \Phi(|\nabla u_k|) &\leqslant \int_{\{|\nabla u_k| \leqslant \tilde{G}\}} \Phi(|\nabla u_k|) = \int_\Omega \Phi(|\nabla u_k|) - \int_{\{|\nabla u_k|>\tilde{G}\}} \Phi(|\nabla u_k|) \\
&\leqslant (1+\sigma)\beta - (1-C_1\sigma)\beta = (C_1+1)\sigma\beta \leqslant \delta
\end{aligned}$$

and we are done. □

LEMMA 5.3. *Let* $\beta > 0$, $\{t_k\} \subset (0,1)$, $t_k \to 0$ *and let* $\{u_k\} \subset W_0L^\Phi(B(R))$ *be radial functions satisfying* $||\nabla u_k||_{L^{\beta\Phi}(B(R))} \leqslant (1+\mathrm{o}(1))\beta^{\frac{1}{n}}$. *If* $\langle m_{t_k}^*, u_k\rangle \to 1$, *then*

$$u_k - m_{t_k} \to 0 \qquad \text{in } W_0L^\Phi(B(R)).$$

*Proof.* The proof is easily obtained applying the uniform convexity of the norm $||\cdot||_{L^{\beta\Phi}(B(R))}$ to the gradients of the functions $u_k$ and $m_{t_k}$. Let us give the details.

First, we infer from (5.3)

$$||\nabla u_k||_{L^{\beta\Phi}(B(R))} \to \beta^{\frac{1}{n}}.$$

Now, since we have $\langle m_{t_k}^*, m_{t_k}\rangle = \int_{B(R)} \Phi_0(|\nabla m_{t_k}|) \to 1$ and $||\nabla m_{t_k}||_{L^{\beta\Phi}(B(R))} \to \beta^{\frac{1}{n}}$ (see (5.1)), we obtain from $\langle m_{t_k}^*, u_k\rangle \to 1$ and $||\nabla u_k||_{L^{\beta\Phi}(B(R))} \to \beta^{\frac{1}{n}}$

$$\left\langle m_{t_k}^*, \frac{\frac{m_{t_k}}{||\nabla m_{t_k}||_{L^{\beta\Phi}(B(R))}} + \frac{u_k}{||\nabla u_k||_{L^{\beta\Phi}(B(R))}}}{2}\right\rangle$$
$$= \frac{1}{2}\left(\left\langle m_{t_k}^*, \frac{m_{t_k}}{||\nabla m_{t_k}||_{L^{\beta\Phi}(B(R))}}\right\rangle + \left\langle m_{t_k}^*, \frac{u_k}{||\nabla u_k||_{L^{\beta\Phi}(B(R))}}\right\rangle\right) \to \beta^{-\frac{1}{n}}.$$

Combining this result with (5.3) we obtain

$$\left|\left|\frac{\frac{\nabla m_{t_k}}{||\nabla m_{t_k}||_{L^{\beta\Phi}(B(R))}} + \frac{\nabla u_k}{||\nabla u_k||_{L^{\beta\Phi}(B(R))}}}{2}\right|\right|_{L^{\beta\Phi}(B(R))} \to 1.$$

Therefore the uniform convexity of the the norm $||\cdot||_{L^{\beta\Phi}(B(R))}$ implies

$$\left|\left|\frac{\nabla m_{t_k}}{||\nabla m_{t_k}||_{L^{\beta\Phi}(B(R))}} - \frac{\nabla u_k}{||\nabla u_k||_{L^{\beta\Phi}(B(R))}}\right|\right|_{L^{\beta\Phi}(B(R))} \to 0.$$

Finally, since $||\nabla m_{t_k}||_{L^{\beta\Phi}(B(R))} \to \beta^{\frac{1}{n}}$ and $||\nabla u_k||_{L^{\beta\Phi}(B(R))} \to \beta^{\frac{1}{n}}$, we have

$$\begin{aligned}
\beta^{-\frac{1}{n}}&||\nabla m_{t_k} - \nabla u_k||_{L^{\beta\Phi}(B(R))} \\
&\leqslant \left|\left|\frac{\nabla m_{t_k}}{\beta^{\frac{1}{n}}} - \frac{\nabla m_{t_k}}{||\nabla m_{t_k}||_{L^{\beta\Phi}(B(R))}}\right|\right|_{L^{\beta\Phi}(B(R))} \\
&\qquad + \left|\left|\frac{\nabla m_{t_k}}{||\nabla m_{t_k}||_{L^{\beta\Phi}(B(R))}} - \frac{\nabla u_k}{||\nabla u_k||_{L^{\beta\Phi}(B(R))}}\right|\right|_{L^{\beta\Phi}(B(R))} \\
&\qquad + \left|\left|\frac{\nabla u_k}{||\nabla u_k||_{L^{\beta\Phi}(B(R))}} - \frac{\nabla u_k}{\beta^{\frac{1}{n}}}\right|\right|_{L^{\beta\Phi}(B(R))} \\
&\to 0.
\end{aligned}$$

Thus, we are done. □

## 6. Proof of Theorem 1.7

*Proof of Theorem 1.7(i).* Assume that $\theta \in (0,1)$ (the case $\theta = 0$ is studied in the proof of Theorem 1.7(iii)) and $\limsup_{k\to\infty} J_{P_\theta}(u_k) > J_{P_\theta}(u)$. Passing to a subsequence we can suppose that the limit exists and $\lim_{k\to\infty} J_{P_\theta}(u_k) > J_{P_\theta}(u)$. Passing to a subsequence again we can also suppose that $u_k \to u$ in $L^\Phi(\Omega)$ and $u_k \to u$ a.e. in $\Omega$. Since the symmetric rearrangement preserves the convergence in Orlicz spaces (see [24, Theorem 1.D]), we can also suppose that $u_k^\# \to u^\#$ in $L^\Phi(B(R))$ and $u_k^\# \to u^\#$ a.e. in $B(R)$.

*Step* 1. In this step we show that passing to a subsequence we can find $\{t_k\} \subset (0,1)$, $t_k \to 0$, such that

$$\liminf_{k\to\infty} \langle m_{t_k}^*, u_k^\# \rangle \geqslant (1-\theta)^{\frac{1}{n}} \tag{6.1}$$

and

$$\liminf_{k\to\infty} \langle m_{t_k}^*, (u_k - u)^\# \rangle \geqslant (1-\theta)^{\frac{1}{n}}. \tag{6.2}$$

Let us prove (6.1). First, let us consider the case that there are $\delta > 0$, $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that

$$\langle m_t^*, u_k^\# \rangle \leqslant (1-\varepsilon)(1-\theta)^{\frac{1}{n}} \qquad \text{for every } t \in (0,\delta) \text{ and every } k \geqslant k_0. \tag{6.3}$$

Therefore by (5.2) with $\beta = 1$, (1.5), (1.7) and $P_\theta = (1-\theta)^{-\frac{\gamma}{n}}$, we have for every $t \in (0,\delta)$ small enough and $k \geqslant k_0$

$$\begin{aligned} \exp_{[\ell]}\Big(K_{\ell,n,\alpha} P_\theta |h_k(Rt)|^\gamma\Big) &\leqslant \exp_{[\ell]}\Big(K_{\ell,n,\alpha} P_\theta |g_t(Rt)|^\gamma \big(|\langle m_t^*, u_k^\# \rangle| + \psi(t)\big)^\gamma\Big) \\ &\leqslant \exp_{[\ell]}\Big(K_{\ell,n,\alpha} |g_t(Rt)|^\gamma \Big(1 - \frac{\varepsilon}{2}\Big)^\gamma\Big) \\ &= \begin{cases} \exp(n(1-\frac{\varepsilon}{2})\log(\frac{1}{t})) & \text{for } \ell = 1 \\ \exp_{[\ell]}((1-\frac{\varepsilon}{2})\log_{[\ell]}(\frac{1}{t})) & \text{for } \ell \geqslant 2. \end{cases} \end{aligned} \tag{6.4}$$

Since $h_k(Rt)$ is bounded for $t$ bounded away from zero (see (2.7)), from (6.4) we easily obtain that the integrals

$$\int_{B(R)} \exp_{[\ell]}\Big(K_{\ell,n,\alpha} P_\theta |u_k|^\gamma\Big) = \omega_{n-1} R^n \int_0^1 t^{n-1} \exp_{[\ell]}\Big(K_{\ell,n,\alpha} P_\theta |h_k(Rt)|^\gamma\Big)\, dt$$

have a common integrable majorant and thus the Lebesgue Dominated Convergence Theorem ensures that $J_{P_\theta}(u_k) \to J_{P_\theta}(u)$, a contradiction. Hence there cannot be $\delta > 0$, $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that (6.3) holds and thus we can pass to a subsequence satisfying (6.1).

We proceed to the proof of (6.2). First, let us introduce the following notation. Given $L > 0$, we define

$$u_L(x) = \min\{|u(x)|, L\} \operatorname{sgn}(u(x)) \qquad \text{and} \qquad u^L(x) = u(x) - u_L(x).$$

Similarly we define $u_k^L$ and $(u_k)_L$, $k\in\mathbb{N}$. It can be easily seen that

$$\int_\Omega \Phi(|\nabla u_k|)=\int_\Omega \Phi(|\nabla u_k^L|)+\int_\Omega \Phi(|\nabla (u_k)_L|),\quad u_k^L\to u^L \text{ a.e. in } \Omega$$
$$\text{and}\quad (u_k)_L\to u_L \text{ a.e. in } \Omega.$$

Moreover $u_k^L$ form a bounded sequence in $W_0L^\Phi(\Omega)$ and thus there is a weakly convergent subsequence. Since $u_k^L$ converge almost everywhere to $u^L$ it is easy to see that

$$u_k^L\rightharpoonup u^L \text{ in } W_0L^\Phi(\Omega)\qquad\text{and}\qquad (u_k)_L\rightharpoonup u_L \text{ in } W_0L^\Phi(\Omega).$$

The proof of (6.2) is obtained establishing the following chain of inequalities

$$\begin{aligned}(1-\theta)^{\frac1n}&\leqslant \liminf_{k\to\infty}\langle m^*_{t_k},u_k^\#\rangle\leqslant \liminf_{k\to\infty}\langle m^*_{t_k},(u_k^L)^\#\rangle\\&\leqslant \liminf_{k\to\infty}\langle m^*_{t_k},(u_k^L-u^L)^\#\rangle+\varepsilon\leqslant \liminf_{k\to\infty}\langle m^*_{t_k},(u_k-u)^\#\rangle+\varepsilon,\end{aligned}\tag{6.5}$$

with $\varepsilon>0$ being an arbitrarily small number and $L$ depending on $\varepsilon$ is specified below.

The first inequality in (6.5) is just (6.1). The second inequality easily follows from $g_{t_k}(Rt_k)\to\infty$ (see (1.7)), (5.2) and

$$u_k^\#-L\leqslant (u_k^L)^\#\leqslant u_k^\#.\tag{6.6}$$

The third inequality follows from Lemma 5.1, since we can make $\int_\Omega\Phi(\nabla u^L|)$ as small as we wish via a choice of sufficiently large $L$. The last inequality follows from (5.2), since $g_{t_k}(Rt_k)\to\infty$ and

$$|u_k^L-u^L|=|u_k-u+(u_L-(u_k)_L)|\leqslant|u_k-u|+|u_L|+|(u_k)_L|\leqslant|u_k-u|+2L.$$

This completes the proof of (6.2).

*Step* 2. In this step we prove

$$\limsup_{k\to\infty}||\nabla(u_k-u)^\#||_{L^{\frac{1}{1-\theta}\Phi}(B(R))}\leqslant 1.\tag{6.7}$$

Fix $\varepsilon>0$. Next, we fix $L>0$ so large that

$$\int_\Omega\Phi(|\nabla u^L|)=\tau,\tag{6.8}$$

where $\tau\in(0,\frac12\min\{\theta,1-\theta\})$ is a small number specified below.

By the Pólya-Szegö inequality (Theorem 2.1) we have

$$\begin{aligned}&||\nabla(u_k-u)^\#||_{L^{\frac{1}{1-\theta}\Phi}(B(R))}\\&\qquad\leqslant||\nabla(u_k-u)||_{L^{\frac{1}{1-\theta}\Phi}(\Omega)}\\&\qquad\leqslant||\nabla((u_k)_L-u_L)||_{L^{\frac{1}{1-\theta}\Phi}(\Omega)}+||\nabla u_k^L||_{L^{\frac{1}{1-\theta}\Phi}(\Omega)}+||\nabla u^L||_{L^{\frac{1}{1-\theta}\Phi}(\Omega)}\\&\qquad=I_1+I_2+I_3.\end{aligned}\tag{6.9}$$

If $\tau$ is small enough, then (2.3) implies that $I_3 < \varepsilon$.

Next, since $(u_k)_L \rightharpoonup u_L$, by the weak lower semicontinuity of the modular we have for $k$ large enough

$$\int_\Omega \Phi(|\nabla(u_k)_L|) \geqslant \int_\Omega \Phi(|\nabla u_L|) - \tau = \int_\Omega \Phi(|\nabla u|) - \int_\Omega \Phi(|\nabla u^L|) - \tau = \theta - 2\tau$$

and thus

$$\int_\Omega \Phi(|\nabla u_k^L|) = \int_\Omega \Phi(|\nabla u_k|) - \int_\Omega \Phi(|\nabla(u_k)_L|) \leqslant 1 - \theta + 2\tau. \tag{6.10}$$

Hence, if $\tau$ is small enough, using (2.4) we obtain $I_2 < 1 + \varepsilon$.

It remains to prove that $I_1 < \varepsilon$. In the proof, we employ both norms $||\cdot||_{L^{\frac{1}{1-\theta}\Phi}(\Omega)}$ and $||\cdot||_{L^\Phi(\Omega)}$. From (6.1) we obtain for $k$ large enough

$$\langle m_{t_k}^*, u_k^\# \rangle \geqslant (1 - \theta - \tau)^{\frac{1}{n}} \tag{6.11}$$

and thus, by (5.2), (5.3) and (6.6) we have for $k$ large enough

$$||\nabla(u_k^L)^\#||_{L^\Phi(B(R))} \geqslant (1 - \theta - 2\tau)^{\frac{1}{n}}. \tag{6.12}$$

Now, by Remark 3.1, there is $\eta \in (0, \frac{1}{2}\min\{\theta, 1-\theta\})$ such that

$$\begin{aligned} &\int_\Omega \Phi(|\nabla(u_k)_L|) \leqslant \theta + 2\eta \quad \text{and} \quad \theta - \eta \leqslant \int_\Omega \Phi(|\nabla u_L|) \\ &\qquad \Longrightarrow \qquad ||\nabla((u_k)_L - u_L)||_{L^{\frac{1}{1-\theta}\Phi}(\Omega)} < \varepsilon. \end{aligned} \tag{6.13}$$

If $\tau < \eta$, then from (6.8) and $\int_\Omega \Phi(|\nabla u|) = \theta$ we see that the second inequality in (6.13) is satisfied and it remains to prove

$$\int_\Omega \Phi(|\nabla(u_k)_L|) \leqslant \theta + 2\eta. \tag{6.14}$$

To prove (6.14), let us start with the proof of

$$\int_\Omega \Phi(|\nabla(u_k^L)^\#|) \geqslant 1 - \theta - 2\eta. \tag{6.15}$$

By Lemma 2.1, there are $G > 0$ and $\delta > 0$ such that

$$\begin{aligned} &\int_{\{|\nabla(u_k^L)^\#| < G\}} \Phi(|\nabla(u_k^L)^\#|) < \delta \quad \text{and} \quad ||\nabla(u_k^L)^\#||_{L^\Phi(B(R))} \geqslant (1 - \theta - \eta)^{\frac{1}{n}} \\ &\qquad \Longrightarrow \qquad \int_\Omega \Phi(|\nabla(u_k^L)^\#|) \geqslant 1 - \theta - 2\eta \end{aligned} \tag{6.16}$$

(the assumptions of Lemma 2.1 concerning $C_1$ and $C_2$ are satisfied since

$$||\nabla(u_k^L)^\#||_{L^\Phi(B(R))} \leqslant ||\nabla u_k^\#||_{L^\Phi(B(R))} \leqslant ||\nabla u_k||_{L^\Phi(\Omega)} \leqslant 1$$

and (6.12) gives us the lower bound). Next, the estimate of the integral on the left hand side of (6.16) follows from Lemma 5.2 providing $\tau$ is small enough (the assumptions are satisfied by (6.10) and (6.11)) and the estimate of the norm $||\nabla(u_k^L)^{\#}||_{L^{\Phi}(B(R))}$ follows from (6.12). Thus, we have proved (6.15).

Now, (6.15) and the Pólya-Szegö inequality (Theorem 2.1) yield for $k$ large enough

$$\int_{\Omega}\Phi(|\nabla(u_k)_L|) = \int_{\Omega}\Phi(|\nabla u_k|) - \int_{\Omega}\Phi(|\nabla u_k^L|) \leqslant 1 - \int_{\Omega}\Phi(|\nabla(u_k^L)^{\#}|) \leqslant \theta + 2\eta$$

and (6.14) is proved. Therefore both inequalities on the left hand side of (6.13) are satisfied and thus we have proved that $I_1 < \varepsilon$. This concludes the proof of (6.7).

*Step* 3. Our aim is to prove

$$(1-\theta)^{-\frac{1}{n}}(u_k - u)^{\#} - m_{t_k} \stackrel{k\to\infty}{\to} 0 \qquad \text{in } W_0L^{\Phi}(B(R)). \tag{6.17}$$

Combining (6.2) and (6.7) with (5.3) we obtain

$$\langle m_{t_k}^*, (1-\theta)^{-\frac{1}{n}}(u_k - u)^{\#}\rangle \stackrel{k\to\infty}{\to} 1$$

and

$$||(1-\theta)^{-\frac{1}{n}}\nabla(u_k - u)^{\#}||_{L^{\frac{1}{1-\theta}\Phi}(B(R))} \stackrel{k\to\infty}{\to} (1-\theta)^{-\frac{1}{n}}.$$

Now, we complete the proof of (6.17) using Lemma 5.3. Thus, we are done. □

*Proof of Theorem 1.7(ii).* Let us suppose that $\xi \in (0,1)$ (the case $\xi = 0$ is studied in the proof of Theorem 1.7(iii)) and $\limsup_{k\to\infty} J_{P_\xi}(u_k) > J_{P_\xi}(u)$. Again, we can suppose that $\lim_{k\to\infty} J_{P_\xi}(u_k)$ exists, $\lim_{k\to\infty} J_{P_\xi}(u_k) > J_{P_\xi}(u)$, $u_k \to u$ in $L^{\Phi}(\Omega)$, $u_k \to u$ a.e. in $\Omega$, $u_k^{\#} \to u^{\#}$ in $L^{\Phi}(B(R))$, $u_k^{\#} \to u^{\#}$ a.e. in $B(R)$.

*Step* 1. The aim of this step is to show that passing to a subsequence we can find $\{t_k\} \subset (0,1)$, $t_k \to 0$, such that

$$\liminf_{k\to\infty}\langle m_{t_k}^*, u_k^{\#}\rangle \geqslant (1-\xi)^{\frac{1}{n}} \tag{6.18}$$

and

$$\liminf_{k\to\infty}\langle m_{t_k}^*, u_k^{\#} - u^{\#}\rangle \geqslant (1-\xi)^{\frac{1}{n}}. \tag{6.19}$$

Inequality (6.18) is proved in the same way as (6.1). Next, (6.19) easily follows from (5.4) and (6.18).

*Step* 2. In this step we prove

$$\limsup_{k\to\infty}||\nabla(u_k^{\#} - u^{\#})||_{L^{\frac{1}{1-\xi}\Phi}(B(R))} \leqslant 1. \tag{6.20}$$

The proof of (6.7) is still valid for radial functions $u_k^{\#}$ and $u^{\#}$ (we replace $\theta$ by $\xi$ and we also use (6.18) instead of (6.1)). From (6.9) we can see that the quantity $||\nabla(u_k^{\#} - u^{\#})||_{L^{\frac{1}{1-\xi}\Phi}(B(R))}$ is still estimated by $I_1 + I_2 + I_3$.

*Step* 3. Our aim is to prove

$$(1-\xi)^{-\frac{1}{n}}(u_k^\# - u^\#) - m_{t_k} \overset{k\to\infty}{\to} 0 \qquad \text{in } W_0L^\Phi(B(R)). \tag{6.21}$$

Combining (6.19) and (6.20) with (5.3) we obtain

$$\langle m_{t_k}^*, (1-\xi)^{-\frac{1}{n}}(u_k^\# - u^\#)\rangle \overset{k\to\infty}{\to} 1$$

and

$$||(1-\xi)^{-\frac{1}{n}}\nabla(u_k^\# - u^\#)||_{L^{\frac{1}{1-\xi}\Phi}(B(R))} \overset{k\to\infty}{\to} (1-\xi)^{-\frac{1}{n}}.$$

Now, we complete the proof of (6.21) using Lemma 5.3. □

*Proof of Theorem 1.7(iii).* In the proof of Theorem 1.7(i), we were using the assumptions (1.8) and (1.9) only when employing the uniform convexity of the space $W_0L^\Phi(\Omega)$ in Step 2 and Step 3. Thus we still have condition (6.1) which reads in our case (recall that $\theta = 0$)

$$\liminf_{k\to\infty}\langle m_{t_k}^*, u_k^\#\rangle \geqslant 1. \tag{6.22}$$

Now, we claim that it is enough to prove

$$\limsup_{k\to\infty} ||\nabla u_k^\#||_{L^{\Phi_0}(B(R))} \leqslant 1. \tag{6.23}$$

Indeed, $\Phi_0$ satisfies (1.8) and (1.9) and thus (6.23) and Lemma 5.3 (see also (5.3)) imply $u_k^\# - m_{t_k} \to 0$ in the Dirichlet norm corresponding to $\Phi_0$. Nevertheless, the Luxemburg norms corresponding to $\Phi_0$ and $\Phi$, respectively, give us the same convergence.

Thus, let us complete the proof establishing (6.23). Fix $\varepsilon > 0$. By (1.6), there is $t_0 > 0$ such that $\Phi_0(t) \leqslant (1+\varepsilon)\Phi(t)$ for every $t > t_0$ and thus for every $k$ we have

$$\int_{\{|\nabla u_k^\#|>t_0\}} \Phi_0(|\nabla u_k^\#|) \leqslant (1+\varepsilon)\int_{\{|\nabla u_k^\#|>t_0\}} \Phi(|\nabla u_k^\#|) \leqslant (1+\varepsilon)\int_{B(R)} \Phi(|\nabla u_k^\#|) \leqslant 1+\varepsilon. \tag{6.24}$$

Next we claim that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\int_{\{|\nabla u_k^\#|\leqslant t_0\}} \Phi(|\nabla u_k^\#|) \leqslant \delta \qquad \Longrightarrow \qquad \int_{\{|\nabla u_k^\#|\leqslant t_0\}} \Phi_0(|\nabla u_k^\#|) \leqslant \varepsilon. \tag{6.25}$$

To prove this, pick $\sigma > 0$ so small that $\Phi_0(\sigma)\mathscr{L}^n(B(R)) < \frac{\varepsilon}{2}$. Since $\Phi(\sigma) > 0$, there is plainly $L > 0$ such that $\Phi_0(t) \leqslant L\Phi(t)$ on $[\sigma, t_0]$. Hence we can set $\delta = \frac{\varepsilon}{2L}$ to obtain

$$\begin{aligned}\int_{\{|\nabla u_k^\#|\leqslant t_0\}} \Phi_0(|\nabla u_k^\#|) &= \int_{\{|\nabla u_k^\#|\leqslant\sigma\}} \Phi_0(|\nabla u_k^\#|) + \int_{\{\sigma<|\nabla u_k^\#|\leqslant t_0\}} \Phi_0(|\nabla u_k^\#|)\\ &\leqslant \int_{B(R)} \Phi_0(\sigma) + L\int_{\{\sigma<|\nabla u_k^\#|\leqslant t_0\}} \Phi(|\nabla u_k^\#|) \leqslant \frac{\varepsilon}{2} + L\frac{\varepsilon}{2L} = \varepsilon.\end{aligned}$$

Finally, since we have $\int_{B(R)} \Phi(|\nabla u_k|) \leqslant 1$ and (6.22), we can use Lemma 5.2 to ensure that for $k$ large enough the left hand side of (6.25) is satisfied. Hence (6.24) and (6.25) yield

$$\int_{B(R)} \Phi_0(|\nabla u_k^{\#}|) = \int_{\{|\nabla u_k^{\#}|>t_0\}} \Phi_0(|\nabla u_k^{\#}|) + \int_{\{|\nabla u_k^{\#}|\leqslant t_0\}} \Phi_0(|\nabla u_k^{\#}|) \leqslant 1+2\varepsilon.$$

Now, (2.4) implies (6.23) and we are done. □

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*Robert Černý*
*Department of Mathematical Analysis*
*Faculty of Mathematics and Physics, Charles University*
*Sokolovská 83, 186 00 Prague 8, Czech Republic*
*e-mail:* rcerny@karlin.mff.cuni.cz