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BURKHOLDER–GUNDY–DAVIS INEQUALITY ON LORENTZ MARTINGALE SPACES

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Abstract. Let $f = (f_n)_{n\geqslant 0}$ be a martingale, $0 < p < \infty$, $1 \leqslant q < \infty$. In this paper we obtain a $L_{p,q}$-version of Burkholder-Gundy-Davis martingale inequality

$$\| S(f) \|_{p,q} \approx \| M(f) \|_{p,q},$$

by means of rearrangement technique.

1. Introduction and Preliminaries

Let $1 \leqslant p < \infty$. The famous Burkholder-Gundy-Davis inequality is

$$\| S(f) \|_{p} \approx \| M(f) \|_{p},$$

see [1–2]. It is well-known that Burkholder-Gundy-Davis inequality is one of the fundamental inequalities in classical martingale H^p theory.

The aim of this paper is to extend Burkholder-Gundy-Davis inequality from the type of L_p-norm to that of $L_{p,q}$-quasinorm. Here we use the rearrangement technique. Let $0 < p < \infty$, $1 \leqslant q < \infty$. we obtain a $L_{p,q}$-version of Burkholder-Gundy-Davis martingale inequality

$$\| S(f) \|_{p,q} \approx \| M(f) \|_{p,q} .$$

For rearrangement technique in martingale setting we refer to [3].

The organization of this paper is divided into two sections. Some basic knowledge, which we will use, is collected in this section. Main result and its proof are given in the next section.

Let (Ω, μ) be a σ-finite measure space, $\mathscr{M}(\Omega)$ the space of all measurable functions on Ω. For $f \in \mathscr{M}(\Omega)$, denote its distribution function by

$$\lambda_f(t) = \mu(x : |f(x)| > t),\ t \geqslant 0,$$

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and its decreasing rearrangement function f^* is defined as

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leqslant t\}, \ \ t \geqslant 0.$$

For $0 < p,q < \infty$, the Lorentz space $L_{p,q}$ is defined as

$$L_{p,q} = L_{p,q}(\Omega, \mathscr{F}, \mu) = \{f : \| f \|_{p,q} < \infty\},$$

where

$$\|f\|_{p,q} = \left(\int_0^\infty (f^*(t))^q t^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, and $\{\mathscr{F}_n\}_{n\geqslant 0}$ a nondecreasing sequence of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F} = \sigma(\bigcup_n \mathscr{F}_n)$. The expectation operator is denoted by $\mathbb{E}$. For a martingale $f = (f_n)_{n\geqslant 0}$ relative to $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_n)_{n\geqslant 0})$, denote its martingale difference by $df_i = f_i - f_{i-1}$ ($i \geqslant 0$, with convention $df_0 = 0$), its maximal function and square function by

$$M_n(f) = \sup_{0\leqslant i\leqslant n} | f_i |, \qquad M(f) = \sup_{i\geqslant 0} | f_i |,$$

$$S_n(f) = (\sum_{i=0}^n |df_i|^2)^{\frac{1}{2}}, \qquad S(f) = (\sum_{i=0}^\infty |df_i|^2)^{\frac{1}{2}}.$$

For $0 < p,q < \infty$, we define Lorentz martingale spaces as follows:

$$H^*_{p,q} = \{f = (f_n)_{n\geqslant 0} : \|f\|_{H^*_{p,q}} = \| M(f) \|_{p,q} < \infty\};$$

$$H^S_{p,q} = \{f = (f_n)_{n\geqslant 0} : \|f\|_{H^S_{p,q}} = \|S(f)\|_{p,q} < \infty\}.$$

Throughout this paper, we denote the set of non-negative integers by $\mathbf{N}$. We use C or C_p (depending only on p) to denote some constant and may be different at each occurrence. The equivalence $a \approx b$ means that $C_1 a \leqslant b \leqslant C_2 a$ for some positive constants C_1 and C_2.

2. A $L_{p,q}$-version of Burkholder-Gundy-Davis inequality

LEMMA 1. *For any martingale $f = (f_n)_{n\geqslant 0}$, there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$(S(f))^*(t) \leqslant (S(f))^*(2t) + C_1(M(f))^* \left(\frac{t}{2}\right), \quad \forall t > 0, \tag{1}$$

$$(M(f))^*(t) \leqslant (M(f))^*(2t) + C_2(S(f))^* \left(\frac{t}{2}\right), \quad \forall t > 0. \tag{2}$$

Proof. For an arbitrage fixed $t>0$, define stopping times

$$\nu=\inf\left\{n\in\mathbf{N}:M_n(f)>(M(f))^*\left(\frac{t}{2}\right)\right\},\quad \tau=\inf\{n\in\mathbf{N}:S_n(f)>(S(f))^*(2t)\}.$$

Then

$$\mathbb{P}(\nu<\infty)=\mathbb{P}\left(M(f)>(Mf)^*\left(\frac{t}{2}\right)\right)\leqslant\frac{t}{2},\quad M_{\nu-1}(f)\leqslant(M(f))^*\left(\frac{t}{2}\right),$$

and

$$\mathbb{P}(\tau<\infty)=\mathbb{P}(S(f)>(S(f))^*(2t))\leqslant 2t,\quad S_{\tau-1}(f)\leqslant(S(f))^*(2t).$$

Now consider a new family of σ-algebras $\{\mathscr{F}_n{}'\}_{n\geqslant 0}$ with $\mathscr{F}_n{}'=\mathscr{F}_{\tau+n}$, and a new process $f^{\tau,\nu-1}=(f_n{}')_{n\geqslant 0}$ with $f_n{}'=f_{\tau+n}^{(\nu-1)}-f_{\tau-1}^{(\nu-1)}$, then $f^{\tau,\nu-1}$ is a martingale with respect to $\{\mathscr{F}_n{}'\}_{n\geqslant 0}$. Now let $C_1=4C$, where C is the constant in the Davis inequality $\mathbb{E}_0(S(f))\leqslant C\mathbb{E}_0(M(f))$. Since

$$\begin{aligned}S(f^{(\nu-1)})-S_{\tau-1}(f^{(\nu-1)})&\leqslant(S(f^{(\nu-1)})^2-S_{\tau-1}(f^{(\nu-1)})^2)^{\frac{1}{2}}\\&=S(f^{\tau,\nu-1}),\end{aligned}$$

then applying Davis inequality we get

$$\begin{aligned}&\mathbb{P}\left(S(f)>(S^{(q)}(f))^*(2t)+C_1(Mf)^*\left(\frac{t}{2}\right)\right)\\
\leqslant{}&\mathbb{P}(\nu<\infty)+\mathbb{P}\left(\nu=\infty,S_{\nu-1}(f)>(S(f))^*(2t)+C(Mf)^*\left(\frac{t}{2}\right)\right)\\
\leqslant{}&\frac{t}{2}+\mathbb{P}\left(\tau<\nu=\infty,S_{\nu-1}(f)-S_{(\nu-1)\wedge(\tau-1)}(f)>C_1(Mf)^*\left(\frac{t}{2}\right)\right)\\
\leqslant{}&\frac{t}{2}+\left(C_1(Mf)^*\left(\frac{t}{2}\right)\right)^{-1}\mathbb{E}[\mathbb{E}(S_{\nu-1}(f)-S_{(\nu-1)\wedge(\tau-1)}(f)\mid\mathscr{F}_\tau)\chi_{\{\tau<\nu\}}]\\
\leqslant{}&\frac{t}{2}+\left(C_1(Mf)^*\left(\frac{t}{2}\right)\right)^{-1}\mathbb{E}[\mathbb{E}(S(f^{\tau,\nu-1})\mid\mathscr{F}_\tau)\chi_{\{\tau<\nu\}}]\\
\leqslant{}&\frac{t}{2}+\left(C_1(Mf)^*\left(\frac{t}{2}\right)\right)^{-1}C\mathbb{E}[\mathbb{E}(M(f^{\tau,\nu-1})\mid\mathscr{F}_\tau)\chi_{\{\tau<\nu\}}]\\
\leqslant{}&\frac{t}{2}+\left(C_1(Mf)^*\left(\frac{t}{2}\right)\right)^{-1}2C\mathbb{E}[\mathbb{E}(M_{\nu-1}(f)\mid\mathscr{F}_\tau)\chi_{\{\tau<\infty\}}]\\
\leqslant{}&\frac{t}{2}+\frac{t}{2}=t.\end{aligned}$$

Hence, we obtain (1).

If define stopping times

$$\nu=\inf\left\{n\in\mathbf{N}:S_n(f)>(S(f))^*\left(\frac{t}{2}\right)\right\},\quad \tau=\inf\{n\in\mathbf{N}:M_n(f)>(M(f))^*(2t)\},$$

then we can prove (2) in a similar way. □

LEMMA 2. [4] *Let* (F,G) *be a pair of non-negative measurable functions on* $(\Omega,\mathscr{F},\mathbb{P})$. *If* (F,G) *satisfies the rearrangement inequality :*

$$F^*(t) \leqslant F^*(2t) + CG^*\left(\frac{t}{2}\right), \quad \forall t > 0.$$

Then with the same C, *we have*

$$F^*(t) \leqslant 2CG^*\left(\frac{t}{2}\right) + \frac{C}{\log 2}\int_t^\infty \frac{G^*(s)}{s}ds, \quad \forall t > 0.$$

LEMMA 3. (Hardy's inequality) [5] *If* $1 \leqslant q < \infty$, $r > 0$ *and* f *is a non-negative function defined on* $(0,\infty)$, *then*

$$\left(\int_0^\infty \left(\int_t^\infty f(u)du\right)^q t^r \frac{dt}{t}\right)^{\frac{1}{q}} \leqslant \frac{q}{r}\left(\int_0^\infty (tf(t))^q t^r \frac{dt}{t}\right)^{\frac{1}{q}}.$$

THEOREM 1. *Let* $0 < p < \infty$, $1 \leqslant q < \infty$. *Then for any martingale* $f = (f_n)_{n\geqslant 0}$ *we have*

$$\| S(f) \|_{p,q} \approx \| M(f) \|_{p,q} . \tag{3}$$

Proof. It follows from (2.1) in Lemma 2.1., Lemma 2.2. and 2.3. that

$$\begin{aligned}
\| S(f) \|_{p,q} &= \left(\int_0^\infty (S(f)^*(t))^q t^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\leqslant C\left(\left(\int_0^\infty \left(M(f)^*\left(\frac{t}{2}\right)\right)^q t^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^\infty \left(\int_t^\infty \frac{M(f)^*(s)}{s} ds\right)^q t^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}}\right) \\
&\leqslant C\left(\int_0^\infty (M(f)^*(t))^q t^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} \\
&= C \| M(f) \|_{p,q} .
\end{aligned}$$

For the converse, we can prove $\| M(f) \|_{p,q} \leqslant C \| S(f) \|_{p,q}$ in a similar way. □

REMARK 1. If $p = q$ and $1 \leqslant p < \infty$ in (3), we obtain the famous Burkholder-Gundy-Davis inequality in classical martingale H^p theory.

COROLLARY 1. *For* $0 < p < \infty$, $1 \leqslant q < \infty$, *we have* $H^S_{p,q} = H^*_{p,q}$.

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