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ON BONNESEN–STYLE ISOPERIMETRIC INEQUALITIES FOR n–SIMPLICES

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Abstract. In this paper, we derive some Bonnesen-style inequalities referring to n-dimensional simplices in the sprit of isoperimetric problems. These inequalities combine various metric quantities of simplices and yield, in several cases, characterizations of regular simplices. Also related reverse Bonnesen-style inequalities are derived.

1. Introduction

As a well known result, for a simple closed curve $\mathscr{C}$ (in the Euclidian plane) of length L enclosing a domain of area A, then the inequality

$$L^2 - 4\pi A \geqslant 0 \tag{1.1}$$

holds, with equality if and only if this curve is a Euclidean circle. The quantity $L^2 - 4\pi A$ is called the *isoperimetric deficit* of $\mathscr{C}$.

As an extension, Bonnesen proves [1] that if $\mathscr{C}$ is convex and there exists a circular annulus of thickness d containing $\mathscr{C}$, then

$$L^2 - 4\pi A \geqslant 4\pi d^2. \tag{1.2}$$

In fact, Fuglede [2] shows that convexity is not a necessary condition.

There is a related isoperimetric inequality known as Bonnesen inequality [3]:

$$L^2 - 4\pi A \geqslant \pi^2 (R - r)^2, \tag{1.3}$$

where R is the circumradius and r is the inradius of the curve $\mathscr{C}$. Note that if the right side of (1.3) equals zero, then $R = r$. This means that $\mathscr{C}$ is a circle and $L^2 - 4\pi A = 0$.

More generally, inequalities of the form

$$L^2 - 4\pi A \geqslant K \tag{1.4}$$

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are called Bonnesen-style isoperimetric inequalities if equality is only attained for the Euclidean circle [3]. See [2–10] for more detailed references.

When the simple closed curve $\mathscr{C}$ is a triangle (in the Euclidean plane) of area S and with side lengths a_1, a_2, a_3, the following inequality is known:

$$P^2 \geqslant 3\sqrt{3}S, \tag{1.5}$$

where $P = \frac{1}{2}(a_1 + a_2 + a_3)$. Equality holds if and only if this triangle is regular.

Inequality (1.5) can be regarded as isoperimetric inequality for triangles.

Let E^n denote the n-dimensional Euclidean space. Let $A_1, A_2, \cdots, A_{n+1}$ denote the vertices of an n-*simplex* Ω_n in the Euclidian space E^n (i.e., Ω_n is the n-dimensional convex hull of $\{A_1, A_2, \cdots, A_{n+1}\}$), and F_i the $(n-1)$-dimensional volume of the facet $f_i = \{A_1, \cdots, A_{i-1}, A_{i+1}, \cdots, A_{n+1}\}$ opposite to the vertex P_i $(i = 1, 2, \cdots, n+1)$. Putting $F = \sum_{i=1}^{n+1} F_i$, thus F yields the *surface area* of the simplex Ω_n. We denote by a_{ij} $(i, j = 1, 2, \cdots, n+1)$ the *edge lengths* of Ω_n (sometimes, we set $a_1, a_2, \cdots, a_{\frac{1}{2}n(n+1)}$ in some order). The sum of all a_{ij} $(1 \leqslant i < j \leqslant n+1)$ is denoted by L, and L is called the *total edge length* of Ω_n. And if all edge lengths are equal, the simplex is called *regular*.

The known Veljan-Korchmaros inequality [11] involving the volume and the edge lengths of Ω_n reads as follows:

$$\prod_{1 \leqslant i < j \leqslant n+1} a_{ij}^{\frac{2}{n+1}} \geqslant \left(\frac{2^n n!^2}{n+1} \right)^{\frac{1}{2}} V, \tag{1.6}$$

with equality if and only if the simplex Ω_n is regular.

By applying the arithmetic-geometric mean inequality to (1.6), we obtain

$$L^{2(n+1)} \geqslant \frac{n^{2(n+1)}(n+1)^{\frac{(n+1)(2n-1)}{n}}}{2^{n+1}} (n! \cdot V)^{\frac{2(n+1)}{n}} \tag{1.7}$$

or

$$L^2 \geqslant \frac{n^2(n+1)^{\frac{2n-1}{n}}}{2} (n! \cdot V)^{\frac{2}{n}}, \tag{1.7$'$}$$

with equality if and only if Ω_n is regular.

The inequality (1.7) or (1.7') may be called *isoperimetric inequality of an n-simplex*. The deficit value between the right-hand side and left-hand side of inequality (1.7) or (1.7') can be regarded as the isopermetric deficit for the n-simplex Ω_n:

$$\Delta_1 = L^{2(n+1)} - \frac{n^{2(n+1)}(n+1)^{\frac{(n+1)(2n-1)}{n}}}{2^{n+1}} (n! \cdot V)^{\frac{2(n+1)}{n}} \tag{1.8}$$

or

$$\Delta_1' = L^2 - \frac{n^2(n+1)^{\frac{2n-1}{n}}}{2} (n! \cdot V)^{\frac{2}{n}}. \tag{1.8$'$}$$

In addition, the *volume* V and the facet areas of the simplex Ω_n satisfies the following inequality:

$$(V)^{\frac{2}{n}} \leqslant [(n-1)!]^{\frac{2}{n-1}} \frac{(n+1)^{\frac{1}{n}}}{n^{\frac{1}{n-1}}} \left(\prod_{i=1}^{n+1} F_i\right)^{2(n^2-1)}, \tag{1.9}$$

with equality if and only if Ω_n is regular (see [12, 13]).

Applying the arithmetic-geometric mean inequality to (1.9), we obtain

$$F^{2(n^2-1)} \geqslant \left[\frac{n\cdot(n+1)^{\frac{1}{n}}}{(n-1)!^2}(n!\cdot V)^{\frac{2(n-1)}{n}}\right]^{n^2-1}, \tag{1.10}$$

or

$$F^{\frac{2}{n-1}} \geqslant \left[\frac{n\cdot(n+1)^{\frac{1}{n}}}{(n-1)!^2}\right]^{\frac{1}{n-1}} (n!\cdot V)^{\frac{2}{n}}, \tag{1.10$'$}$$

with equality if and only if Ω_n is regular.

The inequality (1.10) or (1.10’) may be also called isoperimetric inequality for an n-simplex. The deficit value between the right-hand side and left-hand side of inequality (1.10) or (1.10’) can be regarded as the other isopermetric deficit for the n-simplex Ω_n:

$$\Delta_2 = F^{2(n^2-1)} - \left[\frac{n\cdot(n+1)^{\frac{1}{n}}}{(n-1)!^2}(n!\cdot V)^{\frac{2(n-1)}{n}}\right]^{n^2-1}, \tag{1.11}$$

or

$$\Delta_2' = F^{\frac{2}{n-1}} - \left[\frac{n\cdot(n+1)^{\frac{1}{n}}}{(n-1)!^2}\right]^{\frac{1}{n-1}} (n!\cdot V)^{\frac{2}{n}}. \tag{1.11$'$}$$

2. The Bonnesen-style inequalities for simplices

In this section, we present two forms of the Bonnesen-style inequalities involving the total edge length and the surface area of a simplex.

THEOREM 2.1. *For an n-simplex Ω_n, we have*

$$\Delta_1 \geqslant \frac{1}{2}(n+1)^{2n}R^n\left(\frac{nr}{R}\right)^{\frac{2(n^2-1)}{n}} \sum_{i=1}^{\frac{n(n+1)}{2}} \left(a_i - \sqrt{\frac{2(n+1)}{n}}R\right)^2, \tag{2.1}$$

with equality if and only if Ω_n is regular.

THEOREM 2.2. *For an n-simplex Ω_n, we have*

$$\Delta_2 \geqslant \frac{n^{(n^2-1)(n+4)}(n+1)^{(n+1)(n^2-1)}}{2(n!)^{2(n^2-1)}} R^{2n(n^2-n-1)} \times \left(\frac{nr}{R}\right)^{\frac{2(n^2-n-1)(n^2-1)}{n}} \sum_{i=1}^{\frac{n(n+1)}{2}} \left(a_i - \sqrt{\frac{2(n+1)}{n}}R\right)^2, \tag{2.2}$$

with equality if and only if Ω_n is regular.

COROLLARY 2.3. *Let ABC be a triangle of area S, and with side lengths a_1, a_2, a_3, then*

$$P^6 - 2^3 3^{\frac{9}{2}} S^3 \geqslant \frac{3^4}{2^7} R^2 \left(\frac{2r}{R}\right)^3 \sum_{i=1}^{3} (a_i - \sqrt{3}R)^2. \tag{2.3}$$

Equality is attained if and only if the triangle is regular, where $P = \frac{1}{2}(a_1 + a_2 + a_3)$.

COROLLARY 2.4. *For a tetrahedron $ABCD$, we have*

$$L^8 - 2^{12} 3^{\frac{32}{3}} V^{\frac{8}{3}} \geqslant 2^{11} R^3 \left(\frac{3r}{R}\right)^{\frac{16}{3}} \sum_{i=1}^{6} (a_i - \sqrt{\frac{8}{3}}R)^2, \tag{2.4}$$

$$F^{16} - \frac{3^{16}}{2^{\frac{8}{3}}} V^{\frac{32}{3}} \geqslant 3^{40} 2^{47} \cdot R^{30} \left(\frac{3r}{R}\right)^{\frac{80}{3}} \sum_{i=1}^{6} (a_i - \sqrt{\frac{8}{3}}R)^2. \tag{2.5}$$

Equality is attained if and only if the tetrahedron is regular, where again F is the surface area of $ABCD$.

To prove the above theorems, we need two lemmas.

LEMMA 2.5. ([11]) *For an n-simplex Ω_n, we have*

$$\sum_{i=1}^{\frac{1}{2}n(n+1)} a_i^2 \leqslant (n+1)^2 R^2, \tag{2.6}$$

$$\left(\prod_{i=1}^{\frac{1}{2}n(n+1)} a_i\right)^{\frac{4}{n}} \geqslant \frac{2^{n+1} n!^2}{n} V^2 \cdot R^2, \tag{2.7}$$

$$\left(\prod_{i=1}^{n+1} F_i\right)^{n-1} \geqslant \frac{n^{\frac{3n^2-4}{2}}}{(n+1)^{\frac{(n+1)(n-2)}{2}} n!^n} V^{n^2-n-1} \cdot R, \tag{2.8}$$

$$R \geqslant \left[\frac{n! \cdot n^{\frac{n}{2}}}{(n+1)^{\frac{n+1}{2}}}\right] V^{\frac{1}{n}}. \tag{2.9}$$

Equalities are attained if and only if Ω_n is regular.

LEMMA 2.6. ([12]) *For an n-simplex Ω_n, we have*

$$V \geqslant \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n^2-2}{2n}}}{n!} R^{\frac{1}{n}} r^{\frac{n^2-1}{n}}. \tag{2.10}$$

Equality is attained if and only if Ω_n is regular.

The proof of Theorem 2.1. Applying (2.6), we get by suitable calculation

$$\begin{aligned}
&\sum_{i=1}^{\frac{1}{2}n(n+1)} \left(a_i - \sqrt{\frac{2(n+1)}{n}} R\right)^2 \\
&= \sum_{i=1}^{\frac{1}{2}n(n+1)} a_i^2 + (n+1)^2 R^2 - 2\sqrt{\frac{2(n+1)}{n}} R \sum_{i=1}^{\frac{1}{2}n(n+1)} a_i \\
&\leqslant (n+1)^2 R^2 + (n+1)^2 R^2 - 2\sqrt{\frac{2(n+1)}{n}} R \sum_{i=1}^{\frac{1}{2}n(n+1)} a_i.
\end{aligned} \tag{2.11}$$

By (2.11), (2.9), and the arithmetic-geometric means inequality we get

$$\begin{aligned}
R^2 &\geqslant \frac{1}{(n+1)^2} \sqrt{\frac{2(n+1)}{n}} R \sum_{i=1}^{\frac{1}{2}n(n+1)} a_i + \frac{1}{2(n+1)^2} \sum_{i=1}^{\frac{1}{2}n(n+1)} \left(a_i - \sqrt{\frac{2(n+1)}{n}} R\right)^2 \\
&\geqslant \frac{1}{(n+1)^2} \sqrt{\frac{2(n+1)}{n}} \left[\frac{n! \cdot n^{\frac{n}{2}}}{(n+1)^{\frac{n+1}{2}}}\right]^{\frac{1}{n}} V^{\frac{1}{n}} \cdot \frac{n(n+1)}{2} \prod_{i=1}^{\frac{1}{2}n(n+1)} a_i^{\frac{2}{n(n+1)}} \\
&\quad + \frac{1}{2(n+1)^2} \sum_{i=1}^{\frac{1}{2}n(n+1)} \left(a_i - \sqrt{\frac{2(n+1)}{n}} R\right)^2.
\end{aligned} \tag{2.12}$$

By (2.12) and (1.6) we get

$$R^2 \geqslant \frac{(n!)^{\frac{2}{n}} n}{(n+1)^{\frac{n+1}{n}}} V^{\frac{2}{n}} + \frac{1}{2(n+1)^2} \sum_{i=1}^{\frac{1}{2}n(n+1)} \left(a_i - \sqrt{\frac{2(n+1)}{n}} R\right)^2. \tag{2.13}$$

Using the arithmetic-geometric means inequality, (2.7) and (2.13) we find that

$$\begin{aligned}
L^{2(n+1)} &= \left(\sum_{i=1}^{\frac{n(n+1)}{2}} a_i\right)^{2(n+1)} \geqslant \left(\frac{n(n+1)}{2}\right)^{2(n+1)} \left(\prod_{i=1}^{\frac{n(n+1)}{2}} a_i\right)^{\frac{4}{n}} \\
&\geqslant \left(\frac{n(n+1)}{2}\right)^{2(n+1)} \frac{2^{n+1} n!^2}{n} V^2 \cdot R^2 \\
&\geqslant \left(\frac{n(n+1)}{2}\right)^{2(n+1)} \frac{2^{n+1} n!^2}{n} V^2 \\
&\quad \times \left\{\frac{(n!)^{\frac{2}{n}} n}{(n+1)^{\frac{n+1}{n}}} V^{\frac{2}{n}} + \frac{1}{2(n+1)^2} \sum_{i=1}^{\frac{n(n+1)}{2}} \left(a_i - \sqrt{\frac{2(n+1)}{n}} R\right)^2\right\}
\end{aligned}$$

$$= \frac{n^{2(n+1)}(n+1)^{\frac{(n+1)(2n-1)}{n}}}{2^{n+1}}(n!\cdot V)^{\frac{2(n+1)}{n}}$$
$$+ \frac{n^{2n+1}(n+1)^{2n}}{2^{n+2}}(n!\cdot V)^2 \sum_{i=1}^{\frac{n(n+1)}{2}}\left(a_i - \sqrt{\frac{2(n+1)}{n}}R\right)^2. \tag{2.14}$$

From (2.14) and (2.10) we obtain

$$L^{2(n+1)} \geqslant \frac{n^{2(n+1)}(n+1)^{\frac{(n+1)(2n-1)}{n}}}{2^{n+1}}(n!\cdot V)^{\frac{2(n+1)}{n}} + \frac{1}{2}(n+1)^{2n}R^n\left(\frac{nr}{R}\right)^{\frac{2(n^2-1)}{n}}$$
$$\times \sum_{i=1}^{\frac{n(n+1)}{2}}\left(a_i - \sqrt{\frac{2(n+1)}{n}}R\right)^2. \tag{2.15}$$

Thus equality (2.1) is true. From Lemma 2.5 and Lemma 2.6, it is easy to see that equality holds in (2.1) if and only if Ω_n is regular. The proof is complete. □

The proof of Theorem 2.2. Similar to the proof of Theorem 2.1, by the arithmetic-geometric mean inequality, the inequalities (2.8), (2.13) and (2.10) it follows that

$$F^{2(n^2-1)} = \left(\sum_{i=1}^{n+1}F_i\right)^{2(n^2-1)} \geqslant (n+1)^{2(n^2-1)}\left(\prod_{i=1}^{n+1}F_i\right)^{2(n-1)}$$
$$\geqslant (n+1)^{2(n^2-1)}\left[\frac{n^{\frac{3n^2-4}{2}}}{(n+1)^{\frac{(n+1)(n-2)}{2}}n!^n}\right]^2 V^{2(n^2-n-1)}\cdot R^2$$
$$\geqslant \left[\frac{n\cdot(n+1)^{\frac{1}{n}}}{(n-1)!^2}(n!\cdot V)^{\frac{2(n-1)}{n}}\right]^{n^2-1} + \left[\frac{n^{\frac{3n^2-4}{2}}(n+1)^{\frac{n(n+1)}{2}}}{n!^n}\right]^2 V^{2(n^2-n-1)}$$
$$\times \frac{1}{2(n+1)^2}\sum_{i=1}^{\frac{n(n+1)}{2}}\left(a_i - \sqrt{\frac{2(n+1)}{n}}R\right)^2$$
$$\geqslant \left[\frac{n\cdot(n+1)^{\frac{1}{n}}}{(n-1)!^2}(n!\cdot V)^{\frac{2(n-1)}{n}}\right]^{n^2-1} + \frac{n^{(n^2-1)(n+4)}(n+1)^{(n+1)(n^2-1)}}{2(n!)^{2(n^2-1)}}R^{2n(n^2-n-1)}$$
$$\times \left(\frac{nr}{R}\right)^{\frac{2(n^2-n-1)(n^2-1)}{n}}\sum_{i=1}^{\frac{n(n+1)}{2}}\left(a_i - \sqrt{\frac{2(n+1)}{n}}R\right)^2. \tag{2.16}$$

Thus equality (2.2) is true. From Lemma 2.5 and Lemma 2.6, it is easy to see that equality holds in (2.2) if and only if Ω_n is regular. The proof is complete. □

3. An upper bound on the isoperimetric deficit of simplices

Besides asking for the lower bound on the isoperimetric deficit, there is another question: is there an invariant C of geometric significance such that

$$L^2 - 4\pi A \leqslant C?$$

This is a long standing unsolved problem in geometry. Results for special convex domains can be found in [7] and [13, pp. 119–121].

In this section, we derive an upper bound on the isoperimetric deficit of an n-simplex.

DEFINITION 3.1. The (minimal) *width* of an n-simplex Ω_n is the minimum distance between a pair of parallel supporting hyperplanes of Ω_n and denoted by $\omega(\Omega_n)$; see [14].

G. T. Sallee posed the following problem: Which n-simplex, inscribed in the unit ball of E^n, has largest width? R. Alexander [14] showed (see also [15, pp. 233–244]) that

$$\omega(\Omega_n) \leqslant \alpha_n \cdot R, \tag{3.1}$$

where $\alpha_n = \left[\frac{(n+1)^2}{n\cdot(n+1-z)\cdot z}\right]^{\frac{1}{2}}$, $z = \left[\frac{n+1}{2}\right]$,and equality holds if and only if Ω_n is regular.

Let O and R denote the *circumcenter* and the *circumradius* of an n-simplex, and I and r be its *incenter* and *inradius*, respectively. Further on, we write G for the *barycenter* of such a simplex.

We refer to [16] for the n-dimensional version of the *Euler inequality*

$$R \geqslant nr, \tag{3.2}$$

with equality if and only if the respective simplex Ω_n is regular.

In 1985, Klamkin [17] improved this result and obtained that

$$R^2 \geqslant n^2 r^2 + |OI|^2, \tag{3.3}$$

with equality if and only if the simplex Ω_n is regular.

In 1995, S. G. Yang [18] generalized results from the [16] and [17].

Let $A_1, A_2, \cdots, A_{n+1}$ denote the vertices of an n-simplex Ω_n in E^n with barycenter G, let D be an arbitrary point in E^n, and $R_i = |DA_i|$ $(i = 1, 2, ..., n+1)$. Let P be an arbitrary interior point of Ω_n, and let r_i denote the distance from P to the ith facet f_i of Ω_n. Then we have

$$\frac{1}{n+1}\sum_{i=1}^{n+1} R_i^2 \geqslant n^2 \left(\prod_{i=1}^{n+1} r_i^2\right)^{\frac{1}{n+1}} + |DG|^2. \tag{3.4}$$

Equality holds if and only if the simplex Ω_n is regular. Various further geometric inequalities for an n-simplex are established in [16–20].

Our main results are the following theorems and corollaries.

THEOREM 3.1. *For an n-simplex Ω_n, we have*

$$\Delta_1' \leqslant \frac{n(n+1)^3}{2\alpha_n^2}(\alpha_n^2 R^2 - \omega^2(\Omega_n)), \tag{3.5}$$

$$\Delta_2' \leqslant \frac{f(n)}{\alpha_n^2}(\alpha_n^2 R^2 - \omega^2(\Omega_n)), \tag{3.6}$$

with equality if and only if Ω_n is regular, where $f(n) = \frac{(n+1)^{\frac{n+1}{n-1}}}{n^{\frac{n-2}{n-1}}(n-1)!^{\frac{2}{n-1}}}$.

COROLLARY 3.2. *Let ABC be a triangle of area S, with side lengths a_1, a_2, a_3 and of width $\omega(\triangle)$. Then*

$$\frac{9}{4}R^2 - \omega^2(\triangle) \geqslant \frac{4}{27}(P^2 - 3\sqrt{3}S). \tag{3.7}$$

Equality is attained if and only if the triangle is regular, where $P = \frac{1}{2}(a_1 + a_2 + a_3)$.

COROLLARY 3.3. *For a tetrahedron $ABCD$ we have*

$$\frac{4}{3}R^2 - \omega^2(\Omega_3) \geqslant \frac{1}{96}(L^2 - 72\sqrt[3]{9}V^{\frac{2}{3}}), \tag{3.8}$$

$$\frac{4}{3}R^2 - \omega^2(\Omega_3) \geqslant \frac{\sqrt{3}}{8}(S - 8\sqrt[6]{243}V^{\frac{2}{3}}). \tag{3.9}$$

Equality is attained if and only if the tetrahedron is regular.

THEOREM 3.4. *For an n-simplex Ω_n, we have*

$$\Delta_1' \leqslant \frac{n(n+1)^3}{2}\left[\frac{1}{n+1}\sum_{i=1}^{n+1} R_i^2 - |DG|^2 - n^2\left(\prod_{i=1}^{n+1} r_i^2\right)^{\frac{1}{n+1}}\right], \tag{3.10}$$

$$\Delta_2' \leqslant g(n)\left[\frac{1}{n+1}\sum_{i=1}^{n+1} R_i^2 - |DG|^2 - n^2\left(\prod_{i=1}^{n+1} r_i^2\right)^{\frac{1}{n+1}}\right], \tag{3.11}$$

with equality if and only if Ω_n is regular, where $g(n) = \frac{(n+1)^{\frac{n}{n-1}}}{n^{\frac{n-2}{n-1}}(n-1)!^{\frac{2}{n-1}}}$.

COROLLARY 3.5. *For an n-simplex Ω_n, we have*

$$R^2 - |OG|^2 - n^2 r^2 \geqslant \frac{2}{n(n+1)^3}\Delta_1', \tag{3.12}$$

$$R^2 - |OG|^2 - n^2 r^2 \geqslant \frac{2}{g(n)}\Delta_2', \tag{3.13}$$

with equality if and only if Ω_n is regular.

REMARK. Inequalities (3.5) and (3.6) are improvements of (3.1). Inequalities (3.10) and (3.11) are improvements of (3.4). In addition, (3.12) and (3.13) are sharpenings of *Euler's inequality* (3.2).

To prove Theorem 3.2 and Theorem 3.5, we still need some lemmas.

LEMMA 3.6. ([19]) *Let $\omega(\Omega_n)$ and V denote the width and the volume of the n-simplex Ω_n, respectively. Then*

$$\omega(\Omega) \leqslant \alpha_n \cdot \frac{n^{1/2}(n!)^{1/n}}{(n+1)^{(n+1)/2n}} V^{1/n} \tag{3.14}$$

with equality if and only if Ω_n is regular, where $\alpha_n = \frac{n+1}{n^{1/2}[\frac{n+1}{2}]^{1/2}(n+1-[\frac{n+1}{2}])^{1/2}}$, and $[m]$ stands for the largest integer part of the real number m.

LEMMA 3.7. ([18]) *Let P be an arbitrary interior point of the simplex Ω_n, r_i denote the distance from P to the i-th facet f_i, and V denote the volume of Ω_n. Then we have*

$$V \geqslant \frac{1}{n!} n^{\frac{n}{2}} (n+1)^{\frac{n+1}{2}} \left(\prod_{i=1}^{n+1} r_i \right)^{\frac{n}{n+1}}, \tag{3.15}$$

with equality if and only if Ω_n is regular.

LEMMA 3.8. ([18]) *For an n-simplex Ω_n, we have*

$$\frac{1}{n+1} \sum_{i=1}^{n+1} R_i^2 = \sum_{i=1}^{\frac{n(n+1)}{2}} a_{ij}^2 + (n+1)^2 |DG|^2. \tag{3.16}$$

Proof of Theorem 3.1. Combining (3.14) and (1.7), and applying the arithmetic-geometric mean inequality and (2.6), we can deduce that

$$\begin{aligned}
\frac{n(n+1)^3}{2}\omega_n^2 &\leqslant \frac{n(n+1)^3}{2}\alpha_n^2 V^{\frac{2}{n}} \frac{(n!)^{\frac{2}{n}} \cdot n}{(n+1)^{\frac{n+1}{n}}} \\
&= \alpha_n^2 \frac{n^2(n+1)^{\frac{2n-1}{n}}}{2} (n! \cdot V)^{\frac{2}{n}} \\
&\leqslant \alpha_n^2 L^2 \leqslant \frac{n(n+1)}{2} \alpha_n^2 \cdot \sum_{i=1}^{\frac{n(n+1)}{2}} a_{ij}^2 \\
&= \frac{n(n+1)^3}{2} \alpha_n^2 \frac{1}{(n+1)^2} \sum_{i=1}^{\frac{n(n+1)}{2}} a_{ij}^2 \\
&\leqslant \frac{n(n+1)^3}{2} \alpha_n^2 R^2.
\end{aligned}$$

The chain of inequalities yields (3.5).

Now we prove inequality (3.6). Using (3.14), (3.10) and the power mean inequality, we get

$$\begin{aligned}
\frac{(n+1)^{\frac{n+1}{n-1}}}{n^{\frac{n-2}{n-1}}\cdot(n-1)!^{\frac{2}{n-1}}}\omega_n^2 &\leqslant \frac{(n+1)^{\frac{n+1}{n-1}}}{n^{\frac{n-2}{n-1}}\cdot(n-1)!^{\frac{2}{n-1}}}\alpha_n^2 V^{\frac{2}{n}}\frac{(n!)^{\frac{2}{n}}\cdot n}{(n+1)^{\frac{n+1}{n}}}\\
&= \alpha_n^2\left[\frac{n(n+1)^{\frac{1}{n}}}{(n-1)!^2}\right]^{\frac{1}{n-1}}(n!\cdot V)^{\frac{2}{n}}\\
&\leqslant \alpha_n^2 F^{\frac{2}{n-1}} \leqslant \alpha_n^2(n+1)^{\frac{1}{n-1}}\cdot\left(\sum_{i=1}^{n+1}F_i^2\right)^{\frac{1}{n-1}}\\
&= \frac{(n+1)^{\frac{n+1}{n-1}}}{n^{\frac{n-2}{n-1}}\cdot(n-1)!^{\frac{2}{n-1}}}\alpha_n^2\frac{1}{(n+1)^2}\sum_{i=1}^{\frac{n(n+1)}{2}}a_{ij}^2\\
&\leqslant \frac{(n+1)^{\frac{n+1}{n-1}}}{n^{\frac{n-2}{n-1}}\cdot(n-1)!^{\frac{2}{n-1}}}\alpha_n^2 R^2.
\end{aligned}$$

From the chain of inequalities we obtain (3.6). From (3.14) (1.7), (1.10) and (2.6), it is easy to see that equalities in (3.5) and (3.6) hold if and only if the respective simplices are regular. The proof is complete. □

Proof of Theorem 3.4. Combining (3.15) and (1.7) first, and applying then the the power mean inequality, and (3.16), we can deduce that

$$\begin{aligned}
\frac{n(n+1)^3}{2}n^2\left(\prod_{i=1}^{n+1}r_i^2\right)^{\frac{1}{n+1}} &\leqslant \frac{n(n+1)^3}{2}V^{\frac{2}{n}}\frac{(n!)^{\frac{2}{n}}\cdot n}{(n+1)^{\frac{n+1}{n}}}\\
&= \frac{n^2(n+1)^{\frac{2n-1}{n}}}{2}(n!\cdot V)^{\frac{2}{n}}\\
&\leqslant L^2 \leqslant \frac{n(n+1)}{2}\cdot\sum_{i=1}^{\frac{n(n+1)}{2}}a_{ij}^2\\
&= \frac{n(n+1)^3}{2}\frac{1}{(n+1)^2}\sum_{i=1}^{\frac{n(n+1)}{2}}a_{ij}^2\\
&= \frac{n(n+1)^3}{2}\frac{1}{n+1}\sum_{i=1}^{n+1}R_i^2-|DG|^2.
\end{aligned}$$

From this chain of inequalities we get (3.10).

By (3.15), (1.10), the power mean inequality and (3.16) we can deduce that

$$
\begin{aligned}
g(n)\cdot n^2\left(\prod_{i=1}^{n+1} r_i^2\right)^{\frac{1}{n+1}} &\leqslant \frac{(n+1)^{\frac{n+1}{n-1}}}{n^{\frac{n-2}{n-1}}\cdot(n-1)!^{\frac{2}{n-1}}} V^{\frac{2}{n}} \frac{(n!)^{\frac{2}{n}}\cdot n}{(n+1)^{\frac{n+1}{n}}}\\
&= \left[\frac{n(n+1)^{\frac{1}{n}}}{(n-1)!^2}\right]^{\frac{1}{n-1}} (n!\cdot V)^{\frac{2}{n}}\\
&\leqslant F^{\frac{2}{n-1}} \leqslant (n+1)^{\frac{1}{n-1}}\cdot\left(\sum_{i=1}^{n+1} F_i^2\right)^{\frac{1}{n-1}}\\
&= \frac{(n+1)^{\frac{n+1}{n-1}}}{n^{\frac{n-2}{n-1}}\cdot(n-1)!^{\frac{2}{n-1}}} \frac{1}{(n+1)^2} \sum_{i=1}^{\frac{n(n+1)}{2}} a_{ij}^2\\
&= g(n)\sum_{i=1}^{n+1} R_i^2 - |DG|^2.
\end{aligned}
$$

From this we get (3.11), and the proof is complete. From (3.15), (3.16), (1.7), (1.10) and (2.6), it is easy to see that equalities in (3.10) and (3.11) hold if and only if Ω_n is regular. □

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