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ON THE FUNCTIONAL BLASCHKE–SANTALÓ INEQUALITY

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Abstract. In this paper, using functional Steiner symmetrizations, we show that Meyer and Pajor's proof of the Blaschke-Santaló inequality can be extended to the functional setting.

1. Introduction

For a convex body $K \subset \mathbb{R}^n$ and a point $z \in \mathbb{R}^n$, the polar body K^z of K with respect to z is the convex set defined by $K^z = \{y \in \mathbb{R}^n : \langle y-z, x-z\rangle \leqslant 1 \text{ for every } x \in K\}$. The *Santaló point* $s(K)$ of K is a point for which $V_n(K^{s(K)}) = \min_{z\in int(K)} V_n(K^z)$, where $V_n(K)$ denotes the volume of set K. The Blaschke-Santaló inequality [4, 18, 19] states that $V_n(K)V_n(K^{s(K)}) \leqslant V_n(B_2^n)^2$, where B_2^n is the Euclidean ball.

For a log-concave function $f : \mathbb{R}^n \to [0,\infty)$ and a point $z \in \mathbb{R}^n$, its polar with respect to z is defined by $f^z(y) = \inf_{x\in\mathbb{R}^n} \frac{e^{-\langle x-z, y-z\rangle}}{f(x)}$. The *Santaló point* $s(f)$ of f is the point z_0 satisfying $\int f^{z_0} = \inf_{z\in\mathbb{R}^n} \int f^z$.

The *functional Blaschke-Santaló inequality* of log-concave functions is the analogue of Blaschke-Santaló inequality of convex bodies.

THEOREM 1.1. (Artstein, Klartag, Milman) *Let* $f : \mathbb{R}^n \to [0,+\infty)$ *be a log-concave function such that* $0 < \int f < \infty$. *Then,* $\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} f^{s(f)} \leqslant (2\pi)^n$ *with equality holds exactly for Gaussians.*

When f is even, the functional Blaschke-Santaló inequality follows from an earlier inequality of Ball [2]; and in [9], Fradelizi and Meyer proved something more general (see also [11]). Lutwak and Zhang [13] and Lutwak et al. [14] gave other very different forms of the Blaschke-Santaló inequality. In this paper, we give a more general result than Theorem 1.1, which becomes into a special case of $\lambda = 1/2$ in Theorem 1.2.

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THEOREM 1.2. *Let $f:\mathbb{R}^n\to[0,+\infty)$ be a log-concave function such that $0<\int f<\infty$. Let H be an affine hyperplane and let H_+ and H_- denote two closed half-spaces bounded by H. If $\lambda\in(0,1)$ satisfies $\lambda\int_{\mathbb{R}^n} f=\int_{H_+} f$. Then there exists $z\in H$ such that*

$$\int_{\mathbb{R}^n} f\int_{\mathbb{R}^n} f^z \leqslant \frac{1}{4\lambda(1-\lambda)}(2\pi)^n. \tag{1.1}$$

In [12], Lehec proved a very general functional version for non-negative Borel functions, Theorem 1.2 is a particular case of result of Lehec. Lehec's proof is by induction on the dimension, and the proof is by functional Steiner symmetrizations. In fact, Mayer and Pajor [15] have proved the Blaschke-Santaló inequality for convex bodies, here we show that Meyer and Pajor's proof of the Blaschke-Santaló inequality can be extended to the functional setting. It has recently come to our attention that in a remark of [9], Fradelizi and Meyer expressed the same idea to prove the functional Blaschke-Santaló inequality.

2. Notations and background materials

Let $|\cdot|$ denote the Euclidean norm. Let $\mathrm{int}A$ denote the interior of $A\subset\mathbb{R}^n$. Let $\mathrm{cl}A$ denote the closure of A. Let $\dim A$ denote the dimension of A. A set $C\subset\mathbb{R}^n$ is called a *convex cone* if C is convex and nonempty and if $x\in C$, $\lambda\geqslant 0$ implies $\lambda x\in C$. We define $C^*:=\{x\in\mathbb{R}^n:\langle x,y\rangle\leqslant 0 \text{ for all } y\in C\}$ and call this the *dual cone* of C.

For a non-empty convex set $K\subset\mathbb{R}^n$ and an affine hyperplane H with unit normal vector u, the *Steiner symmetrization* S_HK of K with respect to H is defined as $S_HK:=\{x'+\frac{1}{2}(t_1-t_2)u:\ x'\in P_H(K),\ t_i\in I_K(x') \text{ for } i=1,2\}$, where $P_H(K):=\{x'\in H:\ x'+tu\in K \text{ for some } t\in\mathbb{R}\}$ is the projection of K onto H and $I_K(x'):=\{t\in\mathbb{R}:\ x'+tu\in K\}$.

Let $\bar{\mathbb{R}}=\mathbb{R}\cup\{-\infty,\infty\}$. For a given function $f:\mathbb{R}^n\to\bar{\mathbb{R}}$ and for $\alpha\in\bar{\mathbb{R}}$ we use the abbreviation $\{f=\alpha\}:=\{x\in\mathbb{R}^n:f(x)=\alpha\}$, and $\{f\leqslant\alpha\}$, $\{f<\alpha\}$ etc. are defined similarly. A function $f:\mathbb{R}^n\to\bar{\mathbb{R}}$ is called *proper* if $\{f=-\infty\}=\emptyset$ and $\{f=\infty\}\neq\mathbb{R}^n$. A function ϕ is called *convex* if ϕ is proper and $\phi(\alpha x+(1-\alpha)y)\leqslant\alpha\phi(x)+(1-\alpha)\phi(y)$ for all $x,y\in\mathbb{R}^n$ and for any $0\leqslant\lambda\leqslant 1$. A function f is called *log-concave* if $f=e^{-\phi}$, where ϕ is a convex function. A function $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is called *coercive* if $\lim_{|x|\to+\infty}f(x)=+\infty$. A function f is called *symmetric* about H if for any $x'\in H$ and $t\in\mathbb{R}$, $f(x'+tu)=f(x'-tu)$. A function $f:\mathbb{R}^n\to\mathbb{R}$ is called *unconditional* about z if $f(x_1-z_1,\dots,x_n-z_n)=f(|x_1-z_1|,\dots,|x_n-z_n|)$ for every $(x_1,\dots,x_n)\in\mathbb{R}^n$. If $z=0$, then f is called *unconditional*.

The *effective domain* of convex function ϕ is the nonempty set $\mathrm{dom}\phi:=\{\phi<\infty\}$. The *support* of function f is the set $\mathrm{supp}f:=\{f\neq 0\}$. For log-concave function $f=e^{-\phi}$, it is clear that $\mathrm{supp}f=\mathrm{dom}\phi$. The nonempty set $\mathrm{epi}\phi:=\{(x,r)\in\mathbb{R}^n\times\mathbb{R}:r\geqslant\phi(x)\}$ denote the *epigraph* of convex function ϕ.

For an affine subspace G of $\mathbb{R}^n$, let $G^\perp$ denote the *orthogonal complement* of G, we have $G^\perp=\{x\in\mathbb{R}^n:\langle x,y-y'\rangle=0 \text{ for every } y,y'\in G\}$. The *Santaló point* $s_G(f)$ of f about G is a point satisfying $\int f^{s_G(f)}=\inf_{z\in G}\int f^z$. Let f be a log-concave function such that $0 < \int f < \infty$, and let H_+ and H_- be two half-spaces bounded by an affine hyperplane H; let $0 < \lambda < 1$; we shall say that H is λ*-separating* for f if $\int_{H_+} f \int_{H_-} f = \lambda(1-\lambda)(\int_{\mathbb{R}^n} f)^2$ and when $\lambda = 1/2$, we shall say that H is *medial* for f. For a function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$, its Legendre transform about z is defined by $\mathscr{L}^z\phi(y) = \sup_{x\in\mathbb{R}^n}[\langle x-z, y-z\rangle - \phi(x)]$. If $f(x) = e^{-\phi(x)}$, where $\phi(x)$ is a convex function, then $f^z(y) = e^{-\mathscr{L}^z\phi(y)}$. Since $\mathscr{L}^z(\mathscr{L}^z\phi) = \phi$ for a convex function ϕ, $(f^z)^z = f$. If $z = 0$, we shall use the simpler notation $\mathscr{L}$ for $\mathscr{L}^0$.

Given two functions $f, g : \mathbb{R}^n \to [0,\infty)$, their *Asplund product* is defined by $(f \star g)(x) = \sup_{x_1+x_2=x} f(x_1)g(x_2)$. The λ*-homothety* of a function f is defined as $(\lambda \cdot f)(x) = f^\lambda(\frac{x}{\lambda})$. Then, the classical Prékopa inequality (see Prékopa [16, 17]) can be stated as follows: Given $f, g : \mathbb{R}^n \to [0,+\infty)$ and $0 < \lambda < 1$, $\int (\lambda \cdot f) \star ((1-\lambda)\cdot g) \geqslant (\int f)^\lambda (\int g)^{1-\lambda}$. The following lemma, as a particular case of a result due to Ball [3], was proved by Meyer and Pajor in [15].

LEMMA 2.1. [15] *Let f_0, f_1, $f_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be three functions such that $0 < \int_0^{+\infty} f_i < \infty$, $i = 0,1,2$, they are continuous and suppose that $f_0\left(\frac{2xy}{x+y}\right) \geqslant f_1(x)^{\frac{y}{x+y}} f_2(y)^{\frac{x}{x+y}}$ for every $x, y > 0$. Then one has*

$$\frac{1}{\int_0^{+\infty} f_0(t)dt} \leqslant \frac{1}{2}\left(\frac{1}{\int_0^{+\infty} f_1(t)dt} + \frac{1}{\int_0^{+\infty} f_2(t)dt}\right).$$

3. The functional Steiner symmetrization

The familiar definition of Steiner symmetrization for a nonnegative measurable function f can be stated as following (see [5, 6, 7, 8]):

DEFINITION 3.1. For a measurable function $f : \mathbb{R}^n \to [0,+\infty)$ and an affine hyperplane $H \subset \mathbb{R}^n$, let m denote the Lebesgue measure, if $m(\{f > t\}) < +\infty$ for all $t > 0$, then its Steiner symmetrization is defined as

$$S_H f(x) = \int_0^\infty \mathscr{X}_{S_H\{f>t\}}(x)dt, \tag{3.1}$$

where $\mathscr{X}_A$ denotes the characteristic function of set A.

Next, we give a approach of defining Steiner symmetrization for coercive convex functions by the Steiner symmetrization of epigraphs. A similar functional steiner symmetrization is defined in a remark of AKM's paper [1] and studied in an article by Lehec [10]. The idea of our definition is same as the given definition in a remark at the end of an article by Fradelizi and Meyer [9].

DEFINITION 3.2. For a coercive convex function ϕ and an affine hyperplane $H \subset \mathbb{R}^n$, we define the *Steiner symmetrization* $S_H\phi$ of ϕ with respect to H as a function satisfying

$$\mathrm{epi}(S_H\phi) = S_{\widetilde{H}}(\mathrm{cl\ epi}\phi), \tag{3.2}$$

where $\widetilde{H} = \{(x', s) \in \mathbb{R}^{n+1} : x' \in H\}$ is an affine hyperplane in $\mathbb{R}^{n+1}$.

REMARK 1.

(i) By Definition 3.2, for an integrable log-concave function $f = e^{-\phi}$, the Steiner symmetrization of f can be defined as $S_H f := e^{-(S_H\phi)}$. If we define $S_H f$ by Definition 3.1, then $S_H f$ still satisfies (3.2). Thus, for integrable log-concave functions, the two definitions are essentially same.

(ii) By Definition 3.2, for a given $x' \in H$ and any $s \in \mathbb{R}$, we have

$$V_1\left(\{(S_H\phi)(x'+tu) < s\}\right) = V_1\left(\{\phi(x'+tu) < s\}\right).$$

By the Fubini's theorem, we have

$$\int_{\mathbb{R}} (S_H f)(x'+tu)dt = \int_{\mathbb{R}} f(x'+tu)dt. \tag{3.3}$$

Similarly, $\int_{\mathbb{R}^n} S_H f = \int_{\mathbb{R}^n} f$ is also established.

PROPOSITION 3.3. *For a coercive convex function ϕ and an affine hyperplane $H \subset \mathbb{R}^n$ with outer unit normal vector u, then $S_H\phi$ has the following properties.*

(i) $S_H\phi$ is a closed coercive convex function and symmetric about H.

(ii) Let H_1 and H_2 be two orthogonal hyperplanes in $\mathbb{R}^n$, then $S_{H_2}(S_{H_1}\phi)$ is symmetric about both H_1 and H_2.

(iii) For any given $x' \in H$ and $t \in \mathbb{R}$, let $\phi_1(t) := \phi(x'+tu)$ and $(S\phi_1)(t) := (S_H\phi)(x'+tu)$, then $(S\phi_1)(t)$ satisfies one of the following three cases. 1). $(S\phi_1)(t) = \phi_1(t_1) = \phi_1(t_1 - 2t)$ for some $t_1 \in \mathbb{R}$. 2). $(S\phi_1)(t) = \phi_1(t_0 - 2t) \geqslant \lim_{t\to t_0,\, t<t_0} \phi_1(t)$ for some $t_0 \in \mathbb{R}$. 3). $(S\phi_1)(t) = \phi_1(t_0 + 2t) \geqslant \lim_{t\to t_0,\, t>t_0} \phi_1(t)$ for some $t_0 \in \mathbb{R}$.

Proof. (i) By the fact that ϕ is convex if and only if epiϕ is convex, since ϕ is convex, epiϕ is a convex subset of $\mathbb{R}^{n+1}$. Since the closure of a convex set is convex, and the Steiner symmetrization of a convex set is also convex, by (3.2), epi$(S_H\phi)$ is a convex subset of $\mathbb{R}^{n+1}$. Therefore, $S_H\phi$ is a convex function. By Definition 3.2, it is clear that $S_H\phi$ is closed, coercive and symmetric with respect to H.

(ii) Since epi$(S_{H_2}(S_{H_1}\phi))$ is symmetric about both $\widetilde{H_1}$ and $\widetilde{H_2}$, where $\widetilde{H}_i = \{(x', s) \in \mathbb{R}^{n+1} : x' \in H_i\}$ $(i = 1, 2)$, $S_{H_2}(S_{H_1}\phi)$ is symmetric about both H_1 and H_2.

(iii) If dom$\phi_1 = \mathbb{R}$, by (3.2) in Definition 3.2, we have

$$\text{epi}(S\phi_1) = S_{\widetilde{H}}(\text{cl epi}\phi_1). \tag{3.4}$$

Thus there exists some $t_1 \in \mathbb{R}$ satisfying

$$(S\phi_1)(t) = \phi_1(t_1) = \phi_1(t_1 - 2t). \tag{3.5}$$

If dom$\phi_1 \neq \mathbb{R}$, then there exist eight cases for domϕ_1: 1) $[\alpha, \beta]$; 2) (α, β); 3) $(\alpha, \beta]$; 4) $[\alpha, \beta)$; 5) $(-\infty, \beta]$; 6) $(-\infty, \beta)$; 7) $[\alpha, +\infty)$; 8) $(\alpha, +\infty)$. Here, we only prove our conclusion for dom$\phi_1 = (\alpha, \beta)$. By the same method we can prove our conclusion for other cases.

For $\mathrm{dom}\phi_1 = (\alpha,\beta)$, by Definition 3.2, it is clear that $(S\phi_1)(t) = +\infty$ for $|t| \geqslant \frac{\beta-\alpha}{2}$. If $|t| < \frac{\beta-\alpha}{2}$, let $\lim_{x\to\alpha,\, x>\alpha}\phi_1(x) = b_1$, $\lim_{x\to\beta,\, x<\beta}\phi_1(x) = b_2$, then we consider the following four cases. (a) If $b_1 = b_2 = +\infty$, then by (3.4), there exists some $t_1 \in \mathbb{R}$ satisfying (3.5). (b) If $b_1 < +\infty$, $b_2 = +\infty$, then there exists $\gamma \in (\alpha,\beta)$ such that $\phi_1(\gamma) = b_1$. Then by (3.4), for $|t| < \frac{\gamma-\alpha}{2}$, (3.5) is established, for $|t| \geqslant \frac{\gamma-\alpha}{2}$, we have $(S\phi_1)(t) = \phi_1(\alpha+2t) \geqslant b_1$. (c) If $b_1 = +\infty$, $b_2 < +\infty$, then there exists $\gamma \in (\alpha,\beta)$ such that $\phi_1(\gamma) = b_2$. Then by (3.4), for $|t| < \frac{\beta-\gamma}{2}$, (3.5) is established, for $|t| \geqslant \frac{\gamma-\alpha}{2}$, we have $(S\phi_1)(t) = \phi_1(\beta-2t) \geqslant b_2$. (d) If $b_1 < \infty$, $b_2 < +\infty$, we consider three cases. If $b_1 = b_2$, then (3.5) is established. If $b_1 > b_2$, the proof is same as in (c). If $b_1 < b_2$, the proof is same as in (b). This completes the proof. □

4. The proofs of theorems

In order to prove theorems stated in the introduction, we have to establish the following six lemmas:

LEMMA 4.1. *If f be a log-concave function such that $0 < \int f < \infty$, then the function F defined by $F(z) := \int_{\mathbb{R}^n} f^z(x)dx$ has the following properties.* (*i*) *$F(z)$ is a coercive convex function on $\mathbb{R}^n$ and is strictly convex on* $\mathrm{int\,dom}F$; (*ii*) *If $f(x)$ is even about z_0, then $F(z)$ is also even about z_0.*

Proof. (i) *Step 1.* We shall prove F is coercive. Let $f = e^{-\phi}$, for any given $z \in \mathbb{R}^n$ and $r > 0$, we have

$$F(z) = \int_{\mathbb{R}^n} f^z(x+z)dx \geqslant \int_{rB_2^n} f^z(x+z)dx = \int_{rB_2^n} e^{-\mathscr{L}\phi(x)+\langle x,z\rangle}dx. \tag{4.1}$$

Since $f = e^{-\phi}$ is integrable, there is $\gamma > 0$ and $h \in \mathbb{R}$ such that

$$\phi(x) \geqslant \gamma\sum_{i=1}^{n}|x_i| + h \text{ for any } x \in \mathbb{R}^n. \tag{4.2}$$

Thus, for $y \in \gamma B_\infty^n$, where $B_\infty^n = \{x \in \mathbb{R}^n : |x_i| \leqslant 1, i = 1,\ldots,n\}$, $\mathscr{L}\phi(y) \leqslant \sup_{x\in\mathbb{R}^n}[\langle y,x\rangle - \gamma\sum_{i=1}^n|x_i| - h] \leqslant -h$. Let $rB_2^n \subset \frac{1}{2}\gamma B_\infty^n$, we have $rB_2^n \subset \mathrm{int}(\mathrm{dom}\mathscr{L}\phi)$. Since function $g(x) := e^{-\mathscr{L}\phi(x)}$ is continuous on rB_2^n. Thus, there exists $m > 0$ such that $g(x) \geqslant m$ for any $x \in rB_2^n$. Therefore,

$$\int_{rB_2^n} e^{-\mathscr{L}\phi(x)+\langle x,z\rangle}dx \geqslant m\int_{rB_2^n} e^{\langle x,z\rangle}dx. \tag{4.3}$$

For any $z \in \mathbb{R}^n$ and $|z| \geqslant 1$, let $z' = \frac{r}{2}\frac{z}{|z|}$, we get a closed half-space $H^+ = \{x \in \mathbb{R}^n : \langle x - z', z\rangle \geqslant 0\}$. For any $x \in H^+$, we have $\langle x,z\rangle \geqslant \langle z',z\rangle = \frac{r}{2}|z|$. Therefore,

$$\int_{rB_2^n} e^{\langle x,z\rangle}dx \geqslant \int_{(rB_2^n)\cap H^+} e^{\frac{r|z|}{2}}dx = V_n((rB_2^n)\cap H^+)e^{\frac{r|z|}{2}}. \tag{4.4}$$

Since $V_n((rB_2^n)\cap H^+)$ is a positive constant independent of z, by (4.1), (4.3) and (4.4), $F(z)$ is coercive.

Step 2. We shall prove that F is convex and is strictly convex on $\operatorname{int}\operatorname{dom}F$. First, we prove $F(z)$ is proper. It is clear that $F(z) > -\infty$ for any $z\in\mathbb{R}^n$. The following claim shows that $\{F=\infty\}\neq\mathbb{R}^n$.

CLAIM 1. *For any $z\in\operatorname{int}\operatorname{supp}f$, $F(z)<\infty$.*

Proof of Claim 1. For any $z\in\operatorname{int}\operatorname{supp}f$, there is a closed ball $z+rB_2^n\subset\operatorname{supp}f$. Since $\operatorname{supp}f=\operatorname{dom}\phi$, there is $M\in\mathbb{R}$ such that $M=\sup\{\phi(y): y\in z+rB_2^n\}$. Thus, we have

$$f^z(x)\leqslant\exp\{-\sup_{y\in(z+rB_2^n)}[\langle x-z,y-z\rangle-\phi(y)]\}\leqslant e^M\cdot e^{-r|x-z|^2}.$$

Therefore, $\int_{\mathbb{R}^n}f^z(x)dx\leqslant e^M\int_{\mathbb{R}^n}e^{-r|x-z|^2}dx<\infty.$ □

For any $z_1,z_2\in\mathbb{R}^n$ and $\alpha\in(0,1)$. Let $f=e^{-\phi}$, we have $F(z)=\int_{\mathbb{R}^n}e^{-\mathscr{L}\phi(x)+\langle x,z\rangle}dx$. Since $g_x(z):=e^{-\mathscr{L}\phi(x)+\langle x,z\rangle}$ is a convex function about z, we have

$$F(\alpha z_1+(1-\alpha)z_2)\leqslant\alpha F(z_1)+(1-\alpha)F(z_2). \tag{4.5}$$

If $z_1,z_2\in\operatorname{int}\operatorname{dom}F$ and $z_1\neq z_2$, then inequality (4.5) is a strict inequality. Thus $F(z)$ is strictly convex on $\operatorname{int}\operatorname{dom}F$.

(ii) Since $f(x)$ is even about z_0, $f(z_0+x)=f(z_0-x)$ for any $x\in\mathbb{R}^n$. For any $z\in\mathbb{R}^n$, we have

$$F(z_0+z)=\int_{\mathbb{R}^n}f^{z_0+z}(x)dx=\int_{\mathbb{R}^n}f^{z_0-z}(-x+2z_0)dx=F(z_0-z).$$

This completes the proof. □

REMARK 2. By Lemma 4.1, if f is even about z_0, then $s(f)=z_0$.

LEMMA 4.2. *Let f be a log-concave function such that $0<\int f<\infty$, and let $G\subset\mathbb{R}^n$ be an affine subspace satisfying $G\cap\operatorname{int}\operatorname{supp}f\neq\emptyset$. Then there exists a unique point $z_0\in G$ satisfying the following two equivalent claims. (i) $F(z_0)=\min\{F(z);z\in G\}$, where $F(z):=\int_{\mathbb{R}^n}f^z(x)dx$. (ii) $\operatorname{grad}F(z_0)=\int_{\mathbb{R}^n}xf^{z_0}(x+z_0)dx\in G^\perp$.*

Proof. By Lemma 4.1, F is coercive and strictly convex on $\operatorname{int}\operatorname{dom}F$, thus there is a unique minimal point $z_0=s_G(f)$ on G. Let $f=e^{-\phi}$, then $F(z)=\int_{\mathbb{R}^n}e^{-\mathscr{L}\phi(x)+\langle x,z\rangle}dx$. By the dominated convergence theorem, we have $\operatorname{grad}F(z)=\int_{\mathbb{R}^n}xe^{-\mathscr{L}\phi(x)+\langle x,z\rangle}dx=\int_{\mathbb{R}^n}xf^z(x+z)dx$.

Next, we prove the equivalence of (i) and (ii). Let $\eta_1,\dots,\eta_m$ $(m<n)$ be an orthonormal basis of G and let $\eta_{m+1},\dots,\eta_n$ be an orthonormal basis of $G^\perp$. Let $z=\sum_{i=1}^n z_i\eta_i$, since $z_0=s_G(f)\in G$, we have $\frac{\partial F(z)}{\partial z_i}\Big|_{z=z_0}=\lim_{t\to0}\frac{F(z_0+t\eta_i)-F(z_0)}{t}=$

0, $i=1,\dots,m$. Hence, $\mathrm{grad}F(z_0)\in G^{\perp}$. On the other hand, if $\mathrm{grad}F(z_0)\in G^{\perp}$, then $\frac{\partial F(z)}{\partial z_i}\Big|_{z=z_0}=0$, $i=1,\dots,m$. Since $F(z)$ is strictly convex on $G\cap\mathrm{int\,dom}F$, z_0 is the unique minimal point on G. $\square$

REMARK 3. In Lemma 4.2, if $G=\mathbb{R}^n$, then the lemma shows that the Santaló point $s(f)$ of f is the barycenter of the function $f^{s(f)}$.

LEMMA 4.3. *Let f be a log-concave function such $0<\int f<\infty$. Let $G\subset\mathbb{R}^n$ be an affine subspace satisfying $G\cap\mathrm{int\,supp}f\neq\emptyset$ and $z=s_G(f)$. Let H be an affine hyperplane such that $G\subset H$ and let g be the function defined by $g^z=S_H(f^z)$. Then we have $s_G(g)=z=s_G(f)$.*

Proof. It may be supposed that $z=s_G(f)=0$, $H=\{(x_1,\cdots,x_n)\in\mathbb{R}^n : x_n=0\}$ and $G=\{(x_1,\cdots,x_n)\in\mathbb{R}^n : x_{m+1}=\cdots=x_n=0\}$ for some m, $1\leqslant m\leqslant n-1$. By Lemma 4.2, we have $\int_{\mathbb{R}^n}xf^0(x)dx\in G^{\perp}$. Let $f^0_{x'}(t):=f^0(x'+tu)$ for any $x'\in H$, where u is the unit normal vector of H. Thus, $\int_H x_i\left(\int_{\mathbb{R}}f^0_{x'}(t)dt\right)dx'=0$ for $1\leqslant i\leqslant m$. By $g^0=S_H(f^0)$ and (3.3), for every $x'\in H$, $\int_{\mathbb{R}}f^0_{x'}(t)=\int_{\mathbb{R}}g^0_{x'}(t)$. Thus, $\int_H x_i\left(\int_{\mathbb{R}}g^0_{x'}(t)dt\right)dx'=0$ for $1\leqslant i\leqslant m$, which conversely gives $\int_{\mathbb{R}^n}xg^0(x)dx\in G^{\perp}$. Thus, by Lemma 4.2 again, we obtain $s_G(g)=0=s_G(f)$. $\square$

LEMMA 4.4. *For a log-concave function f such that $0<\int f<\infty$, if f is symmetric about some affine hyperplane H, then, for any $z\in H$, f^z is also symmetric about H.*

Proof. Let u be the unit normal vector of H. For any $x',y'\in H$ and $s,t\in\mathbb{R}$, since $f(x'+su)=f(x'-su)$, we have

$$\begin{aligned} f^z(y'+tu) &= \inf_{x'+su\in\mathbb{R}^n}\frac{\exp\{-\langle y'+tu-z,x'+su-z\rangle\}}{f(x'+su)}\\ &= \inf_{x'+su\in\mathbb{R}^n}\frac{\exp\{-\langle y'-z-tu,x'-z-su\rangle\}}{f(x'-su)}=f^z(y'-tu). \end{aligned}$$

This completes the proof. $\square$

LEMMA 4.5. *Let f be a log-concave function such that $0<\int f<\infty$ and let H be an affine hyperplane satisfying $H\cap\mathrm{int\,supp}f\neq\emptyset$ and $z\in H\cap\mathrm{int\,supp}f$; let λ, $0<\lambda<1$ such that H is λ-separating for f^z. Then*

$$\int_{\mathbb{R}^n}(S_Hf)^z\geqslant 4\lambda(1-\lambda)\int_{\mathbb{R}^n}f^z.$$

Proof. It may be supposed that $z=0$ and $H=\{(x_1,\dots,x_n):x_n=0\}$. For $y'\in H$ and $s\in\mathbb{R}$, let (y',s) denote $y'+su$, where u is a unit normal vector of H. For f^0 and $s\in\mathbb{R}$, we define a new function

$$f^0_{(s)}(y'):=f^0(y',s),\ \text{for any } y'\in H.$$

Next we shall prove that for any $y' \in H$ and $s,t > 0$

$$\left(\frac{t}{s+t}\cdot f^0_{(s)}\right) \star \left(\frac{s}{s+t}\cdot f^0_{(-t)}\right)(y') \leqslant (S_H f)^0_{(\frac{2st}{s+t})}(y'). \tag{4.6}$$

CLAIM 2. *For any $x' \in H$ and $w \in \mathbb{R}$, if $(S_H f)(x'+wu) > 0$, then there is some $w_1 \in \mathbb{R}$ such that $(S_H f)(x'+wu) \leqslant f(x'+w_1 u)$ and $(S_H f)(x'+wu) \leqslant f(x'+(w_1-2w)u)$.*

Proof of Claim 2. Let $f = e^{-\phi}$, since $(S_H f)(x'+wu) > 0$, then $(S_H\phi)(x'+wu) < +\infty$. By Proposition 3.3(iii), there is $w_1 \in \mathbb{R}$ such that $(S_H\phi)(x'+wu) \geqslant \phi(x'+w_1 u)$ and $(S_H\phi)(x'+wu) \geqslant \phi(x'+(w_1-2w)u)$, here we assume $\phi(x'+w_1 u)$ or $\phi(x'+(w_1-2w)u)$ equals the limit in Proposition 3.3(iii), which doesn't affect our proof. Hence the claim follows. $\square$

For any $y'_1,\, y'_2 \in H$ such that $y' = y'_1 + y'_2$, we have

$$\begin{aligned}
(S_H f)^0_{(\frac{2st}{s+t})}(y') &= \inf_{(x',w)\in H\times\mathbb{R}} \frac{\exp\{-\langle (y',\frac{2st}{s+t}),(x',w)\rangle\}}{(S_H f)(x',w)} \\
&\geqslant \inf_{(x',w)\in H\times\mathbb{R}} \frac{\exp\{-\langle (y',\frac{2st}{s+t}),(x',w)\rangle\}}{f(x',w_1)^{\frac{t}{s+t}} f(x',w_1-2w)^{\frac{s}{s+t}}} \\
&\geqslant \inf_{(x',w)\in H\times\mathbb{R}} \frac{\exp\{-\frac{t}{s+t}\langle (\frac{s+t}{t}y'_1,s),(x',w_1)\rangle\}}{f(x',w_1)^{\frac{t}{s+t}}} \\
&\quad \times \inf_{(x',w)\in H\times\mathbb{R}} \frac{\exp\{-\frac{s}{s+t}\langle (\frac{s+t}{s}y'_2,-t),(x',w_1-2w)\rangle\}}{f(x',w_1-2w)^{\frac{s}{s+t}}} \\
&\geqslant f^0\left(\frac{s+t}{t}y'_1,s\right)^{\frac{t}{s+t}} f^0\left(\frac{s+t}{s}y'_2,-t\right)^{\frac{s}{s+t}},
\end{aligned}$$

where the first inequality is by Claim 2, and the second inequality is by $\inf(AB) \geqslant (\inf A)(\inf B)$, and last inequality is by the definition of the polar of functions. Since y'_1 and y'_2 are arbitrary, we get (4.6).

Let $F_0(w) = \int_H (S_H f)^0_{(w)}$, $F_1(s) = \int_H f^0_{(s)}$ and $F_2(t) = \int_H f^0_{(-t)}$. By the Prékopa inequality and (4.6), we have

$$F_0\left(\frac{2st}{s+t}\right) \geqslant F_1(s)^{\frac{t}{s+t}} F_2(t)^{\frac{s}{s+t}} \text{ for every } s,t > 0.$$

Now, by Proposition 3.3(i) and Lemma 4.4, $(S_H f)^0$ is symmetric about H, we have $\int_0^{+\infty} F_0 = \frac{1}{2}\int_{\mathbb{R}^n}(S_H f)^0$ and since H is λ-separating for f^0, we have $\left(\int_0^{+\infty} F_1\right)\left(\int_0^{+\infty} F_2\right) = \lambda(1-\lambda)\left(\int_{\mathbb{R}^n} f^0\right)^2$. Since $F_0,\ F_1,\ F_2 : [0,+\infty) \to \mathbb{R}^+$ satisfy the hypothesis of Lemma 2.1, and by definitions of F_1 and F_2, one has $\int_0^{+\infty} F_1 + \int_0^{+\infty} F_2 = \int_{\mathbb{R}^n} f^0$, thus, by Lemma 2.1

$$\frac{2}{\int_{\mathbb{R}^n}(S_H f)^0} \leqslant \frac{1}{2}\left(\frac{1}{\int_0^{+\infty} F_1} + \frac{1}{\int_0^{+\infty} F_2}\right) = \frac{1}{2\lambda(1-\lambda)\int_{\mathbb{R}^n} f^0}.$$

This gives the desired inequality. □

LEMMA 4.6. *If* f *is an integrable, unconditional, log-concave function, then* $\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} f^0 \leqslant (2\pi)^n$.

Proof. Let $f_1 = f$, $f_2 = f^0$ and $f_3 = e^{-\frac{|x|^2}{2}}$, then f_1, f_2 and f_3 are unconditional. Thus we have $\int_{\mathbb{R}^n} f_j = 2^n \int_{\mathbb{R}^n_+} f_j$, $j = 1,2,3$. For $(y_1,\dots,y_n) \in \mathbb{R}^n$, we define $g_i(y_1,\dots,y_n) = f_i(e^{y_1},\dots,e^{y_n})e^{\sum_{i=1}^n y_i}$. We get $\int_{\mathbb{R}^n_+} f_j = \int_{\mathbb{R}^n} g_j$, and for every $s,t \in \mathbb{R}^n$, $g_1(s)g_2(t) \leqslant g_3\left(\frac{s+t}{2}\right)^2$. Hence $\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} f^0 \leqslant (2\pi)^n$ follows from Prékopa inequality. □

Proof of Theorem 1.2. We proceed by n successive Steiner symmetrizations until we get an unconditional log-concave function. Let $u_1 \in S^{n-1}$, u_1 orthogonal to $H = H_1$ and let $(u_i)_{i=2}^n \subset S^{n-1}$ such that $(u_1,\dots,u_n)$ form an orthonormal basis for $\mathbb{R}^n$. Let $z_1 = s_{H_1}(f)$ and define a log-concave function f_1 by the identity $f_1^{z_1} = S_{H_1}(f^{z_1})$. Then $\int f_1^{z_1} = \int f^{z_1}$. By Proposition 3.3(i) and Lemma 4.4, f_1 is symmetric about H_1 and by Lemma 4.5, applied to f^{z_1}, $z = z_1$ and $H = H_1$, λ-separating for $f = (f^{z_1})^{z_1}$, we get $\int_{\mathbb{R}^n} f_1 \geqslant 4\lambda(1-\lambda)\int_{\mathbb{R}^n} f$ and thus $\int f_1 \int f_1^{z_1} \geqslant 4\lambda(1-\lambda)\int f \int f^{z_1}$. Choose now the hyperplane H_2, orthogonal to u_2, and medial for f_1 and define $z_2 = s_{(H_1\cap H_2)}(f_1)$. By Lemma 4.3 we have $z_1 = s_{H_1}(f) = s_{H_1}(f_1)$, we get $\int f_1^{z_2} = \min_{z\in H_1\cap H_2}\int f_1^z \geqslant \min_{z\in H_1}\int f_1^z = \int f_1^{z_1}$. We define now a new log-concave function f_2 by the identity $f_2^{z_2} = S_{H_2}(f_1^{z_2})$. By Proposition 3.3(ii) and Lemma 4.4, f_2 is symmetric about both H_1 and H_2. Since H_2 is medial for f_1, we get by Lemma 4.5 applied to $f_1^{z_2}$, $z = z_2$ and $H = H_2$ that $\int f_2 \geqslant \int f_1$. Moreover, we have $\int f_2^{z_2} = \int S_{H_2}(f_1^{z_2}) = \int f_1^{z_2} \geqslant \int f_1^{z_1}$. It follows that $\int f_2 \int f_2^{z_2} \geqslant \int f_1 \int f_1^{z_1}$.

We continue this procedure by choosing hyperplanes $H_2,\dots,H_n$, points $z_2,\dots,z_n$, and defining log-concave functions $f_2,\dots,f_n$ such that for $2 \leqslant i \leqslant n$, we have (i) H_i is medial for f_{i-1} and orthogonal to u_i; (ii) $z_i = s_{(H_1\cap H_2\cap\dots\cap H_i)}(f_{i-1})$; (iii) $f_i^{z_i} = S_{H_i}(f_{i-1}^{z_i})$. From (ii) (iii) and Lemma 4.3, we have $z_i = s_{(H_1\cap\dots\cap H_i)}(f_{i-1}) = s_{(H_1\cap\dots\cap H_i)}(f_i)$. Choosing H_{i+1}, z_{i+1}, f_{i+1} according to (i) (ii) (iii), we get thus $\int f_{i+1}^{z_{i+1}} = \int S_{H_{i+1}}(f_i^{z_{i+1}})$ $= \int f_i^{z_{i+1}} \geqslant \int f_i^{s_{(H_1\cap\dots\cap H_i)}(f_i)} = \int f_i^{z_i}$. Now, Lemma 4.5 applied to $f_i^{z_{i+1}}$, $z = z_{i+1}$ and H_{i+1}, medial for $f_i = (f_i^{z_{i+1}})^{z_{i+1}}$, gives $\int f_{i+1} \geqslant \int f_i$. Thus, $\int f_i \int f_i^{z_i}$ is an increasing sequence, for $2 \leqslant i \leqslant n$. Therefore, we have $4\lambda(1-\lambda)\int f \int f^{z_1} \leqslant \int f_1 \int f_1^{z_1} \leqslant \dots \leqslant \int f_n \int f_n^{z_n}$. From Proposition 3.3(ii), f_n is an unconditional function about z_n and $z_n \in H_1 \cap H_2 \cap \dots \cap H_n$ is a center of symmetry for f_n. By Lemma 4.6, we have $\int f \int f^{z_1} \leqslant \frac{(2\pi)^n}{4\lambda(1-\lambda)}$, this concludes the proof. □

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