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REVERSE HARDY–TYPE INEQUALITIES FOR SUPREMAL OPERATORS WITH MEASURES

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Abstract. In this paper we characterize the validity of the inequalities

$$\|g\|_{p,(a,b),\lambda} \leqslant c\|u(x)\|g\|_{\infty,(x,b),\mu}\|_{q,(a,b),\nu}$$

and

$$\|g\|_{p,(a,b),\lambda} \leqslant c\|u(x)\|g\|_{\infty,(a,x),\mu}\|_{q,(a,b),\nu}$$

for all non-negative Borel measurable functions g on the interval $(a,b) \subseteq \mathbb{R}$, where $0 < p \leqslant +\infty$, $0 < q \leqslant +\infty$, λ, μ and ν are non-negative Borel measures on (a,b), and u is a weight function on (a,b).

1. Introduction

In [1], authors make a comprehensive study of general inequalities of the form

$$\|gw\|_{p,(a,b),\mu} \leqslant c\,\big\|u(x)\|g\|_{1,(x,b),\mu}\big\|_{q,(a,b),\nu}, \qquad g \in B^+(I) \tag{1.1}$$

and

$$\|gw\|_{p,(a,b),\mu} \leqslant c\,\big\|u(x)\|g\|_{1,(a,x),\mu}\big\|_{q,(a,b),\nu}, \qquad g \in B^+(I), \tag{1.2}$$

involving non-negative Borel measures μ, ν and λ, with complete proofs and estimates for the best constants c, provided that $0 < p \leqslant 1$ and $0 < q \leqslant +\infty$. In addition to the extra generality and the filling gaps in previous works on these inequalities, the approach used in [1] unifies the continuous and discrete problems, so that the integral and series inequalities follow as particular cases. The general inequalities involving three Borel measures λ, μ and ν

$$\|g\|_{p,(a,b),\lambda} \leqslant c\,\big\|u(x)\|g\|_{1,(x,b),\mu}\big\|_{q,(a,b),\nu}, \qquad g \in B^+(I) \tag{1.3}$$

and

$$\|g\|_{p,(a,b),\lambda} \leqslant c\,\big\|u(x)\|g\|_{1,(a,x),\mu}\big\|_{q,(a,b),\nu}, \qquad g \in B^+(I), \tag{1.4}$$

are reduced to either to (1.1) or (1.2).

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The object of this paper is to characterize the inequalities

$$\|gw\|_{p,(a,b),\mu} \leqslant c \|u(x)\|g\|_{\infty,(x,b),\mu}\|_{q,(a,b),\nu} \tag{1.5}$$

and

$$\|gw\|_{p,(a,b),\mu} \leqslant c \|u(x)\|g\|_{\infty,(a,x),\mu}\|_{q,(a,b),\nu} \tag{1.6}$$

for all non-negative Borel measurable functions g on the interval $(a,b) \subseteq \mathbb{R}$, where $0 < p \leqslant +\infty$, $0 < q \leqslant +\infty$, μ and ν are non-negative Borel measures on (a,b). Note that we do not need the restriction $0 < p \leqslant 1$, which is important when one consider the reverse Hardy inequalities. The general inequalities (involving three non-negative Borel measures λ, μ and ν) are reduced either to (1.5) or (1.6).

The main results of the present paper are Theorems 3.4 and 4.1. Our method is based on a discretization techniques for function norms developed in [1].

The paper is organized as follows. We start with notation and preliminary results in Section 2. Necessary and sufficient conditions for the validity of inequalities (1.5) and (1.6) can be found in Sections 3 and 4, respectively. Finally, in Section 5 we show that the results from Sections 3 and 4 can be used to characterize the validity of inequalities mentioned at the Abstract of this paper.

2. Notation and preliminaries

Throughout the paper, we always denote by c a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $a \lesssim b$ ($b \gtrsim a$), we mean that $a \leqslant cb$, where $c > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are *equivalent*. We use the abbreviation LHS$(*)$ (RHS$(*)$) for the left (right) hand side of the relation $(*)$.

We adopt the following usual conventions.

CONVENTION 2.1. (i) We put $1/(\pm\infty) = 0$, $0 \cdot (\pm\infty) = 0$, $0/0 = 0$.
(ii) We denote by

$$p' := \begin{cases} \frac{p}{1-p} & \text{if} \quad 0 < p < 1, \\ +\infty & \text{if} \quad p = 1, \\ \frac{p}{p-1} & \text{if} \quad 1 < p < +\infty, \\ 1 & \text{if} \quad p = +\infty. \end{cases}$$

(iii) If g is a monotone function on $I := (a,b) \subseteq \mathbb{R}$, then by $g(a)$ and $g(b)$ we mean the limits $\lim_{x\to a+} g(x)$ and $\lim_{x\to b-} g(x)$, respectively.

Let μ be a non-negative Borel measure on I. We denote by $B^+(I)$ the set of all non-negative Borel measurable functions on I. If E is a nonempty Borel measurable subset of I and f is a Borel measurable function on E, then we put

$$\|f\|_{p,E,\mu} := \left(\int_E |f(y)|^p d\mu \right)^{1/p}, \quad \text{if} \quad 0 < p < +\infty,$$

$$\|f\|_{\infty,E,\mu} := \sup\{\alpha : \ \mu(\{y \in E : \ |f(y)| \geqslant \alpha\}) > 0\}.$$

In this paper, u, v and w will denote weights, that is, non-negative Borel measurable functions on I.

Let $\emptyset \neq \mathscr{Z} \subseteq \overline{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty, +\infty\}$, $0 < q \leqslant +\infty$ and $\{w_k\} = \{w_k\}_{k\in\mathbb{Z}}$ be a sequence of positive numbers. We denote by $\ell^q(\{w_k\}, \mathscr{Z})$ the following discrete analogue of a weighted Lebesgue space: if $0 < q < +\infty$, then

$$\ell^q(\{w_k\}, \mathscr{Z}) = \left\{ \{a_k\}_{k\in\mathscr{Z}} : \|\{a_k\}\|_{\ell^q(\{w_k\},\mathscr{Z})} := \left(\sum_{k\in\mathscr{Z}} |a_k w_k|^q \right)^{1/q} < +\infty \right\},$$

and

$$\ell^\infty(\{w_k\}, \mathscr{Z}) = \left\{ \{a_k\}_{k\in\mathscr{Z}} : \|\{a_k\}\|_{\ell^\infty(\{w_k\},\mathscr{Z})} := \sup_{k\in\mathscr{Z}} |a_k w_k| < +\infty \right\}.$$

If $w_k = 1$ for all $k \in \mathscr{Z}$, we write simply $\ell^q(\mathscr{Z})$ instead of $\ell^q(\{w_k\}, \mathscr{Z})$. When $N, M \in \overline{\mathbb{Z}}$, $N \leqslant M$ and $\mathscr{Z} = \{N, N+1, \ldots, M-1, M\}$, we will sometimes use notation $\ell^q(N,M)$ instead of $\ell^q(\mathscr{Z})$.

We shall use the following inequality, which is a simple consequence of the discrete Hölder inequality:

$$\|\{a_k b_k\}\|_{\ell^q(\mathscr{Z})} \leqslant \|\{a_k\}\|_{\ell^r(\mathscr{Z})} \|\{b_k\}\|_{\ell^p(\mathscr{Z})}, \tag{2.1}$$

where $1/r = (1/q - 1/p)_+$.[1]

DEFINITION 2.2. Let $N, M \in \overline{\mathbb{Z}}$, $N < M$. A positive almost non-increasing sequence $\{\tau_k\}_{k=N}^M$ (that is, there exists $K \geqslant 1$ such that $\tau_{n+1} \leqslant K\tau_n$) is called *almost geometrically decreasing* if there are $\alpha \in (1, +\infty)$ and $L \in \mathbb{N}$ such that

$$\alpha\, \tau_k \leqslant \tau_{k-L} \quad \text{for all} \quad k \in \{N+L, \ldots, M\}.$$

A positive almost non-decreasing sequence $\{\sigma_k\}_{k=N}^M$ (that is, there exists $K \geqslant 1$ such that $\sigma_n \leqslant K\sigma_{n+1}$) is called *almost geometrically increasing* if there are $\alpha \in (1, +\infty)$ and $L \in \mathbb{N}$ such that

$$\sigma_k \geqslant \alpha \sigma_{k-L} \quad \text{for all} \quad k \in \{N+L, \ldots, M\}.$$

REMARK 2.3. Definition 2.2 implies that if $0 < q < +\infty$, then the following three statements are equivalent:

(i) $\{\tau_k\}_{k=N}^M$ is an almost geometrically decreasing sequence;
(ii) $\{\tau_k^q\}_{k=N}^M$ is an almost geometrically decreasing sequence;
(iii) $\{\tau_k^{-q}\}_{k=N}^M$ is an almost geometrically increasing sequence.

We quote some known results. Proofs can be found in [5] and [6].

[1] For any $a \in \mathbb{R}$ denote by $a_+ = a$ when $a > 0$ and $a_+ = 0$ when $a \leqslant 0$.

LEMMA 2.4. *Let* $q \in (0,+\infty]$, $N, M \in \overline{\mathbb{Z}}$, $N \leqslant M$, $\mathscr{Z} = \{N, N+1, \dots, M-1, M\}$ *and let* $\{\tau_k\}_{k=N}^M$ *be an almost geometrically decreasing sequence. Then*

$$\left\| \left\{ \tau_k \sum_{m=N}^{k} a_m \right\} \right\|_{\ell^q(\mathscr{Z})} \approx \|\{\tau_k a_k\}\|_{\ell^q(\mathscr{Z})} \tag{2.2}$$

and

$$\left\| \left\{ \tau_k \sup_{N \leqslant m \leqslant k} a_m \right\} \right\|_{\ell^q(\mathscr{Z})} \approx \|\{\tau_k a_k\}\|_{\ell^q(\mathscr{Z})} \tag{2.3}$$

for all non-negative sequences $\{a_k\}_{k=N}^M$.

Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and if the natural embedding of X in Y is continuous.

The following two lemmas are discrete version of the classical Landau resonance theorems. Proofs can be found, for example, in [3].

PROPOSITION 2.5. ([3, Proposition 4.1]) *Let* $0 < p, q \leqslant +\infty$, $\emptyset \neq \mathscr{Z} \subseteq \overline{\mathbb{Z}}$ *and let* $\{v_k\}_{k \in \mathscr{Z}}$ *and* $\{w_k\}_{k \in \mathscr{Z}}$ *be two sequences of positive numbers. Assume that*

$$\ell^p(\{v_k\}, \mathscr{Z}) \hookrightarrow \ell^q(\{w_k\}, \mathscr{Z}). \tag{2.4}$$

Then

$$\|\{w_k v_k^{-1}\}\|_{\ell^r(\mathscr{Z})} \leqslant c,$$

where $1/r = (1/q - 1/p)_+$ *and* c *stands for the norm of embedding* (2.4).

Now we recall some basic facts on discretization of function norms from [1].

LEMMA 2.6. ([1, Lemma 3.1]) *Let* φ *be a non-negative, non-decreasing, finite and right-continuous function on* (a,b). *There is a strictly increasing sequence* $\{x_k\}_{k=N}^{M+1}$, $-\infty \leqslant N \leqslant M \leqslant +\infty$, *with elements from the closure of the interval* (a,b), *such that:*

(i) *if* $N > -\infty$, *then* $\varphi(x_N) > 0$; $\varphi(x) = 0$ *for every* $x \in (a, x_N)$; *if* $M < +\infty$, *then* $x_{M+1} = b$;

(ii) $\varphi(x_{k+1}-) \leqslant 2\varphi(x_k)$ *if* $N \leqslant k \leqslant M$;

(iii) $2\varphi(x_k-) \leqslant \varphi(x_{k+1})$ *if* $N < k < M$.

DEFINITION 2.7. ([1, Definition 3.2]) Let φ be a non-negative, non-decreasing, finite and right-continuous function on (a,b). A strictly increasing sequence $\{x_k\}_{k=N}^{M+1}$, $-\infty \leqslant N < M \leqslant +\infty$, is said to be *a discretizing sequence of the function* φ if it satisfies the conditions (i) – (iii) of Lemma 2.6.

REMARK 2.8. ([1, Remark 3.3]) We shall use the following *convention*: if $N = -\infty$, then we put $x_N = \lim_{k \to -\infty} x_k$. It is clear that if $N = -\infty$ and $x_N > a$, then $\varphi(x) = 0$ for all $x \in (a, x_N)$ (cf. condition (i) of Lemma 2.6).

Let φ be a non-negative, non-decreasing, finite and right-continuous function on (a,b). Using a discretizing sequence $\{x_k\}_{k=N}^{M+1}$ of φ, we define the sequence of intervals $\{J_k\}_{k=N}^{M}$ as follows:

$$J_i = (x_i, x_{i+1}], \quad \text{if} \quad N \leqslant i < M, \quad \text{and} \quad J_M = (x_M, b) \quad \text{if} \quad M < \infty. \tag{2.5}$$

THEOREM 2.9. ([1, Corollary 3.6 and Corollary 3.7]) *Let $0 < q \leqslant +\infty$. Suppose that μ and ν are non-negative Borel measures on $I = (a,b)$. Let $u \in B^+(I)$ be such that the function $\|u\|_{q,(a,t],\nu} < +\infty$, $t \in I$. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of $\varphi(t) = \|u\|_{q,(a,t+],\nu} := \lim_{s\to t+} \|u\|_{q,(a,s],\nu}$, $t \in I$, then*

$$\Big\| u(x) \|g\|_{\infty,(x,b),\mu} \Big\|_{q,I,\nu} \approx \Big\| \|g\|_{\infty,J_k,\mu} \|u\|_{q,(a,x_k+],\nu} \Big\|_{\ell^q(N,M)} \tag{2.6}$$

for all $g \in B^+(I)$, where $\{J_k\}_{k=N}^{M}$ is defined by (2.5).

REMARK 2.10. Lemma 2.6 (iii), implies that $\{\|u\|_{q,(a,x_k+],\nu}\}_{k=N}^{M}$ in Theorem 2.9 is an almost geometrically increasing sequence. (We can take $\alpha = L = 2$ in Definition 2.2).

REMARK 2.11. Let $q < +\infty$. Then

$$\|u\|_{q,(a,x+],\nu} = \|u\|_{q,(a,x],\nu} \qquad \text{for all} \qquad x \in I.$$

In this paper we shall need the Lebesgue-Stieltjes integral. To this end, we recall some basic facts.

Let φ be non-decreasing and finite function on the interval $I := (a,b) \subseteq \mathbb{R}$. We assign to φ the function λ defined on subintervals of I by

$$\lambda([\alpha,\beta]) = \varphi(\beta+) - \varphi(\alpha-), \tag{2.7}$$
$$\lambda([\alpha,\beta)) = \varphi(\beta-) - \varphi(\alpha-), \tag{2.8}$$
$$\lambda((\alpha,\beta]) = \varphi(\beta+) - \varphi(\alpha+), \tag{2.9}$$
$$\lambda((\alpha,\beta)) = \varphi(\beta-) - \varphi(\alpha+). \tag{2.10}$$

The function λ is a non-negative, additive and regular function of intervals. Thus (cf. [8], Chapter 10), it admits a unique extension to a non-negative Borel measure λ on I. The Lebesgue-Stieltjes integral $\int_I f\,d\varphi$ is defined as $\int_I f\,d\lambda$.

If $J \subseteq I$, then the Lebesgue-Stieltjes integral $\int_J f\,d\varphi$ is defined as $\int_J f\,d\lambda$. We shall also use the Lebesgue-Stieltjes integral $\int_J f\,d\varphi$ when φ is a non-increasing and finite on the interval I. In such a case we put

$$\int_J f\,d\varphi := -\int_J f\,d(-\varphi).$$

If φ is a non-decreasing, finite and right-continuous function on $I = (a,b)$ and J is a subinterval of I of the form (α,β), $[\alpha,\beta)$ or $(\alpha,\beta]$, then the formulae (2.10), (2.8) and (2.9) imply that

$$\int_{(\alpha,\beta)} d\varphi = \varphi(\beta-) - \varphi(\alpha), \tag{2.11}$$

$$\int_{[\alpha,\beta)} d\varphi = \varphi(\beta-) - \varphi(\alpha-), \tag{2.12}$$

$$\int_{(\alpha,\beta]} d\varphi = \varphi(\beta) - \varphi(\alpha). \tag{2.13}$$

In this paper the role of the function φ will be played by a function h which will be *non-decreasing* and *right-continuous* or *non-increasing* and *left-continuous* on I. At the first case, the associated Borel measure λ will be determined by (cf. (2.9))

$$\lambda((\alpha,\beta]) = h(\beta) - h(\alpha) \qquad \text{for any} \qquad (\alpha,\beta] \subset I \tag{2.14}$$

(since the Borel subsets of I can be generated by subintervals $(\alpha,\beta] \subset I$).

Considering inequalities (1.5) and (1.6), in the case when $0 < p < q \leqslant +\infty$ and $1/r = 1/p - 1/q$, we shall write conditions characterizing the validity of inequalities in a compact form involving $\int_{(a,b)} f\,dh$. To this end, we adopt the following conventions from [1].

CONVENTION 2.12. Let $I = (a,b) \subseteq \mathbb{R}$, $f : I \to [0,\infty]$ and $h : I \to [-\infty, 0]$. Assume that h is non-decreasing and right-continuous on I. If $h : I \to (-\infty, 0]$, then the symbol $\int_I f\,dh$ means the usual Lebesgue-Stieltjes integral. However, if $h = -\infty$ on some subinterval (a,c) with $c \in I$, then we define $\int_I f\,dh$ only if $f = 0$ on $(a,c]$ and we put

$$\int_I f\,dh = \int_{(c,b)} f\,dh.$$

CONVENTION 2.13. Let $I = (a,b) \subseteq \mathbb{R}$, $f : I \to [0,\infty]$ and $h : I \to [0,\infty]$. Assume that h is non-decreasing and left-continuous on I. If $h : I \to [0,\infty)$, then the symbol $\int_I f\,dh$ means the usual Lebesgue-Stieltjes integral. However, if $h = +\infty$ on some subinterval (c,b) with $c \in I$, then we define $\int_I f\,dh$ only if $f = 0$ on $[c,b)$ and we put

$$\int_I f\,dh = \int_{(a,c)} f\,dh.$$

3. Reverse Hardy-type inequalities for supremal operators

In this section we characterize inequality (1.5). We start with the following discretization lemma.

LEMMA 3.1. *Assume that $0 < p,q \leqslant +\infty$. Let μ and ν be non-negative Borel measures on $I = (a,b) \subseteq \mathbb{R}$. Let $w \in B^+(I)$ and let $u \in B^+(I)$ satisfy $\|u\|_{q,(a,t],\nu} < \infty$ for all $t \in I$ and $u \neq 0$ a.e. on (a,b). If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of $\varphi(t) := \|u\|_{q,(a,t+],\nu}$, then inequality (1.5) holds for all $g \in B^+(I)$ if and only if*

$$A := \left\| \left\{ \|w\|_{p,J_k,\mu} \, \|u\|_{q,(a,x_k+],\nu}^{-1} \right\} \right\|_{\ell^\rho(N,M)} < \infty, \tag{3.1}$$

and

$$w = 0 \quad \mu - a.e. \text{ in } (a, x_N] \quad \text{if} \quad x_N > a, \tag{3.2}$$

where $1/\rho := (1/p - 1/q)_+$.

The best possible constant c *in* (1.5) *satisfies* $c \approx A$.

Proof. By Theorem 2.9,

$$\big\| u(x) \|g\|_{\infty,(x,b),\mu} \big\|_{q,I,\nu} \approx \big\| \big\{ \|g\|_{\infty,J_k,\mu} \|u\|_{q,(a,x_k+],\nu} \big\} \big\|_{\ell^q(N,M)}, \tag{3.3}$$

for all $g \in B^+(I)$, where $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function $\varphi(t) = \|u\|_{q,(a,t+],\nu}$, $t \in (a,b)$, and $\{J_k\}_{k=N}^{M}$ is defined by (2.5). By Lemma 2.6 (cf. also Remark 2.8),

$$\begin{aligned} &\text{if} \quad x_N > a, \quad \text{then} \quad \|u\|_{q,(a,x_N),\nu} = 0; \\ &\text{if} \quad M < +\infty, \quad \text{then} \quad x_{M+1} = b; \end{aligned} \tag{3.4}$$

$$\|u\|_{q,(a,x_{k+1}),\nu} \leqslant 2\|u\|_{q,(a,x_k+],\nu} \quad \text{if} \quad N \leqslant k \leqslant M; \tag{3.5}$$

$$2\|u\|_{q,(a,x_k),\nu} \leqslant \|u\|_{q,(a,x_{k+1}+],\nu} \quad \text{if} \quad N < k < M. \tag{3.6}$$

Sufficiency. Let (3.1) and (3.2) hold. Since

$$\|gw\|_{p,(a,b),\mu} = \big\| \big\{ \|gw\|_{p,J_k,\mu} \big\} \big\|_{\ell^p(N,M)}, \quad \text{for any} \quad g \in B^+(I), \tag{3.7}$$

and

$$\|gw\|_{p,J_k,\mu} \leqslant \|g\|_{\infty,J_k,\mu} \|w\|_{p,J_k,\mu}, \quad N \leqslant k \leqslant M, \tag{3.8}$$

on using (2.1) and (3.3), we have that

$$\begin{aligned} \|gw\|_{p,(a,b),\mu} &\leqslant \big\| \big\{ \|g\|_{\infty,J_k,\mu} \|w\|_{p,J_k,\mu} \big\} \big\|_{\ell^p(N,M)} \\ &\leqslant \Big\| \Big\{ \|w\|_{p,J_k,\mu} \|u\|_{q,(a,x_k+],\nu}^{-1} \Big\} \Big\|_{\ell^\rho(N,M)} \big\| \big\{ \|g\|_{\infty,J_k,\mu} \|u\|_{q,(a,x_k+],\nu} \big\} \big\|_{\ell^q(N,M)} \\ &\approx \Big\| \Big\{ \|w\|_{p,J_k,\mu} \|u\|_{q,(a,x_k+],\nu}^{-1} \Big\} \Big\|_{\ell^\rho(N,M)} \big\| u(x) \|g\|_{\infty,(x,b),\mu} \big\|_{q,I,\nu} \end{aligned}$$

Consequently, $c \lesssim A$.

Necessity. We now prove necessity. The validity of inequality (1.5) on $B^+(I)$ and (3.3) imply that

$$\big\| \big\{ \|gw\|_{p,J_k,\mu} \big\} \big\|_{\ell^p(N,M)} \lesssim c \, \big\| \big\{ \|g\|_{\infty,J_k,\mu} \|u\|_{q,(a,x_k+],\nu} \big\} \big\|_{\ell^q(N,M)} \tag{3.9}$$

for all $g \in B^+(I)$.

Let $g_k \in B^+(I)$, $N \leqslant k \leqslant M$, be functions such that

$$\operatorname{supp} g_k \subset J_k, \quad \|g_k\|_{\infty,J_k,\mu} = 1 \quad \text{and} \quad \|gw\|_{p,J_k,\mu} \gtrsim \|w\|_{p,J_k,\mu}. \tag{3.10}$$

Then we define the test function g by

$$g = \sum_{k=N}^{M} a_k g_k, \tag{3.11}$$

where $\{a_k\}$ is a sequence of non-negative numbers. Consequently, (3.9) yields

$$\left\| \{a_k \|w\|_{p,J_k,\mu}\} \right\|_{\ell^p(N,M)} \lesssim c \left\| \{a_k \|u\|_{q,(a,x_k+],\nu}\} \right\|_{\ell^q(N,M)}, \tag{3.12}$$

and, by Proposition 2.5, we arrive at

$$A = \left\| \left\{ \|w\|_{p,J_k,\mu} \, \|u\|_{q,(a,x_k+],\nu}^{-1} \right\} \right\|_{\ell^\rho(N,M)} \lesssim c. \tag{3.13}$$

On the other hand, assuming that $x_N > a$, testing (1.5) with $g = \chi_{(a,x_N]}$ and using (3.4), we arrive at $\|w\|_{p,(a,x_N],\mu} = 0$, which implies (3.2). $\square$

The following lemma is true.

LEMMA 3.2. *Assume that $0 < q \leqslant p \leqslant +\infty$. Let μ and ν be non-negative Borel measures on $I = (a,b) \subseteq \mathbb{R}$. Let $w \in B^+(I)$ and let $u \in B^+(I)$ satisfy $\|u\|_{q,(a,t],\nu} < \infty$ for all $t \in I$ and $u \neq 0$ a.e. on (a,b). If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of $\varphi(t) = \|u\|_{q,(a,t+],\nu}$, $t \in I$, then*

$$A = \left\| \left\{ \|w\|_{p,J_k,\mu} \, \|u\|_{q,(a,x_k+],\nu}^{-1} \right\} \right\|_{\ell^\infty(N,M)} < \infty, \tag{3.14}$$

and (3.2) hold if and only if

$$A_1 := \left\| \|w\|_{p,(a,x],\mu} \|u\|_{q,(a,x),\nu}^{-1} \right\|_{\infty,(a,b),\mu} < \infty. \tag{3.15}$$

Moreover, $A \approx A_1$.

Proof. Sufficiency. Assume that $A_1 < \infty$. This condition and (3.4) imply that

$$\|w\|_{p,(a,x_N],\mu} = 0 \quad \text{if} \quad x_N > a. \tag{3.16}$$

Consequently, (3.2) holds.

Applying (3.5), we get that

$$\begin{aligned} A = \sup_{N \leqslant k \leqslant M} \|w\|_{p,J_k,\mu} \|u\|_{q,(a,x_k+],\nu}^{-1} &\leqslant 2 \sup_{N \leqslant k \leqslant M} \|w\|_{p,J_k,\mu} \|u\|_{q,(a,x_{k+1}),\nu}^{-1} \\ &\leqslant 2 \sup_{N \leqslant k \leqslant M} \|w\|_{p,(a,x_{k+1}] \cap I,\mu} \|u\|_{q,(a,x_{k+1}),\nu}^{-1} \\ &\leqslant 2 A_1. \end{aligned}$$

Necessity. Assume that (3.14) and (3.2) hold. Therefore, on using (2.5),

$$A_1 = \sup_{N \leqslant k \leqslant M} \left\| \|w\|_{p,(a,x],\mu} \|u\|_{q,(a,x),\nu}^{-1} \right\|_{\infty,J_k,\mu}$$

and hence

$$A_1 \leqslant \sup_{N\leqslant k\leqslant M} \|w\|_{p,(a,x_{k+1}]\cap I,\mu} \left\| \|u\|^{-1}_{q,(a,x),\nu} \right\|_{\infty,J_k,\mu}$$
$$\leqslant \sup_{N\leqslant k\leqslant M} \|w\|_{p,(a,x_{k+1}]\cap I,\mu} \|u\|^{-1}_{q,(a,x_k+],\nu}.$$

Applying (3.2) again, on using the fact that $\{\|u\|^{-1}_{q,(a,x_k+],\nu}\}_{k=N}^{M}$ is almost geometrically decreasing and Lemma 2.4, we obtain that

$$A_1 \lesssim \sup_{N\leqslant k\leqslant M} \|w\|_{p,J_k,\mu} \|u\|^{-1}_{q,(a,x_k+],\nu} = A. \quad \square$$

To prove our main statement we need the following lemma.

LEMMA 3.3. *Assume that* $0<p<q\leqslant+\infty$ *and* $1/r=1/p-1/q$. *Let* μ *and* ν *be non-negative Borel measures on* $I=(a,b)\subseteq\mathbb{R}$. *Let* $w\in B^+(I)$ *and let* $u\in B^+(I)$ *satisfy* $\|u\|_{q,(a,t],\nu}<\infty$ *for all* $t\in I$ *and* $u\neq 0$ *a.e. on* (a,b). *If* $\{x_k\}_{k=N}^{M+1}$ *is a discretizing sequence of* $\varphi(t)=\|u\|_{q,(a,t+],\nu}$, $t\in I$, *then*

$$A = \left\| \left\{ \|w\|_{p,J_k,\mu} \|u\|^{-1}_{q,(a,x_k+],\nu} \right\} \right\|_{\ell^r(N,M)} < \infty, \tag{3.17}$$

and (3.2) *hold if and only if*

$$A_2 := \left(\int_{(a,b)} \|w\|^r_{p,(a,x],\mu} d\left(-\|u\|^{-r}_{q,(a,x+],\nu} \right) \right)^{1/r} + \|w\|_{p,(a,b),\mu} \|u\|^{-1}_{q,(a,b),\nu} < \infty. \tag{3.18}$$

Moreover, $A\approx A_2$.

Proof. Let $\{x_k\}_{k=N}^{M+1}$ be a discretizing sequence of the function $\varphi(t)=\|u\|_{q,(a,t+],\nu}$, $t\in(a,b)$, and $\{J_k\}_{k=N}^{M}$ is defined by (2.5). By Lemma 2.6 (cf. also Remark 2.8), (3.4)-(3.6) hold.

Sufficiency. Assume that $A_2<\infty$. This condition, (3.4) and Convention 2.12 imply that (3.2) holds. By (3.6),

$$2\|u\|_{q,(a,x_{k+1}),\nu} \leqslant \|u\|_{q,(a,x_{k+2}+],\nu} \leqslant \|u\|_{q,(a,x_{k+3}),\nu} \qquad \text{if} \qquad N<k+1<M.$$

Therefore,

$$\|u\|^{-r}_{q,(a,x_{k+3}),\nu} \leqslant 2^{-r} \|u\|^{-r}_{q,(a,x_{k+1}),\nu},$$

which yields

$$\|u\|^{-r}_{q,(a,x_{k+1}),\nu} - \|u\|^{-r}_{q,(a,x_{k+3}),\nu} \geqslant (1-2^{-r})\|u\|^{-r}_{q,(a,x_{k+1}),\nu} \qquad \text{if} \qquad N\leqslant k\leqslant M-2.$$

Assume that $N\leqslant M-2$. On using (3.5) and the last estimate, we arrive at

$$A^r \lesssim \sum_{k=N}^{M} \|w\|^r_{p,J_k,\mu} \|u\|^{-r}_{q,(a,x_{k+1}),\nu}$$
$$\lesssim \sum_{k=N}^{M-2} \|w\|^r_{p,J_k,\mu} \left(\|u\|^{-r}_{q,(a,x_{k+1}),\nu} - \|u\|^{-r}_{q,(a,x_{k+3}),\nu} \right)$$

$$+\|w\|^r_{p,J_{M-1},\mu}\left(\|u\|^{-r}_{q,(a,x_M),\nu}-\|u\|^{-r}_{q,(a,b),\nu}\right)$$
$$+\|w\|^r_{p,J_{M-1},\mu}\|u\|^{-r}_{q,(a,b),\nu}+\|w\|^r_{p,J_M,\mu}\|u\|^{-r}_{q,(a,b),\nu}. \tag{3.19}$$

Now, by (2.12) with $\varphi(t)=-\|u\|^{-r}_{q,(a,t+],\nu}$, $t\in I$, and $[\alpha,\beta)=[x_{k+1},x_{k+3})$, $N\leqslant k\leqslant M-2$, or $[\alpha,\beta)=[x_M,b)$, we obtain that

$$\begin{aligned}
A^r &\lesssim \sum_{k=N}^{M-2}\|w\|^r_{p,J_k,\mu}\int_{[x_{k+1},x_{k+3})}d\left(-\|u\|^{-r}_{q,(a,t+],\nu}\right)\\
&\quad+\|w\|^r_{p,J_{M-1},\mu}\int_{[x_M,b)}d\left(-\|u\|^{-r}_{q,(a,t+],\nu}\right)+2\|w\|^r_{p,(a,b),\mu}\|u\|^{-r}_{q,(a,b),\nu}\\
&\lesssim \sum_{k=N}^{M-2}\int_{[x_{k+1},x_{k+3})}\|w\|^r_{p,(a,t],\mu}\,d\left(-\|u\|^{-r}_{q,(a,t+],\nu}\right)\\
&\quad+\int_{[x_M,b)}\|w\|^r_{p,(a,t],\mu}\,d\left(-\|u\|^{-r}_{q,(a,t+],\nu}\right)+2\|w\|^r_{p,(a,b),\mu}\|u\|^{-r}_{q,(a,b),\nu}\\
&\leqslant \int_{(a,b)}\|w\|^r_{p,(a,t],\mu}\,d\left(-\|u\|^{-r}_{q,(a,t+],\nu}\right)+2\|w\|^r_{p,(a,b),\mu}\|u\|^{-r}_{q,(a,b),\nu}\\
&\lesssim A_2^r
\end{aligned}$$

(note that we have used (3.2) and Convention 2.12), that is,

$$A\lesssim A_2. \tag{3.20}$$

If $N>M-2$, then (3.20) can be proved more simply and we omit the proof.

Necessity. Now assume that $A<\infty$ and (3.2) holds. On using (3.2), together with (2.13) and (2.11), we have that

$$\begin{aligned}
A_2^r &\approx \sum_{k=N}^{M}\int_{J_k}\|w\|^r_{p,(a,x],\mu}d\left(-\|u\|^{-r}_{q,(a,x+],\nu}\right)+\|w\|^r_{p,(a,b),\mu}\|u\|^{-r}_{q,(a,b),\nu}\\
&\leqslant \sum_{k=N}^{M-1}\|w\|^r_{p,(a,x_{k+1}],\mu}\int_{J_k}d\left(-\|u\|^{-r}_{q,(a,x+],\nu}\right)\\
&\quad+\|w\|^r_{p,(a,b),\mu}\int_{(x_M,b)}d\left(-\|u\|^{-r}_{q,(a,x+],\nu}\right)+\|w\|^r_{p,(a,b),\mu}\|u\|^{-r}_{q,(a,b),\nu}\\
&=\sum_{k=N}^{M-1}\|w\|^r_{p,(a,x_{k+1}],\mu}\left(\|u\|^{-r}_{q,(a,x_k+],\nu}-\|u\|^{-r}_{q,(a,x_{k+1}+],\nu}\right)\\
&\quad+\|w\|^r_{p,(a,b),\mu}\left(\|u\|^{-r}_{q,(a,x_M+],\nu}-\|u\|^{-r}_{q,(a,b),\nu}\right)+\|w\|^r_{p,(a,b),\mu}\|u\|^{-r}_{q,(a,b),\nu}\\
&\lesssim \sum_{k=N}^{M-1}\|w\|^r_{p,(a,x_{k+1}],\mu}\|u\|^{-r}_{q,(a,x_k+],\nu}+\|w\|^r_{p,(a,b),\mu}\|u\|^{-r}_{q,(a,x_M+],\nu}.
\end{aligned} \tag{3.21}$$

Thus, using (3.2) again, we arrive at

$$A_2^r \lesssim \sum_{k=N}^{M} \|w\|_{p,(a,x_{k+1}]\cap I,\mu}^r \|u\|_{q,(a,x_k+],\nu}^{-r} = \sum_{k=N}^{M} \left(\sum_{i=N}^{k} \|w\|_{p,J_i,\mu}^r \right) \|u\|_{q,(a,x_k+],\nu}^{-r}$$

Now, the fact that $\{\|u\|_{q,(a,x_k+],\nu}^{-r}\}_{k=N}^{M}$ is almost geometrically decreasing and Lemma 2.4 imply that

$$A_2 \lesssim \left(\sum_{k=N}^{M} \|w\|_{p,J_k,\mu}^r \|u\|_{q,(a,x_k+],\nu}^{-r} \right)^{1/r} = A. \tag{3.22}$$

Combining (3.20) and (3.22), we get $A \approx A_2$. □

Now we are in position to prove our first main result.

THEOREM 3.4. *Assume that* $0 < p, q \leqslant +\infty$. *Let* μ *and* ν *be non-negative Borel measures on* $I = (a,b) \subseteq \mathbb{R}$. *Let* $w \in B^+(I)$ *and let* $u \in B^+(I)$ *satisfy* $\|u\|_{q,(a,t],\nu} < \infty$ *for all* $t \in I$ *and* $u \neq 0$ *a.e. on* (a,b).

(i) *Let* $0 < q \leqslant p \leqslant +\infty$. *Then inequality* (1.5) *holds for all* $g \in B^+(I)$ *if and only if*

$$A_1 = \left\| \|w\|_{p,(a,x],\mu} \|u\|_{q,(a,x),\nu}^{-1} \right\|_{\infty,(a,b),\mu} < \infty. \tag{3.23}$$

The best possible constant c *in* (1.5) *satisfies* $c \approx A_1$.

(ii) *Let* $0 < p < q < +\infty$ *and* $1/r = 1/p - 1/q$. *Then inequality* (1.5) *holds for all* $g \in B^+(I)$ *if and only if*

$$A_2 = \left(\int_{(a,b)} \|w\|_{p,(a,x],\mu}^r d\left(-\|u\|_{q,(a,x],\nu}^{-r} \right) \right)^{1/r} + \|w\|_{p,(a,b),\mu} \|u\|_{q,(a,b),\nu}^{-1} < \infty. \tag{3.24}$$

The best possible constant c *in* (1.5) *satisfies* $c \approx A_2$.

(iii) *Let* $0 < p < +\infty$, $q = +\infty$. *Then inequality* (1.5) *holds for all* $g \in B^+(I)$ *if and only if*

$$A_3 = \left(\int_{(a,b)} \left(\frac{w(x)}{\|u\|_{\infty,(a,x),\nu}} \right)^p d\mu(x) \right)^{1/p} \tag{3.25}$$
$$\approx \left(\int_{(a,b)} \|w\|_{p,(a,x],\mu}^p d\left(-\|u\|_{\infty,(a,x+],\nu}^{-p} \right) \right)^{1/p} + \|w\|_{p,(a,b),\mu} \|u\|_{\infty,(a,b),\nu}^{-1} < \infty.$$

The best possible constant c *in* (1.5) *satisfies* $c \approx A_3$.

Proof. (i) Let $0 < q \leqslant p \leqslant +\infty$. The statement follows by Lemmas 3.1 and 3.2.

(ii) Let $0 < p < q < +\infty$. The statement follows by Lemmas 3.1 and 3.3.

(iii) Let $0 < p < q = +\infty$. The statement follows by Lemmas 3.1, 3.3 and an integration by parts formula. □

REMARK 3.5. Let $q < +\infty$. Since

$$\|u\|_{q,(a,x+],v} = \|u\|_{q,(a,x],v} \qquad \text{for all} \qquad x \in I,$$

the cases (ii) and (iii) can be combined:

(ii)′ Let $0 < p < q \leqslant +\infty$ and $1/r = 1/p - 1/q$. Then inequality (1.5) holds for all $g \in B^+(I)$ if and only if

$$A_2' := \left(\int_{(a,b)} \|w\|_{p,(a,x],\mu}^r d\left(-\|u\|_{q,(a,x+],v}^{-r} \right) \right)^{1/r} + \|w\|_{p,(a,b),\mu} \|u\|_{q,(a,b),v}^{-1} < \infty.$$

The best possible constant c in (1.5) satisfies $c \approx A_2'$.

REMARK 3.6. Note that inequality (1.5) can be easily characterized by a more simply argumentations when $q = +\infty$. Exchanging essential suprema, we have that

$$\|u(x)\|g\|_{\infty,(x,b),\mu}\|_{\infty,(a,b),v} = \|\|u\|_{\infty,(a,x),v} g(x)\|_{\infty,(a,b),\mu}, \qquad g \in B^+(I). \tag{3.26}$$

Consequently, (1.5) is nothing else the description of the embeddings of weighted $L^\infty(\mu)$ to weighted $L^p(v)$ (see, for instance, [2, Proposition 6.13]). Indeed: on using (3.26), for the best constant c in (1.5) we have that

$$\begin{aligned} c &= \sup_{g \not\sim 0} \frac{\|gw\|_{p,(a,b),\mu}}{\|u(x)\|g\|_{\infty,(x,b),\mu}\|_{\infty,(a,b),v}} \\ &= \sup_{g \not\sim 0} \frac{\|gw\|_{p,(a,b),\mu}}{\|\|u\|_{\infty,(a,x),v} g(x)\|_{\infty,(a,b),\mu}} \\ &= \left(\int_{(a,b)} \left(\frac{w(x)}{\|u\|_{\infty,(a,x),v}} \right)^p d\mu(x) \right)^{1/p}, \end{aligned}$$

when $0 < p < +\infty$, and

$$\begin{aligned} c &= \left\| w(x) \|u\|_{\infty,(a,x),v}^{-1} \right\|_{\infty,(a,b),\mu} \\ &= \left\| \|w\|_{\infty,(a,x],\mu} \|u\|_{\infty,(a,x),v}^{-1} \right\|_{\infty,(a,b),\mu}, \end{aligned}$$

when $p = +\infty$. In the last equality, exchanging essential suprema, we have used that

$$\begin{aligned} \left\| w(x) \|u\|_{\infty,(a,x),v}^{-1} \right\|_{\infty,(a,b),\mu} &= \left\| w(x) \left\| \|u\|_{\infty,(a,t),v}^{-1} \right\|_{\infty,[x,b),\mu} \right\|_{\infty,(a,b),\mu} \\ &= \left\| \left\| w(x) \|u\|_{\infty,(a,t),v}^{-1} \right\|_{\infty,[x,b),\mu} \right\|_{\infty,(a,b),\mu} \\ &= \left\| \left\| w(x) \chi_{[x,b)}(t) \|u\|_{\infty,(a,t),v}^{-1} \right\|_{\infty,(a,b),\mu} \right\|_{\infty,(a,b),\mu} \end{aligned}$$

$$= \left\| \left\| w(x)\chi_{[x,b)}(t) \right\|_{\infty,(a,b),\mu} \|u\|^{-1}_{\infty,(a,t),\nu} \right\|_{\infty,(a,b),\mu}$$
$$= \left\| \|w\|_{\infty,(a,t],\mu} \|u\|^{-1}_{\infty,(a,t),\nu} \right\|_{\infty,(a,b),\mu}.$$

(see, [1, Remark 4.2]).

4. Reverse inequality for the dual operator

The aim of this section is to characterize the inequality (1.6).

THEOREM 4.1. *Assume that* $0 < p, q \leqslant +\infty$. *Let* μ *and* ν *be non-negative Borel measures on* $I = (a,b) \subseteq \mathbb{R}$. *Let* $w \in B^+(I)$ *and let* $u \in B^+(I)$ *satisfy* $\|u\|_{q,[t,b),\nu} < \infty$ *for all* $t \in I$ *and* $u \neq 0$ *a.e. on* (a,b).

(i) *Let* $0 < q \leqslant p \leqslant +\infty$. *Then inequality* (1.6) *holds for all* $g \in B^+(I)$ *if and only if*

$$B_1 := \left\| \|w\|_{p,[x,b),\mu} \|u\|^{-1}_{q,(x,b),\nu} \right\|_{\infty,(a,b),\mu} < \infty. \tag{4.1}$$

The best possible constant c *in* (1.6) *satisfies* $c \approx B_1$.

(ii) *Let* $0 < p < q < +\infty$ *and* $1/r = 1/p - 1/q$. *Then inequality* (1.6) *holds for all* $g \in B^+(I)$ *if and only if*

$$B_2 := \left(\int_{(a,b)} \|w\|^r_{p,[x,b),\mu} d\left(\|u\|^{-r}_{q,[x,b),\nu} \right) \right)^{1/r} + \|w\|_{p,(a,b),\mu} \|u\|^{-1}_{q,(a,b),\nu} < \infty. \tag{4.2}$$

The best possible constant c *in* (1.6) *satisfies* $c \approx B_2$.

(iii) *Let* $0 < p < +\infty$, $q = +\infty$. *Then inequality* (1.6) *holds for all* $g \in B^+(I)$ *if and only if*

$$B_3 := \left(\int_{(a,b)} \left(\frac{w(x)}{\|u\|_{\infty,(x,b),\nu}} \right)^p d\mu(x) \right)^{1/p} \tag{4.3}$$
$$\approx \left(\int_{(a,b)} \|w\|^p_{p,[x,b),\mu} d\left(\|u\|^{-p}_{\infty,[x-,b),\nu} \right) \right)^{1/p} + \|w\|_{p,(a,b),\mu} \|u\|^{-1}_{\infty,(a,b),\nu} < \infty,$$

where

$$\|u\|_{\infty,[x-,b),\nu} := \lim_{s \to t-} \|u\|_{\infty,[s,b),\nu}.$$

The best possible constant c *in* (1.6) *satisfies* $c \approx B_3$.

REMARK 4.2. Note that the proof of Theorem 4.1 is similar to the proof of Theorems 5.1 and 5.4 from [1]. For the sake of completeness we give complete proof here.

Proof. If λ is a non-negative Borel measure on I, we denote by $\tilde{\lambda}$ a non-negative Borel measure on $\tilde{I} := (-b,-a)$ defined by

$$\tilde{\lambda} := \lambda(-E), \qquad \text{where} \qquad -E := \{-x : x \in E\}.$$

Similarly, if $h \in B^+(I)$, then the function $\tilde{h} \in B^+(\tilde{I})$ is given by

$$\tilde{h} := h(-x), \qquad x \in \tilde{I}.$$

It is clear that

$$\int_E h\,d\lambda = \int_{-E} \tilde{h}\,d\tilde{\lambda}, \tag{4.4}$$

and

$$\|h\|_{\infty,E,\lambda} = \|\tilde{h}\|_{\infty,-E,\tilde{\lambda}}, \tag{4.5}$$

for any Borel subset E of I. In particular,

$$\begin{aligned} \|gw\|_{p,(a,b),\mu} &= \|\tilde{g}\tilde{w}\|_{p,(-b,-a),\tilde{\mu}}, \\ \big\|u(x)\|g\|_{\infty,(a,x),\mu}\big\|_{q,(a,b),\nu} &= \big\|\tilde{u}(x)\|g\|_{\infty,(a,-x),\mu}\big\|_{q,(-b,-a),\tilde{\nu}}, \\ &= \big\|\tilde{u}(x)\|\tilde{g}\|_{\infty,(x,-a),\tilde{\mu}}\big\|_{q,(-b,-a),\tilde{\nu}}. \end{aligned}$$

Consequently, inequality (1.6) holds for all $g \in B^+(I)$ if and only if the inequality

$$\|\tilde{g}\tilde{w}\|_{p,(-b,-a),\tilde{\mu}} \leqslant c\,\big\|\tilde{u}(x)\|\tilde{g}\|_{\infty,(x,-a),\tilde{\mu}}\big\|_{q,(-b,-a),\tilde{\nu}} \tag{4.6}$$

holds for all $\tilde{g} \in B^+(\tilde{I})$.

(i) Let $0 < q \leqslant p \leqslant +\infty$. Since $\|\tilde{u}\|_{q,(-b,x],\tilde{\nu}} = \|u\|_{q,[-x,b),\nu}$ if $x \in (-b,-a)$, we deduce from Theorem 3.4 that inequality (1.6) holds on $B^+(I)$ if and only if

$$\sup_{x \in (-b,-a)} \|\tilde{w}\|_{p,(-b,x],\tilde{\mu}}\, \|\tilde{u}\|^{-1}_{q,(-b,x],\tilde{\nu}} < \infty. \tag{4.7}$$

However, using (4.4) and (4.5), we see that condition (4.7) coincides with (4.1).

(ii) Let $0 < p < q < +\infty$ and $1/r = 1/p - 1/q$. By Theorem 3.4, inequality (1.6) holds for all $g \in B^+(I)$ if and only if

$$\begin{aligned} A_2 := &\left(\int_{(-b,-a)} \|\tilde{w}\|^r_{p,(-b,x],\tilde{\mu}}\, d\left(-\|\tilde{u}\|^{-r}_{q,(-b,x],\tilde{\nu}}\right)\right)^{1/r} \\ &+ \|\tilde{w}\|_{p,(-b,-a),\tilde{\mu}} \|\tilde{u}\|^{-1}_{q,(-b,-a),\tilde{\nu}} < \infty. \end{aligned} \tag{4.8}$$

It is clear that

$$\|\tilde{w}\|_{p,(-b,-a),\tilde{\mu}} = \|w\|_{p,(a,b),\mu} \qquad \text{and} \qquad \|\tilde{u}\|_{q,(-b,-a),\tilde{\nu}} = \|u\|_{q,(a,b),\nu}. \tag{4.9}$$

Moreover, by the definition of the Lebesgue-Stieltjes integral,

$$\int_{(-b,-a)} \|\tilde{w}\|^r_{p,(-b,x],\tilde{\mu}} d\left(-\|\tilde{u}\|^{-r}_{q,(-b,x],\tilde{\nu}}\right) = \int_{(-b,-a)} \|\tilde{w}\|^r_{p,(-b,x],\tilde{\mu}} d\tilde{\lambda} =: D, \tag{4.10}$$

where $\tilde{\lambda}$ is the non-negative Borel measure associated to the non-decreasing and right-continuous function $\tilde{\varphi}(x) := -\|\tilde{u}\|^{-r}_{q,(-b,x],\tilde{\nu}}$, $x \in (-b,-a)$, that is,

$$\tilde{\lambda}((\tilde{\alpha},\tilde{\beta}]) = \tilde{\varphi}(\tilde{\beta}) - \tilde{\varphi}(\tilde{\alpha}) \qquad \text{for any} \qquad (\tilde{\alpha},\tilde{\beta}] \subset (-b,-a).$$

Since, by (4.4),

$$\|\tilde{w}\|^r_{p,(-b,x],\tilde{\mu}} = \|w\|^r_{p,[x,b),\mu} \qquad \text{for all} \qquad t \in (-b,-a),$$

we obtain from (4.4) that

$$D = \int_{(-b,-a)} \|\tilde{w}\|^r_{p,(-b,x],\tilde{\mu}} d\tilde{\lambda} = \int_{(a,b)} \|w\|^r_{p,[x,b),\mu}\, d\lambda, \tag{4.11}$$

where $\lambda(E) = \tilde{\lambda}(-E)$ if E is a Borel subset of I. In particular, if $[\alpha,\beta) \subset (a,b)$, then

$$\begin{aligned}\lambda([\alpha,\beta)) = \tilde{\lambda}((-\beta,-\alpha]) &= \tilde{\varphi}(-\alpha) - \tilde{\varphi}(-\beta)\\ &= -\|\tilde{u}\|^{-r}_{q,(-b,-\alpha],\tilde{\nu}} + \|\tilde{u}\|^{-r}_{q,(-b,-\beta],\tilde{\nu}}\\ &= -\|u\|^{-r}_{q,[\alpha,b),\nu} + \|u\|^{-r}_{q,[\beta,b),\nu}.\end{aligned}$$

That means that the non-negative Borel measure λ is associated to the non-decreasing and left-continuous function φ given on (a,b) by

$$\varphi(x) := \|u\|^{-r}_{q,[x,b),\nu}, \qquad x \in (a,b).$$

Consequently,

$$\int_{(a,b)} \|w\|^r_{p,[x,b),\mu}\, d\lambda = \int_{(a,b)} \|w\|^r_{p,[x,b),\mu}\, d\left(\|u\|^{-r}_{q,[x,b),\nu}\right). \tag{4.12}$$

The result now follows from (4.8)-(4.12).

(iii) Let $0 < p < +\infty$, $q = +\infty$. By Theorem 3.4, inequality (1.6) holds for all $g \in B^+(I)$ if and only if

$$\int_{(-b,-a)} \left(\frac{\tilde{w}(x)}{\|\tilde{u}\|_{\infty,(-b,x),\tilde{\nu}}}\right)^p d\tilde{\mu}(x) < \infty. \tag{4.13}$$

By (4.5) and (4.4), we have that $\|\tilde{u}\|_{\infty,(-b,x),\tilde{\nu}} = \|u\|_{\infty,(-x,b),\nu}$

$$\begin{aligned}\int_{(-b,-a)} \left(\frac{\tilde{w}(x)}{\|\tilde{u}\|_{\infty,(-b,x),\tilde{\nu}}}\right)^p d\tilde{\mu}(x) &= \int_{(-b,-a)} \left(\frac{\tilde{w}(x)}{\|u\|_{\infty,(-x,b),\nu}}\right)^p d\tilde{\mu}(x)\\ &= \int_{(a,b)} \left(\frac{w(x)}{\|u\|_{\infty,(x,b),\nu}}\right)^p d\mu(x),\end{aligned}$$

and we see that (4.13) coincides with (4.3).

The equivalency in (4.3) can be shown by the same argumentations as in the case (i) and (ii) or by an integration by parts formula. □

REMARK 4.3. Let $q < +\infty$. Then

$$\|u\|_{q,[x-,b),\nu} := \lim_{s\to t-} \|u\|_{q,[s,b),\nu} = \|u\|_{q,[x,b),\nu} \qquad \text{for all} \qquad x \in I,$$

which allows us to combine the cases (ii) and (iii) of Theorem 4.1:

(ii)$'$ Let $0 < p < q \leqslant +\infty$ and $1/r = 1/p - 1/q$. Then inequality (1.6) holds for all $g \in B^+(I)$ if and only if

$$B_2' := \left(\int_{(a,b)} \|w\|^r_{p,[x,b),\mu} d\left(\|u\|^{-r}_{q,[x-,b),\nu} \right) \right)^{1/r} + \|w\|_{p,(a,b),\mu} \|u\|^{-1}_{q,(a,b),\nu} < \infty.$$

The best possible constant c in (1.6) satisfies $c \approx B_2'$.

5. Inequalities involving three measures

Now, we consider the inequality

$$\|g\|_{p,(a,b),\lambda} \leqslant c \|u(x)\|g\|_{\infty,S_x,\mu}\|_{q,(a,b),\nu}, \qquad g \in B^+(I), \tag{5.1}$$

for all non-negative Borel measurable functions g on the interval $(a,b) \subseteq \mathbb{R}$, where $0 < p \leqslant +\infty$, $0 < q \leqslant +\infty$, λ, μ and ν are non-negative Borel measures on (a,b), u is a weight function on I and either $S_x = (a,x)$ or $S_x = (x,b)$ for all $x \in I$.

By the same way as it has been done in [1], it is easy to show that in order to characterize the validity of (5.1) it is enough to characterize the validity of the inequality

$$\|gw\|_{p,(a,b),\mu} \leqslant c \|u(x)\|g\|_{\infty,S_x,\mu}\|_{q,(a,b),\nu}, \qquad g \in B^+(I). \tag{5.2}$$

THEOREM 5.1. *Assume that $0 < p, q \leqslant +\infty$. Let λ, μ and ν be non-negative Borel measures on $I = (a,b) \subseteq \mathbb{R}$ and let $u \in B^+(I)$. Then inequality (5.1) holds for all $g \in B^+(I)$ if and only if the measure λ is absolutely continuous with respect to μ and inequality (5.2) with $w := (d\lambda/d\mu)^{1/p}$ holds for all $g \in B^+(I)$.*

Proof. Assume that (5.1) holds for all $g \in B^+(I)$. Let $E \subseteq I$ be such that $\mu(E) = 0$ and put $g = \chi_E$. Then $\|g\|_{\infty,S_x,\mu} = 0$ for all $x \in I$. Therefore, the right-hand side of (5.1) is zero, which implies that $\lambda(E) = 0$, when $0 < p < +\infty$, for $\lambda(E) = \|g\|^p_{p,(a,b),\lambda} = 0$, and when $p = +\infty$, for $\|g\|_{\infty,(a,b),\lambda} = \|\chi_E\|_{\infty,(a,b),\lambda} = 0$. Hence, λ is absolutely continuous with respect to μ, and, by the Radon-Nykodym theorem, there is $v \in B^+(I)$ such that $d\lambda = v d\mu$. Putting $w = v^{1/p}$, we have that $d\lambda = w^p d\mu$. Consequently, for any $g \in B^+(I)$, we can rewrite the left-hand side of (5.1) as

$$\|g\|_{p,(a,b),\lambda} = \|gw\|_{p,(a,b),\mu},$$

and our claim follows. □

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