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A NEW ESTIMATE OF THE DIFFERENCE AMONG QUASI–ARITHMETIC MEANS

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Abstract. In the 1960s Cargo and Shisha proved some majorizations for the distance among quasi-arithmetic means (defined as $f^{-1}(\sum_{i=1}^{n} w_i f(a_i))$ for any continuous, strictly monotone function $f\colon I \to \mathbb{R}$, where I is an interval, and $(a_1,\dots,a_n)$ is a vector with entries in I, $(w_1,\dots,w_n)$ is a sequence of corresponding weights $w_i > 0$, $\sum w_i = 1$).

Nearly thirty years later, in 1991, Páles presented an iff condition for a sequence of quasi-arithmetic means to converge to another QA mean. It was closely related with the three-parameter operator $(f(x)-f(y))/(f(x)-f(z))$.

The author presented recently an estimate for the distance among such quasi-arithmetic means whose underlying functions satisfy some smoothness conditions. Used was the operator $f \mapsto f''/f'$ introduced in the 1940s by Mikusiński and Łojasiewicz. It is natural to look for similar estimate(s) in the case of the underlying functions *not* being smooth. For instance, by the way of using Páles' operator. This is done in the present note. Moreover, the result strengthens author's earlier estimates.

1. Introduction

One of the most popular families of means encountered in literature is the family of *quasi-arithmetic means*. Such a mean is defined for any continuous, strictly monotone function $f\colon U \to \mathbb{R}$, U – an interval. When $a = (a_1,\dots,a_n)$ is an arbitrary sequence of points in U and $w = (w_1,\dots,w_n)$ is a sequence of corresponding *weights* ($w_i > 0$, $\sum w_i = 1$) then the mean $\mathfrak{M}_f(a,w)$ is defined by the equality

$$\mathfrak{M}_f(a,w) := f^{-1}\left(\sum_{i=1}^{n} w_i f(a_i)\right).$$

This family of means was introduced in the beginning of the 1930s in a series of nearly simultaneous papers [3, 4, 6] as a generalization of the well-known *power means*.

In the 1960s Cargo and Shisha [1, 2] introduced a metric among quasi-arithmetic means. Namely, if f and g are both continuous, strictly monotone and have the same domain, then one can define a distance

$$\rho(\mathfrak{M}_f,\mathfrak{M}_g) := \sup\{\left|\mathfrak{M}_f(a,w) - \mathfrak{M}_g(a,w)\right| : a \text{ and } w \text{ admissible}\}.$$

They also furnished some majorizations for $\rho(\mathfrak{M}_f,\mathfrak{M}_g)$. One of their results is the proposition below; hereafter $\|\cdot\|_p$ denotes the standard L^p norm ($1 \leqslant p \leqslant \infty$).

In the present note, if not otherwise stated, the intervals are arbitrary.

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PROPOSITION 1. ([2], Theorem 4.2) *Let U be an interval, $g \in \mathscr{C}(U)$ be strictly monotone, $f \in \mathscr{C}^1(U)$, $\inf|f'| > 0$. Then $\rho(\mathfrak{M}_f, \mathfrak{M}_g) \leqslant \frac{2\|f-g\|_\infty}{\inf|f'|}$.*

Departing from another observation in [2], the author proved in [10] an alternative estimate for the distance between two quasi-arithmetic means satisfying certain smoothness conditions. An important tool for that was the operator A, $A_f := f''/f'$ introduced by Mikusiński[1] in [5]. The relevant result was read

PROPOSITION 2. ([10], Theorem 3) *Let U be a closed, bounded interval and $f, g \in C^2(U)$ have nowhere vanishing first derivatives. Then*

$$\rho(\mathfrak{M}_f, \mathfrak{M}_g) \leqslant |U| \exp(2\left\|A_f\right\|_1) \sinh\left(2\left\|A_g - A_f\right\|_1\right).$$

REMARK 1. Note that the left hand side is symmetric with respect to f and g, while the right one is not. One could clearly symmetrize using the min function. Nevertheless, this operation will be omitted to keep the notation compact. The same remark applies to Theorem 2 as well.

The proposition above has an important for the present paper

COROLLARY 1. ([10], Corollary 3) *Let U be a closed, bounded interval and $f \in \mathscr{C}^2(U)$, $f' \neq 0$ everywhere. Moreover, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\mathscr{C}^2(U)$ with nowhere vanishing first derivatives, satisfying $A_{f_n} \to A_f$ in $L^1(U)$. Then $\mathfrak{M}_{f_n} \to \mathfrak{M}_f$ uniformly with respect to a and w.*

Note that the implication converse to that in Corollary 1 does not hold. This might already be observed in the following

EXAMPLE 1. Let $U = [0, 2\pi]$, $f_n(x) = x + n^{-2}\sin(nx)$, $n \geqslant 2$ and $f(x) = x$ for $x \in U$. Then, by Proposition 1, $\rho(\mathfrak{M}_f, \mathfrak{M}_{f_n}) \leqslant 2n^{-2}$. On the other hand, it can be straightforwardly proved that $\left\|A_{f_n} - A_f\right\|_1 = 2n\ln(n+1) \geqslant 4\ln 3$ for every $n \geqslant 2$.

This drawback is implied by the fact that "the first norm does not see cancellations of integrals". On the other hand, in [10] a couple of additional monotonicity assumptions was made from the very beginning. Namely, the mapping $n \mapsto A_{f_n}(x)$ was assumed to be either increasing for every x or else decreasing for every x. Hence there was no point *there* to care about the traps signalled an instant ago.

The situation in the present paper looks differently. As we will see, in order to handle examples like the one above, it is more convenient to use another norm, $\|\cdot\|_*$, which will be proposed in section 3.1.

Historically, some of results presented above (especially Corollary 1) correspond with an earlier result of Páles [9]. Namely, using the three-parameter operator

$$P_f(x,y,z) := \frac{f(x) - f(y)}{f(x) - f(z)}$$

[1] Mikusiński and, independently, Łojasiewicz, proved that comparability of quasi-arithmetic means might be easily expressed in terms of operator A. Besides, in the mathematical economy, the *negative* of this operator happened to be called the *Arrow-Pratt measure of risk aversion*.

defined on $\{(x,y,z) \in U^3 \colon x \neq z\} =: \Delta$, he proved the following

PROPOSITION 3. ([9], Corollary 1) *Let U be an interval, f and f_n, $n \in \mathbb{N}$, be continuous, strictly monotone functions defined on U.*
Then $\Big(\mathfrak{M}_{f_n} \to \mathfrak{M}_f$ *pointwise* $\Big) \iff \Big(P_{f_n} \to P_f$ *pointwise on* $\Delta \Big)$.

During the 15th International Conference on Functional Equations and Inequalities held in 2013, Páles himself asked about possible generalizations, of the $(\Leftarrow)$ part of his theorem, in the spirit of Proposition 2 and Corollary 1 (cf. [8, pp. 121–122]). In other words, he asked for a majorization of $\rho(\mathfrak{M}_f, \mathfrak{M}_g)$ in terms of P. On the other hand, to face the problem presented in Example 1, it is natural to look for possible strengthening of Proposition 2 and Corollary 1.

In this paper we are going to propose such an estimate which not only implies the $(\Leftarrow)$ part of Páles' result but also leads to a handy strengthening of Proposition 2; compare Corollary 2 and Theorem 2, respectively.

2. Main result

The main idea is to use the elementary fact that on compact sets the pointwise convergence of monotone functions coincides with the uniform one. However, Δ is not compact (even if U is). Therefore, finding suitable compact subsets of Δ seemed to be of utmost importance in the search for an estimate for the distance among means.

We observe that, when x approaches z, the operator P becomes unbounded. So it is natural to consider those points of Δ for which x and z are separated one from the other. For any $\alpha > 0$ define

$$\Delta_\alpha := \{(x,y,z) \in U^3 \colon |x-z| \geqslant \alpha\} \subset \Delta.$$

We are going to prove the following

THEOREM 1. *Let U be an interval, f and g be two continuous, strictly monotone functions defined on U, and $\alpha > 0$. Then $\|P_f - P_g\|_{\infty,\Delta_\alpha} < 1$ implies $\rho(\mathfrak{M}_f, \mathfrak{M}_g) < \alpha$.*

Before starting proof, it will be handy to recall some basic properties of the operator P. Namely, for any f,

$$P_f(x,y,z) + P_f(z,y,x) = 1 \qquad \text{for all } (x,y,z) \in \Delta, \tag{1}$$

$$\sum_i w_i P_f\Big(\mathfrak{M}_f(a,w), a_i, z\Big) = 0 \qquad \text{for all } a,\ w, \text{ and admissible } z. \tag{2}$$

Proof of Theorem 1. Fix any $a \in U^n$ with corresponding weights w and write shortly

$$F := \mathfrak{M}_f(a,w) \qquad \text{and} \qquad G := \mathfrak{M}_g(a,w).$$

It is sufficient to find such an $i \in \{1,\dots,n\}$ that $(F, a_i, G) \notin \Delta_\alpha$. Then, by the very definition of Δ_α, $|F-G| < \alpha$.

Suppose conversely that $(F,a_i,G)\in\Delta_\alpha$ for all $i\in\{1,\dots,n\}$. In particular,

$$\left|P_f(F,a_i,G)-P_g(F,a_i,G)\right|<1 \quad \text{for all } i\in\{1,\dots,n\}.$$

Hence, upon using (1) and (2), one obtains

$$\begin{aligned}-1&<\sum_{k=1}^{n} w_k\Big(P_f(F,a_k,G)-P_g(F,a_k,G)\Big)\\ &=\sum_{k=1}^{n} w_kP_f(F,a_k,G)+\sum_{k=1}^{n} w_k\Big(-1+P_g(G,a_k,F)\Big)\\ &=-1+\sum_{k=1}^{n} w_kP_g(G,a_k,F)=-1.\end{aligned}$$

This contradiction ends the proof. □

3. Applications

COROLLARY 2. *Let U be an interval, f and f_n, $n\in\mathbb{N}$, be strictly monotone functions defined on U, $P_{f_n}\to P_f$ pointwise in Δ. Then $\mathfrak{M}_{f_n}(a,w)\to\mathfrak{M}_f(a,w)$ for every fixed a and w.*

Moreover, if U is compact then $\mathfrak{M}_{f_n}\to\mathfrak{M}_f$ uniformly with respect to a and w in their respective ranges.

Proof. Fix: any $a\in U^n$ with corresponding weights w, a compact interval $K\subseteq U$ such that $a\in K^n$, and a positive constant α. We are going to prove that $P_{f_n}\to P_f$ uniformly in $\Delta_\alpha\cap K^3$ and then use Theorem 1.

Fix first $p,q\in K$, $p\neq q$. Then note that P and $\mathfrak{M}$ do not change under affine transformations of f and f_n, $n\in\mathbb{N}$. So (like it is usually done in dealing with quasi-arithmetic means) assume that $f(p)=f_n(p)=0$ and $f(q)=f_n(q)=1$.

One then has $f_n(\cdot)=P_{f_n}(p,\cdot,q)$ and $f(\cdot)=P_f(p,\cdot,q)$. So, by the assumption, $f_n\to f$ pointwise in K. But f and f_n, $n\in\mathbb{N}$ are continuous and strictly monotone. Hence one knows (cf., e.g., [7, 11]) that this convergence is uniform in K.

Then $f_n(x)-f_n(y)\to f(x)-f(y)$ uniformly in $\Delta_\alpha\cap K^3$, as functions of (x,y,z). Similarly $f_n(x)-f_n(z)\to f(x)-f(z)$ uniformly in the same set, as functions of (x,y,z).

Now, to prove that $P_{f_n}\to P_f$ uniformly in $\Delta_\alpha\cap K^3$, it is only needed to guarantee that $f(x)-f(z)$, as a function of (x,y,z), is bounded away from 0 in $\Delta_\alpha\cap K^3$. But it is a continuous, non-vanishing function defined on a compact set.

Therefore, there exists an integer n_α such that

$$\left\|P_{f_n}-P_f\right\|_{\infty,\Delta_\alpha\cap K^3}<1 \quad \text{for all } n>n_\alpha.$$

Hence, by Theorem 1, one obtains

$$\rho(\mathfrak{M}_{f_n}|_K,\mathfrak{M}_f|_K)<\alpha \quad \text{for all } n>n_\alpha,$$

where $\mathfrak{M}_g|_K$ stands for the mean defined for a function g and vectors taking values in the relevant Cartesian products of K. This is more than needed in corollary's first statement.

As for the 'moreover' statement, one just takes $K = U$ in the above. $\square$

COROLLARY 3. *Let U be a compact interval, f and f_n, $n \in \mathbb{N}$, be strictly monotone functions defined on U. Then the following conditions are equivalent:*

(i) $P_{f_n} \to P_f$ *pointwise in* Δ,

(ii) $\mathfrak{M}_{f_n} \to \mathfrak{M}_f$ *pointwise,*

(iii) $\mathfrak{M}_{f_n} \to \mathfrak{M}_f$ *uniformly with respect to a and w.*

Obviously (iii) $\Rightarrow$ (ii). Moreover, by Proposition 3, (i) $\iff$ (ii), while, by Corollary 2, (i) $\Rightarrow$ (iii).

COROLLARY 4. *If, in Theorem 1, the assumed inequality is not sharp, $\left\|P_f - P_g\right\|_{\infty,\Delta_\alpha} \leqslant 1$, then $\rho(\mathfrak{M}_f, \mathfrak{M}_g) \leqslant \alpha$.*

3.1. Strengthening of Proposition 2

Now we are going to propose some solution to the problem hinted at in Example 1. Recalling, that problem arose from the fact that the closeness of functions does not imply closeness of their derivatives. Therefore, Proposition 2 is completely useless in that example. Hence, in proposition's strengthening, instead of using the first norm, one needs to define some other norm omitting that drawback of the L^1 norm. Let U be an interval, $f\colon U \to \mathbb{R}$ be an arbitrary continuous function, and the 'oscillation' norm be defined by

$$\|f\|_* := \sup_{a,b \in U} \left| \int_a^b f(x)dx \right| .$$

We are going to prove that Proposition 2 might be strengthened to

THEOREM 2. *Let U be a closed, bounded interval and $f, g \in C^2(U)$ have nowhere vanishing first derivatives. Then*

$$\rho(\mathfrak{M}_f, \mathfrak{M}_g) \leqslant |U| \exp \left\|A_f\right\|_* \left(\exp \left\|A_f - A_g\right\|_* - 1 \right).$$

It might be proved that the right hand side of the above inequality can be majorized by the one appeared in Proposition 2. This theorem also has a corollary, which is a strengthening of Corollary 1, using $\|\cdot\|_*$ instead of L^1, but it will be worded nowhere in this paper. This time the drawback discussed in the beginning of the section does not appear; cf. Example 2 later on. Moreover, $\|\cdot\|_* \leqslant \|\cdot\|_1$, hence the above theorem holds if one replaces $\|\cdot\|_*$ by $\|\cdot\|_1$; Remark 1 is applicable here.

Proof of Theorem 2. Fix any $(x,y,z)\in\Delta$. We would like to majorize the value of $\left|P_f(x,y,z)-P_g(x,y,z)\right|$. Let us suppose, with no loss of generality, that

$$f(s)=\int_x^s \exp\left(\int_x^t A_f(u)du\right)dt,$$
$$g(s)=\int_x^s \exp\left(\int_x^t A_g(u)du\right)dt.$$

Then

$$\begin{aligned} f(y)-f(x) &= \int_x^y \exp\left(\int_x^t A_f(u)du\right)dt \\ &= \int_x^y \exp\left(\int_x^t A_f(u)-A_g(u)du\right)\exp\left(\int_x^t A_g(u)du\right)dt \\ &= \int_x^y \exp\left(\int_x^t A_f(u)-A_g(u)du\right)g'(t)dt. \end{aligned}$$

By the mean value theorem, there exists $\xi\in I$ such that

$$\begin{aligned} f(y)-f(x) &= \exp\left(\int_x^\xi A_f(u)-A_g(u)du\right)\int_x^y g'(t)dt \\ &= \exp\left(\int_x^\xi A_f(u)-A_g(u)du\right)(g(y)-g(x)). \end{aligned}$$

Similarly,

$$f(z)-f(x)=\exp\left(\int_x^\eta A_f(u)-A_g(u)du\right)(g(z)-g(x)) \text{ for some } \eta\in I.$$

Therefore,

$$P_f(x,y,z)=\exp\left(\int_\eta^\xi A_f(u)-A_g(u)du\right)P_g(x,y,z).$$

So

$$\exp\left(-\left\|A_f-A_g\right\|_*\right)\left|P_g(x,y,z)\right|\leqslant\left|P_f(x,y,z)\right|\leqslant\exp\left(\left\|A_f-A_g\right\|_*\right)\left|P_g(x,y,z)\right|.$$

But $\operatorname{sign}P_f(x,y,z)=\operatorname{sign}P_g(x,y,z)$ for any admissible x, y and z. Hence one obtains

$$\left|P_f(x,y,z)-P_g(x,y,z)\right|\leqslant\left|P_f(x,y,z)\right|(\exp\left\|A_f-A_g\right\|_*-1). \tag{3}$$

Now we are going to majorize the value of $\left|P_f(x,y,z)\right|$. But

$$\left|P_f(x,y,z)\right|=\left|\frac{f(x)-f(y)}{f(x)-f(z)}\right|=\left|\frac{x-y}{x-z}\right|\frac{f'(p)}{f'(q)} \text{ for some } p,q\in U.$$

Moreover $|x-y|\leqslant|U|$ and

$$\left|\ln\left(\frac{f'(p)}{f'(q)}\right)\right|=\left|\int_q^p A_f(x)dx\right|\leqslant\left\|A_f\right\|_*.$$

So

$$\left|P_f(x,y,z)\right| \leqslant \frac{|U|}{|x-z|} \exp \left\|A_f\right\|_*.$$

Hence, continuing the inequality (3),

$$\left|P_f(x,y,z) - P_g(x,y,z)\right| \leqslant \frac{|U|}{|x-z|} \exp \left\|A_f\right\|_* \left(\exp \left\|A_f - A_g\right\|_* - 1\right).$$

Therefore, Corollary 4 with proper α immediately gives

$$\rho(\mathfrak{M}_f, \mathfrak{M}_g) \leqslant |U| \exp \left\|A_f\right\|_* \left(\exp \left\|A_f - A_g\right\|_* - 1\right). \quad \square$$

EXAMPLE 2. Let us take U, f and f_n like in Example 1. Then $A_f \equiv 0$ so $\left\|A_f\right\|_* = 0$,

$$\left\|A_{f_n} - A_f\right\|_* = \sup_{a,b \in [0,2\pi]} \int_a^b \frac{-n \sin nx}{n + \cos nx} = \ln\left(\frac{n+1}{n-1}\right).$$

So, by Theorem 2, $\rho(\mathfrak{M}_f, \mathfrak{M}_{f_n}) \leqslant \frac{4\pi}{n-1}$. This estimate is still much worse then one could expect (it is $\mathscr{O}(n^{-1})$ against $\mathscr{O}(n^{-2})$ ascertained in Example 1) but it is better then the one implied by Proposition 2 ($\mathscr{O}(1)$).

REFERENCES

[1] G. T. CARGO AND O. SHISHA, *On comparable means*, Pacific J. Math. **14** (3), 1964, 1053–1058.

[2] G. T. CARGO AND O. SHISHA, *A metric space connected with generalized means*, J. Approx. Theory **2** (2), 1969, 207–222.

[3] B. DE FINETTI, *Sul concetto di media*, G. Ist. It. Att. **2** (3), 1931, 369–396.

[4] A. KOLMOGOROFF, *Sur la notion de la moyenne*, Rend. Accad. dei Lincei **12** (6), 1930, 388–391.

[5] J. G. MIKUSIŃSKI, *Sur les moyennes de la forme* $\psi^{-1}[\sum q\psi(x)]$, Studia Math. **10** (1), 1948, 90–96.

[6] M. NAGUMO, *Über eine Klasse der Mittelwerte*, Japan. J. Math. **7** (1), 1930, 71–79.

[7] M. NOSARZEWSKA, *On uniform convergence in some classes of functions*, Fund. Math. **39** (1), 1952, 38–52.

[8] ZS. PÁLES, *Report of Meeting 15th International Conference on Functional Equations and Inequalities*, Ann. Univ. Paedagog. Crac. Stud. Math. **XII** (1), 2013, 121–122.

[9] ZS. PÁLES, *On the convergence of Means*, J. Math. Anal. Appl. **156** (1), 1991, 52–60.

[10] P. PASTECZKA, *When is a Family of Generalized Means a Scale?*, Real Anal. Exchange **38** (1), 2013, 193–210.

[11] G. PÓLYA, *Wohlbekannt ist folgender Satz von Dini*, Arxiv der Mathematic und Phisik **28** (1), 1920, 174.

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