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PRODUCTS OF NONCOMMUTATIVE CALDERÓN–LOZANOVSKIĬ SPACES

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Abstract. Let $\mathscr{M}$ be a semifinite von Neumann algebra with a normal semifinite faithful trace τ. We show that the noncommutative Calderón-Lozanovskiĭ spaces $E_\varphi(\mathscr{M})$ can be written in the form $E_\varphi(\mathscr{M}) = E_{\varphi_2}(\mathscr{M}) \cdot E_{\varphi_1}(\mathscr{M})$, if at least one of the following conditions holds:

(i) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for all arguments,

(ii) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for large arguments and $\mathscr{M} \hookrightarrow E(\mathscr{M})$,

(iii) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for small arguments and $E(\mathscr{M}) \hookrightarrow \mathscr{M}$.

Here $E_{\varphi_2}(\mathscr{M}) \cdot E_{\varphi_1}(\mathscr{M})$ denote the product of the noncommutative Calderón-Lozanovskiĭ spaces $E_{\varphi_1}(\mathscr{M})$ and $E_{\varphi_2}(\mathscr{M})$.

1. Introduction

A function $\varphi : [0,\infty) \longrightarrow [0,\infty]$ is said to be a Young function if φ is convex, non-decreasing with $\varphi(0) = 0$. For any Young function φ and any symmetric function space $(E, \|\cdot\|_E)$, we define the Calderón-Lozanovskiĭ space E_φ by

$$E_\varphi = \{f \in L_0 : \varphi(\lambda|f|) \in E \quad \text{for some } \lambda > 0\}.$$

For every $f \in E_\varphi$ the following functional is finite

$$\|f\|_\varphi = \inf\left\{\lambda > 0 : \rho_\varphi\left(\frac{f}{\lambda}\right) \leqslant 1\right\},$$

where

$$\rho_\varphi(f) = \begin{cases} \|\varphi(|f|)\|_E, & \text{if } \varphi(|f|) \in E, \\ \infty, & \text{otherwise.} \end{cases} \tag{1.1}$$

If E is a symmetric function space with the Fatou property and φ is a Young function, then $(E_\varphi, \|\cdot\|_\varphi)$ is a symmetric function space and then E_φ is a special case of a general Calderón-Lozanovskiĭ construction $\Psi(E,F)$ (see [8, 14]), where E is a symmetric function space and $F = L^\infty$. If $E = L^1$, then E_φ is the classical Orlicz space L^φ equipped with the Luxemburg-Nakano norm. If E is a Lorentz space Λ_ω, then E_φ is the Orlicz-Lorentz space $\Lambda_{\varphi,\omega}$, equipped with the Luxemburg-Nakano norm. Recently, P. Kolwicz, K. Lesnik, L. Maligranda [10] proved that $E_\varphi = E_{\varphi_2} \cdot E_{\varphi_1}$, if at least one of the following conditions holds:

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(i) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for all arguments,

(ii) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for large arguments and $L^\infty \hookrightarrow E$,

(iii) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for small arguments and $E \hookrightarrow L^\infty$.

Here we define noncommutative Calderón-Lozanovskiĭ space by $E_\varphi(\mathscr{M}) = \{x \in L_0(\mathscr{M}) : \mu(x) \in E_\varphi\}$. The main result of this paper is the noncommutative analogue of the product spaces of the classical Calderón-Lozanovskiĭ spaces.

The paper is organized as follows: Section 2 consists of some preliminaries and notations, including the noncommutative Calderón-Lozanovskiĭ spaces and their elementary properties. Section 3 presents some results about $M(E_{\varphi_1}(\mathscr{M}), E_\varphi(\mathscr{M}))$. In Section 4 we deal with the product space of the noncommutative Calderón-Lozanovskiĭ spaces. Section 5 is devoted to the normability of the product space $E_{\varphi_1}(\mathscr{M}) \cdot E_{\varphi_2}(\mathscr{M})$.

2. Preliminaries

In this section, we gather some of the elements of noncommutative integration in semifinite von Neumann algebras and Banach function spaces. Our main references are [1, 13, 19, 21]. Let (Ω, Σ, ν) be a complete σ-finite measure space and $L_0(\Omega)$ be the space of all classes of ν-measurable real-valued functions defined on Ω. Let $f \in L_0(\Omega)$. Recall that the distribution function of f is defined as

$$d_f(s) = \nu(\{t \in \Omega : |f(t)| > s\}), \ \ s > 0$$

and its nonincreasing rearrangement is defined as

$$f^*(t) = \inf\{s > 0 : d_f(s) \leqslant t\}, \ \ t > 0.$$

A (quasi-)Banach space $E = (E, \|\cdot\|_E)$ is said to be a (quasi-)Banach ideal space on Ω if E is a linear subspace of $L_0(\Omega)$ and satisfies the so-called ideal property, which means that if $f \in E$, $g \in L_0(\Omega)$ and $|g(t)| \leqslant |f(t)|$ for ν-almost all $t \in \Omega$, then $g \in E$ and $\|g\|_E \leqslant \|f\|_E$. We will also assume that a (quasi-)Banach ideal space on Ω is saturated, i.e., every $A \in \Sigma$ with $\nu(A) > 0$ has a subset $B \in \Sigma$ of finite positive measure for which $\chi_B \in E$. The last statement is equivalent with the existence of a weak unit, i.e., an element $f \in E$ such that $f(t) > 0$ for each $t \in \Omega$. If the measure space (Ω, Σ, ν) is non-atomic we shall speak about (quasi-)Banach function space.

By a symmetric function space on I, where $I = (0,1)$ or $(0,\infty)$ with the Lebesgue measure m, we mean a Banach ideal space $E = (E, \|\cdot\|_E)$ with the additional property that for any two equimeasurable functions $f \sim g$, $f, g \in L_0(I)$ (that is, they have the same distribution functions $d_f(t) = d_g(t)$, $t > 0$) and $f \in E$ we have $g \in E$ and $\|f\|_E = \|g\|_E$. In particular, $\|f\|_E = \|f^*\|_E$. A symmetric function space E on I is said to have the Fatou property if $0 < f_n \in E$, $f_n \uparrow_n f \in L_0(I)$ and $\sup_n \|f_n\|_E < \infty$ imply that $f \in E$ and $\|f_n\|_E \uparrow_n \|f\|_E$. For any $0 < s < \infty$, we define the dilation operator D_s on $L_0(0,\infty)$ by

$$D_s(f)(t) = f\left(\frac{t}{s}\right), \ \ 0 < t < \infty.$$

Similarly, the dilation operator D_s on $L_0(0,1)$ is defined by setting

$$D_s(f)(t) = \begin{cases} f(\frac{t}{s}), & t \leqslant \min\{1,s\}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

If E is a symmetric function space on I, then D_s is a bounded linear operator.

For two ideal (quasi-)Banach spaces E and F on I the symbol $E \hookrightarrow^C F$ means that the embedding $E \subseteq F$ is continuous and $\|f\|_F \leqslant C\|f\|_E$ for all $f \in E$. In the case when the embedding $E \hookrightarrow^C F$ holds with some (unknown) constant $C > 0$ we simply write $E \hookrightarrow F$. Moreover, $E = F (E \equiv F)$ means that the spaces are the same and the (quasi-)norms are equivalent (equal).

Any non-trivial symmetric function space E on I (i.e., $E \neq 0$) is an intermediate space between the spaces $L^1(I)$ and $L^\infty(I)$. More precisely,

$$L^1(I) \cap L^\infty(I) \hookrightarrow E \hookrightarrow L^1(I) + L^\infty(I).$$

A symmetric function space E on I has the majorant property if for all $f \in L_0(I)$, $g \in E$, the condition $\int_0^t f^*(t)dt \leqslant \int_0^t g^*(t)dt$, $t \in I$ implies that $f \in E$ and $\|f\|_E \leqslant \|g\|_E$. Every symmetric function space with the Fatou property have the majorant property. More information about symmetric spaces on I can be found in [1, 13, 9].

A function $\varphi : [0,\infty) \to [0,\infty]$ is said to be a Young function if φ is convex, non-decreasing with $\varphi(0) = 0$. We suppose that φ is neither identically zero nor identically infinity on $(0,\infty)$.

For any Young function φ and any symmetric function spaces $(E, \|\cdot\|_E)$, we define the Calderón-Lozanovskiĭ space E_φ by

$$E_\varphi = \{f \in L_0 : \varphi(\lambda|f|) \in E \ \text{ for some } \lambda > 0\},$$

which is a symmetric function space on I with the so called Luxemburg-Nakano norm defined by

$$\|f\|_\varphi = \inf\Big\{\lambda > 0 : \rho_\varphi\Big(\frac{f}{\lambda}\Big) \leqslant 1\Big\},$$

where $\rho_\varphi(\cdot)$ is the same as (2.1). We refer to [8, 9, 10, 14] for details on the Calderón-Lozanovskiĭ space.

Throughout the present paper, $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ will denote a von Neumann algebra on some Hilbert space $\mathscr{H}$, that is, $\mathscr{M}$ is an $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ containing 1 that is closed for the weak operator topology. A trace τ on the von Neumann algebra $\mathscr{M}$ is a map $\tau : \mathscr{M}^+ \to [0,\infty]$ which is additive, positively homogeneous and unitarily invariant, that is, $\tau(x) = \tau(u^*xu)$ for all $a \in \mathscr{M}^+$ and unitary operators $u \in \mathscr{M}$, where $\mathscr{M}^+ = \{x \in \mathscr{M} : x \geqslant 0\}$. A trace $\tau : \mathscr{M}^+ \to [0,\infty]$ is called

(i) faithful if for all $x \in \mathscr{M}^+$, $\tau(x) = 0$ implies that $x = 0$;

(ii) semifinite if for every $x \in \mathscr{M}^+$ with $\tau(x) > 0$, there exists $0 \leqslant y \leqslant x$ such that $0 < \tau(y) < \infty$;

(iii) normal if $x_i \uparrow_i x$ in $\mathscr{M}^+$ implies that $0 \leqslant \tau(x_i) \uparrow_i \tau(x)$.

A von Neumann algebra $\mathscr{M}$ equipped with a faithful, normal semifinite trace is said to be a semifinite von Neumann algebra. In what follows, $\mathscr{M}$ shall always denote a semifinite von Neumann algebra, equipped with a fixed faithful, normal, semifinite trace τ.

We denote the projection lattice of $\mathscr{M}$ by $P(\mathscr{M})$. A closed densely defined linear operator x in $\mathscr{H}$ with domain $D(x)$ is said to be affiliated with $\mathscr{M}$ if and only if $u^*xu = x$ for all unitary operators u which belong to the commutant of $\mathscr{M}$. When x is affiliated with $\mathscr{M}$, x is said to be τ-measurable if for every $\varepsilon > 0$ there exists $e \in P(\mathscr{M})$ such that $e(\mathscr{H}) \subseteq D(x)$ and $\tau(e^{\perp}) < \varepsilon$ (where for any projection e, we let $e^{\perp} = 1 - e$). The set of all τ-measurable operators will be denoted by $L_0(\mathscr{M})$. The set $L_0(\mathscr{M})$ is an $*$-algebra with sum and product being the respective closure of the algebraic sum and product. The measure topology in $L_0(\mathscr{M})$ is the vector space topology defined via the neighbourhood base $\{V(\varepsilon,\delta) : \varepsilon,\delta > 0\}$, where $V(\varepsilon,\delta) = \{x \in L_0(\mathscr{M}) : \tau(e_{(\varepsilon,\infty)}(|x|)) \leqslant \delta\}$ and $e_{(\varepsilon,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (ε,∞). With respect to the measure topology, $L_0(\mathscr{M})$ is a complete topological $*$-algebra.

For $x \in L_0(\mathscr{M})$ we define

$$\lambda_t(x) = \tau(e_{(t,\infty)}(|x|)) \qquad \text{and} \qquad \mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leqslant t\},$$

where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (t,∞). The function $t \longrightarrow \lambda_t(x)$ is called the distribution function of x and $t \longrightarrow \mu_t(x)$ is the generalized singular number of x. We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $t \longrightarrow \lambda_t(x)$ and $t \longrightarrow \mu_t(x)$, respectively (cf. [6]).

Let E be a symmetric function space on I. We define

$$E(\mathscr{M}) = \{x \in L_0(\mathscr{M}) : \mu(x) \in E\},$$

$$\|x\|_{E(\mathscr{M})} = \|\mu(x)\|_E.$$

Then the noncommutative symmetric function space $(E(\mathscr{M}), \|\cdot\|_{E(\mathscr{M})})$ is a Banach space (cf. [4, 18]). As usual, we put $L^\infty(\mathscr{M}) = \mathscr{M}$ and denote by $\|\cdot\|$ the usual operator norm.

In particular, we define the noncommutative Calderón-Lozanovskiĭ space $E_\varphi(\mathscr{M})$ (φ is a Young function) by

$$E_\varphi(\mathscr{M}) = \{x \in L_0(\mathscr{M}) : \mu(x) \in E_\varphi\},$$

where the functional $\|\cdot\|_\varphi$ on $E_\varphi(\mathscr{M})$ is defined by $\|x\|_\varphi = \|\mu(x)\|_{E_\varphi}$. That is we define the noncommutative Calderón-Lozanovskiĭ space $E_\varphi(\mathscr{M})$ by

$$E_\varphi(\mathscr{M}) = \{x \in L_0(\mathscr{M}) : \varphi(\mu(\lambda x)) \in E \ \text{ for some } \lambda > 0\},$$

which is a noncommutative symmetric function space with the so called Luxemburg-Nakano norm defined by

$$\|x\|_\varphi = \inf\left\{\lambda > 0 : \rho^\varphi_{\mathscr{M}}\left(\frac{x}{\lambda}\right) \leqslant 1\right\},$$

where $\rho^{\varphi}_{\mathscr{M}}(x) := \rho_{\varphi}(\mu(x))$.

Let E and F be two symmetric function spaces on I with norms $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively. The space of pointwise multipliers is defined as follows:

$$M(E,F) = \{f \in L_0(I) : fg \in F \text{ for all } g \in E\}$$

and $M(E,F)$ is a Banach space with the norm $\|f\|_{M(E,F)} = \sup\{\|fg\|_F : \|g\|_E \leqslant 1\}$ (cf. [16]).

Pointwise multipliers between Calderón-Lozanovskiĭ spaces as well as between some other Banach ideal spaces were investigated by several authors, see [9, 10, 12, 16, 15]. A plausible definition of noncommutative pointwise multipliers space is defined as follows:

DEFINITION 1. Let E and F be two symmetric function spaces on I. Then $E(\mathscr{M})$ and $F(\mathscr{M})$ are two noncommutative symmetric function spaces. We define noncommutative pointwise multipliers space $M(E(\mathscr{M}),F(\mathscr{M}))$ by

$$M(E(\mathscr{M}),F(\mathscr{M})) = \{x \in L_0(\mathscr{M}) : \ xy \in F(\mathscr{M}) \text{ for every } y \in E(\mathscr{M})\}.$$

We define a functional $\|\cdot\|_{\mathscr{M}}$ on $M(E(\mathscr{M}),F(\mathscr{M}))$ by

$$\|x\|_{\mathscr{M}} = \sup\{\|xy\|_{F(\mathscr{M})} : \ y \in E(\mathscr{M}), \ \|y\|_{E(\mathscr{M})} \leqslant 1\}.$$

We define the right-continuous inverse φ^{-1} of φ as $\varphi^{-1}(t) = \inf\{s \geqslant 0 : \varphi(s) > t\}$ for $t \in [0,\infty)$ with $\varphi^{-1}(\infty) = \lim_{t\to\infty} \varphi^{-1}(t)$. We write $a_\varphi = \sup\{t \geqslant 0, \varphi(t) = 0\}$ and $b_\varphi = \sup\{t > 0, \varphi(t) < \infty\}$, then $0 \leqslant a_\varphi \leqslant b_\varphi \leqslant \infty$ and $a_\varphi < \infty$, $b_\varphi > 0$, since a Young function is neither identically zero nor identically infinity on $(0,\infty)$.

Throughout this paper we assume that $a_\varphi \neq b_\varphi$. We write $\varphi(b_\varphi) = \lim_{s\to b_\varphi^-} \varphi(s)$ if $b_\varphi < \infty$ and write $\varphi > 0$ when $a_\varphi = 0$ and $\varphi < \infty$ if $b_\varphi = \infty$. The function φ is continuous and nondecreasing on $[0,b_\varphi)$ and strictly increasing on $[a_\varphi,b_\varphi)$. It follows that $\varphi(a_\varphi) = 0$ and

(1) If $\varphi \in \mathscr{Y}_1$, then φ^{-1} is continuous on $[0,\infty)$ and

$$\varphi^{-1}(t) = \begin{cases} a_\varphi & \text{if } t = 0, \\ s, & \text{if } t \in (0,\infty), \text{ and } t = \varphi(s) \text{ where } s \in (a_\varphi,\infty), \\ \infty, & \text{if } t = \infty. \end{cases}$$

(2) If $\varphi \in \mathscr{Y}_2$, then φ^{-1} is continuous on $[0,\infty)$ and

$$\varphi^{-1}(t) = \begin{cases} a_\varphi & \text{if } t = 0, \\ s, & \text{if } t \in (0,\infty), \text{ and } t = \varphi(s) \text{ where } s \in (a_\varphi,b_\varphi), \\ b_\varphi, & \text{if } t = \infty. \end{cases}$$

(3) If $\varphi \in \mathscr{Y}_3$, then φ^{-1} is continuous on $[0,\infty)$ and

$$\varphi^{-1}(t) = \begin{cases} a_\varphi & \text{if } t = 0, \\ s, & \text{if } t \in (0, \varphi(b_\varphi)), \text{ and } t = \varphi(s) \text{ where } s \in (a_\varphi, b_\varphi), \\ b_\varphi, & \text{if } t \geqslant \varphi(b_\varphi), \end{cases}$$

where the set of Young functions $\mathscr{Y}_i$, $i = 1,2,3$ are defined by

$$\mathscr{Y}_1 = \{\varphi : b_\varphi = \infty\},$$
$$\mathscr{Y}_2 = \{\varphi : b_\varphi < \infty \text{ and } \varphi(b_\varphi) = \infty\},$$
$$\mathscr{Y}_3 = \{\varphi : b_\varphi < \infty \text{ and } \varphi(b_\varphi) < \infty\}.$$

Further details can be found in [9, 11]. Next, we will use the following relation between Young functions: we say $\psi_1 \prec \psi_2$ for all arguments [for large arguments] (for small arguments) means that there exists a constant $c > 0$ [there exists a constant $c > 0$, $t_0 > 0$] (there exists a constant $c > 0$, $t_0 > 0$) such that the inequality $\psi_1(t) \leqslant c\psi_2(t)$ holds for all $t > 0$ [for all $t \geqslant t_0$] (for all $0 < t < t_0$), respectively.

Let $x \in L_0(\mathscr{M})$. Recall that any Young function φ is continuous and nondecreasing on $[0, b_\varphi)$. If $b_\varphi = \infty$, then for any $s > 0$, we always have $\varphi(\frac{1}{s}|x|) \in L_0(\mathscr{M})$ and $\mu(\varphi(\frac{1}{s}|x|)) = \varphi(\frac{1}{s}\mu(|x|))$. If $b_\varphi < \infty$, we can always give meaning to $\varphi(\mu(x))$. However $\varphi(|x|)$ may not even exist as an element of $L_0(\mathscr{M})$. Let $x \in L_0(\mathscr{M})$ with $\varphi(|x|) \in L_0(\mathscr{M})$. It follows from Lemma 2.1 of [11] that $\mu(\varphi(|x|)) = \varphi(\mu(|x|))$. On the other hand, if $\varphi \in \mathscr{Y}_1 \cup \mathscr{Y}_2 \cup \mathscr{Y}_3$, then φ^{-1} is continuous and nondecreasing on $[0,\infty)$. Applying Lemma 2.5 (iv) of [6] we get $\varphi^{-1}(|x|) \in L_0(\mathscr{M})$ and $\mu(\varphi^{-1}(|x|)) = \varphi^{-1}(\mu(|x|))$.

PROPOSITION 1. *Let $(E, \|\cdot\|_E)$ be a symmetric function space on I with Fatou property and let φ be a Young function. Then the following properties are satisfied:*

(i) *If $e \in P(\mathscr{M})$ and $\tau(e) < \infty$, then $e \in E_\varphi(\mathscr{M})$.*

(ii) *If $x \in E_\varphi(\mathscr{M})$, then $\rho^\varphi_{\mathscr{M}}(x) \leqslant 1$ if and only if $\|x\|_\varphi \leqslant 1$.*

(iii) *The space $(E_\varphi(\mathscr{M}), \|\cdot\|_\varphi)$ is a noncommutative symmetric function space with Fatou property.*

Proof. (i): Since $\mu(e) = \chi_{(0,\tau(e))}$ and $0 \leqslant a_\varphi < b_\varphi \leqslant \infty$, there exists $\lambda_0 \in (0,\infty)$, such that $\varphi(\frac{\mu(e)}{\lambda_0}) = \varphi(\frac{1}{\lambda_0})\chi_{(0,\tau(e))} < \infty$. Thus, the result follows from $\chi_{(0,\tau(e))} \in E$ and $\tau(e) < \infty$.

(ii): Let $x, y \in E_\varphi(\mathscr{M})$ and $\alpha + \beta = 1$ with $\alpha \geqslant 0$, $\beta \geqslant 0$. By Theorem 4.4 of [6], we have

$$\begin{aligned} \int_0^t \varphi(\mu_s(\alpha x + \beta y))ds &\leqslant \int_0^t \varphi(\mu_s(\alpha x) + \mu_s(\beta y))ds \\ &\leqslant \int_0^t \alpha\varphi(\mu_s(x)) + \beta\varphi(\mu_s(y))ds, \quad t > 0. \end{aligned}$$

Since E is a symmetric function space with the Fatou property, then E have the majorant property. This tells us that

$$\rho_{\mathscr{M}}^{\varphi}(\alpha x+\beta y) = \|\varphi(\mu_s(\alpha x+\beta y))\|_E \leqslant \|\alpha\varphi(\mu_s(x))+\beta\varphi(\mu_s(y)\|_E \\ \leqslant \alpha\|\varphi(\mu_s(x))\|_E+\beta\|\varphi(\mu_s(y)\|_E = \alpha\rho_{\mathscr{M}}^{\varphi}(x)+\beta\rho_{\mathscr{M}}^{\varphi}(y).$$

Thus, the result follows immediately from Lemma 2.2 of [8] and the definition of $\rho_{\mathscr{M}}^{\varphi}(\cdot)$ and $\|\cdot\|_{\varphi}$.

(*iii*): Since E_φ is a symmetric function space, Theorem 4.5 of [4] shows that $E_\varphi(\mathscr{M})$ is a noncommutative symmetric function space. On the other hand, the Fatou property follows immediately from Lemma 2.2 of [8] and Proposition 1.7 of [5]. Indeed, if $0 \leqslant x_n \uparrow x \in L_0(\mathscr{M})$ with $\sup_n \|x_n\|_\varphi < \infty$, then by Proposition 1.7 of [5], we have $\mu(x_n) \uparrow \mu(x)$ and $\sup_n \|\mu(x_n)\|_{E_\varphi} < \infty$. On the other hand, it follows from Lemma 2.2 of [8] that E_φ have the Fatou property. This tells us that $\mu(x) \in E_\varphi$ and $\|\mu(x_n)\|_{E_\varphi} \uparrow \|\mu(x)\|_{E_\varphi}$. That is $x \in E_\varphi(\mathscr{M})$ and $\|x_n\|_\varphi \uparrow \|x\|_\varphi$. □

A similar discussion to the proof of Proposition 2.2 of [11], leads to the following proposition.

PROPOSITION 2. *Let $(E,\|\cdot\|_E)$ be a symmetric function space on $(0,\tau(1))$ with Fatou property and let φ be a Young function and $x \in L_0(\mathscr{M})$. Then there exists some $\alpha > 0$ such that $\rho_\varphi(\alpha\mu(x)) < \infty$ if and only if there exists some $\beta > 0$ such that $\varphi(\beta|x|) \in L_0(\mathscr{M})$ and $\|\varphi(\beta|x|)\|_{E(\mathscr{M})} < \infty$. Moreover,*

$$\|\mu(x)\|_{E_\varphi} = \inf\Big\{s>0 : \varphi\Big(\frac{1}{s}|x|\Big) \in L_0(\mathscr{M}), \Big\|\varphi\Big(\frac{1}{s}|x|\Big)\Big\|_{E(\mathscr{M})} \leqslant 1\Big\}.$$

Proof. The validity of this result for the case $b_\varphi = \infty$ follows from Lemma 2.1 of [11]. Hence let $b_\varphi < \infty$. If now there exists $\beta > 0$ such that $\varphi(\beta|x|) \in L_0(\mathscr{M})$, then it follows from Lemma 2.1 of [11] that $\|\varphi(\beta|x|)\|_{E(\mathscr{M})} = \|\varphi(\beta\mu(x))\|_E$. This implies that $\rho_\varphi(\beta\mu(x)) < \infty$. Conversely, suppose that $\rho_\varphi(\alpha\mu(x)) < \infty$ for some $\alpha > 0$. If for some $t_0 > 0$ we had $\alpha\mu_{t_0}(x) > b_\varphi$, then $\alpha\mu_t(x) \geqslant \alpha\mu_{t_0}(x) > b_\varphi$ for all $0 \leqslant t \leqslant t_0$, which would force

$$\|\varphi(\alpha\mu_t(x))\|_E \geqslant \|\varphi(\alpha\mu_t(x))\chi_{[0,t_0]}\|_E = \|\infty\chi_{[0,t_0]}\|_E = \infty.$$

Thus we must have $\alpha\mu_t(x) \leqslant b_\varphi$, $t > 0$. This means that

$$\alpha\|x\| = \lim_{t\to 0}\alpha\mu_t(x) \leqslant b_\varphi.$$

So in this case we clearly have that $x \in \mathscr{M}$ with $\varphi(\frac{\alpha}{1+\varepsilon}|x|) \in \mathscr{M} \subseteq L_0(\mathscr{M})$. By Lemma 2.1 of [11], we have

$$\Big\|\varphi\Big(\frac{\alpha}{1+\varepsilon}|x|\Big)\Big\|_{E(\mathscr{M})} = \Big\|\varphi\Big(\frac{\alpha}{1+\varepsilon}\mu(|x|)\Big)\Big\|_E \leqslant \|\varphi(\alpha\mu(|x|))\|_E < \infty.$$

To see the second claim, observe that the Lemma 2.1 of [11] ensures that

$$\left\{s>0:\varphi\left(\frac{1}{s}|x|\right)\in L_0(\mathscr{M}),\left\|\varphi\left(\frac{1}{s}|x|\right)\right\|_{E(\mathscr{M})}\leqslant 1\right\}\subseteq\left\{s>0:\left\|\varphi\left(\frac{1}{s}\mu(|x|)\right)\right\|_{E}\leqslant 1\right\}.$$

Hence

$$\begin{aligned}\|\mu(x)\|_{E_\varphi}&=\inf\left\{s>0:\left\|\varphi\left(\frac{1}{s}\mu(|x|)\right)\right\|_{E}\leqslant 1\right\}\\&\leqslant\inf\left\{s>0:\varphi\left(\frac{1}{s}|x|\right)\in L_0(\mathscr{M}),\left\|\varphi\left(\frac{1}{s}|x|\right)\right\|_{E(\mathscr{M})}\leqslant 1\right\}.\end{aligned}$$

To see that equality holds, let $\varepsilon>0$ be given, and select $s_0>0$ so that

$$\|\mu(x)\|_{E_\varphi}\leqslant s_0\leqslant(1+\varepsilon)\|\mu(x)\|_{E_\varphi}\ \text{ and }\ \left\|\varphi\left(\frac{1}{s_0}\mu(x)\right)\right\|_{E}\leqslant 1.$$

It follows from the above case that

$$\varphi\left(\frac{1}{(1+\varepsilon)s_0}|x|\right)\in L_0(\mathscr{M})\ \text{ and }\ \left\|\frac{1}{(1+\varepsilon)s_0}|x|\right\|_{E(\mathscr{M})}\leqslant\left\|\varphi\left(\frac{1}{s_0}\mu(|x|)\right)\right\|_{E}\leqslant 1.$$

Thus

$$\inf\left\{s>0:\varphi\left(\frac{1}{s}|x|\right)\in L_0(\mathscr{M}),\left\|\varphi\left(\frac{1}{s}|x|\right)\right\|_{E(\mathscr{M})}\leqslant 1\right\}\leqslant(1+\varepsilon)s_0\leqslant(1+\varepsilon)^2\|\mu(x)\|_{E_\varphi}.$$

Since $\varepsilon>0$ was arbitrary, we have

$$\inf\left\{s>0:\varphi\left(\frac{1}{s}|x|\right)\in L_0(\mathscr{M}),\left\|\varphi\left(\frac{1}{s}|x|\right)\|_{E(\mathscr{M})}\leqslant 1\right\}\leqslant\|\mu(x)\|_{E_\varphi}.\right.$$

This implies the desired result. □

PROPOSITION 3. *Let E,E_1,E_2 and F,F_1,F_2 be symmetric function spaces.*

(i) *If $x\in M(E(\mathscr{M}),F(\mathscr{M}))$, then $|x|\in M(E(\mathscr{M}),F(\mathscr{M}))$ and $\|x\|_{\mathscr{M}}=\||x|\|_{\mathscr{M}}$.*

(ii)

$$\begin{aligned}\|x\|_{\mathscr{M}}&=\sup\{\|xy\|_{F(\mathscr{M})}:0\leqslant y\in E(\mathscr{M}),\|y\|_{E(\mathscr{M})}\leqslant 1\}\\&=\sup\{\|xy\|_{F(\mathscr{M})}:0\leqslant y\in E(\mathscr{M}),\|y\|_{E(\mathscr{M})}=1\}.\end{aligned}$$

(iii) *If $0<x\leqslant z$ and $x,z\in M(E(\mathscr{M}),F(\mathscr{M}))$, then $\|x\|_{\mathscr{M}}\leqslant\|z\|_{\mathscr{M}}$.*

(iv) *If $M(E,F)\neq\{0\}$ and $e\in P(\mathscr{M})$ with $\tau(e)<\infty$, then*

$$e\in M(E(\mathscr{M}),F(\mathscr{M})).$$

(v) *If $\lim_{t\to\infty}\mu_t(x)=0$ holds for all $x\in M(E(\mathscr{M}),F(\mathscr{M}))$, then the injection $M(E(\mathscr{M}),F(\mathscr{M}))\hookrightarrow L_0(\mathscr{M})$ is continuous.*

Proof. (i): Let $x = u|x|$ be the polar decomposition of x and let $y \in E(\mathcal{M})$ with $\|y\|_{E(\mathcal{M})} \leqslant 1$. It follows that

$$\|xy\|_{F(\mathcal{M})} = \|u|x|y\|_{F(\mathcal{M})} \leqslant \||x|y\|_{F(\mathcal{M})},$$
$$\||x|y\|_{F(\mathcal{M})} = \|u^*xy\|_{F(\mathcal{M})} \leqslant \|xy\|_{F(\mathcal{M})}.$$

Hence $|x| \in M(E(\mathcal{M}),F(\mathcal{M}))$ and $\|x\|_{\mathcal{M}} = \||x|\|_{\mathcal{M}}$.

(ii): Let $y \in E(\mathcal{M})$ with $\|y\|_{E(\mathcal{M})} \leqslant 1$ and let $y = v|y|$ be the polar decomposition of y. Therefore, $\mu(xy) = \mu(x|y^*|v) \leqslant \mu(x|y^*|)$, and so $\|xy\|_{F(\mathcal{M})} \leqslant \|x|y^*|\|_{F(\mathcal{M})}$. Combining this with $|y^*| \geqslant 0$, we have

$$\|x\|_{\mathcal{M}} \leqslant \sup\{\|xy\|_{F(\mathcal{M})} : 0 \leqslant y \in E(\mathcal{M}), \|y\|_{E(\mathcal{M})} \leqslant 1\},$$

which implies that $\|x\|_{\mathcal{M}} = \sup\{\|xy\|_{F(\mathcal{M})} : 0 \leqslant y \in E(\mathcal{M}), \|y\|_{E(\mathcal{M})} \leqslant 1\}$. On the other hand, by a simple computation, we derive

$$\begin{aligned}&\sup\{\|xy\|_{F(\mathcal{M})} : 0 \leqslant y \in E(\mathcal{M}), \|y\|_{E(\mathcal{M})} \leqslant 1\}\\ &= \sup\{\|xy\|_{F(\mathcal{M})} : 0 \leqslant y \in E(\mathcal{M}), \|y\|_{E(\mathcal{M})} = 1\}.\end{aligned}$$

(iii): If $0 \leqslant x \leqslant z$, then there exists $u \in \mathcal{M}$ with $\|u\| \leqslant 1$ such that $x = uz$. Thus, $\mu(xy) = \mu(uzy) \leqslant \mu(zy)$ and therefore, (iii) holds.

(iv): Let $e \in P(\mathcal{M})$ with $\tau(e) < \infty$. By Proposition 2.3 of [9] and the fact $M(E,F) \neq \{0\}$, we get $\chi_{[0,\tau(e))} \in M(E,F)$. By Lemma 2.5 of [6], we obtain

$$\mu_t(ey) \leqslant \mu_{\frac{t}{2}}(e)\mu_{\frac{t}{2}}(y) = D_2(\mu_t(e)\mu_t(y)) = D_2(\chi_{[0,\tau(e))}(t)\mu_t(y)), y \in E(\mathcal{M}).$$

Consequently,

$$\begin{aligned}\|e\|_{\mathcal{M}} &= \sup\{\|\mu(ey)\|_F : y \in E(\mathcal{M}), \|y\|_{E(\mathcal{M})} \leqslant 1\}\\ &\leqslant \|D_2\|_{F\to F} \sup\{\|\mu(y)\chi_{[0,\tau(e))}\|_F : \mu(y) \in E, \|\mu(y)\|_E \leqslant 1\}\\ &\leqslant \|D_2\|_{F\to F} \sup\{\|f\chi_{[0,\tau(e))}\|_F : f \in E, \|f\|_E \leqslant 1\}\\ &= \|D_2\|_{F\to F} \|\chi_{[0,\tau(e))}\|_{M(E,F)} < \infty.\end{aligned}$$

That is $e \in M(E(\mathcal{M}),F(\mathcal{M}))$.

(v): A similar discussion to the proof of Theorem 7.1 of [18] shows that (v) holds. Indeed, for $x \in M(E(\mathcal{M}),F(\mathcal{M}))$ with $\|x\|_{\mathcal{M}} \leqslant 1$. Let $x = u|x|$ be the polar decomposition of x and $|x| = \int_0^\infty \lambda de_\lambda$ be the spectral decomposition of $|x|$. Then $|x|$ admits the Schmidt decomposition $|x| = \int_0^\infty \mu_t(x)d\widetilde{e}_t$, where $\widetilde{e}_t = e_{\mu_t(x)-0}$, $t > 0$ and $e_{0-0} = 1$ [cf. [17]]. Given $\delta > 0$, it is easy to see that $\mu_t(x) \geqslant \mu_\delta(x)\chi_{[\frac{\delta}{2},\delta)}$, thus $|x| = \int_0^\infty \mu_t(x)d\widetilde{e}_t \geqslant \mu_\delta(x)q$, where $q = \int_0^\infty \chi_{[\frac{\delta}{2},\delta)}d\widetilde{e}_t$. It is clear that $\tau(q) < \infty$, which means that $q \in M(E(\mathcal{M}),F(\mathcal{M}))$. Hence, $\|x\|_{\mathcal{M}} = \||x|\|_{\mathcal{M}} \geqslant \mu_\delta(x)\|q\|_{\mathcal{M}}$ and $\|q\|_{\mathcal{M}}^{-1} \geqslant \mu_\delta(x)$, which complete the proof. □

PROPOSITION 4. *Let $E(\mathcal{M})$ and $F(\mathcal{M})$ be two noncommutative symmetric function spaces. If $\lim_{t\to\infty}\mu_t(x) = 0$ holds for all $x \in M(E(\mathcal{M}),F(\mathcal{M}))$, then $M(E(\mathcal{M}),F(\mathcal{M}))$ is a Banach space with the norm*

$$\|x\|_{\mathcal{M}} = \sup\{\|xy\|_{F(\mathcal{M})} : y \in E(\mathcal{M}), \|y\|_{E(\mathcal{M})} \leqslant 1\}.$$

Proof. It is clear that $\|\cdot\|_{\mathscr{M}}$ is subadditive, homogenous and positive. If $\|x\|_{\mathscr{M}} = \||x|\|_{\mathscr{M}} = 0$, then $\|xy\|_{F(\mathscr{M})} = \||x|y\|_{F(\mathscr{M})} = 0$ for every $y \in E(\mathscr{M})$ with $\|y\|_{E(\mathscr{M})} \leqslant 1$. That is $\|xy\|_{F(\mathscr{M})} = \||x|y\|_{F(\mathscr{M})} = 0$ for every $y \in E(\mathscr{M})$. Let $e_{(\frac{1}{n},\infty)}(|x|)$ be the spectral projection of $|x|$ associated with the interval $(\frac{1}{n},\infty)$, $n = 1,2,\cdots$. Since $\tau(e_{(\frac{1}{n},\infty)}(|x|)) < \infty$, then $e_{(\frac{1}{n},\infty)}(|x|) \in E(\mathscr{M})$, and so $|x|e_{(\frac{1}{n},\infty)}(|x|) = 0$, $n = 1,2,\cdots$. Since $|x|e_{(\frac{1}{n},\infty)}(|x|) \to |x|$ in the measure topology, we have $|x| = 0$, i.e., $x = 0$. The proof of Theorem 8.11 of [7] shows that it is sufficient to prove the noncommutative form of the Riesz-Fischer theorem, i.e., we have an estimate

$$\|\sum_{n=1}^{\infty} x_n\|_{\mathscr{M}} \leqslant \sum_{n=1}^{\infty} \|x_n\|_{\mathscr{M}}, x_n \geqslant 0, \ \ n = 1,2,3\cdots,$$

whenever the right-hand side is finite. Let $\sum_{n=1}^{\infty} \|x_n\|_{\mathscr{M}} < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges in $L_0(\mathscr{M})$ to some x. Indeed, set $z_n = \sum_{k=1}^{n} x_k$, it is clear that $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $M(E(\mathscr{M}),F(\mathscr{M}))$, then by (v) of Proposition 3, $\{z_n\}_{n=1}^{\infty}$ converges in $L_0(\mathscr{M})$ to some x. Note that $\sum_{n=1}^{\infty} \|x_n y\|_{\mathscr{M}} \leqslant \sum_{n=1}^{\infty} \|x_n\|_{\mathscr{M}} < \infty$ and

$$\begin{aligned}\|(\sum_{n=1}^{\infty} x_n)y\|_{F(\mathscr{M})} = \|\sum_{n=1}^{\infty} x_n y\|_{F(\mathscr{M})} &\leqslant \sum_{n=1}^{\infty} \|x_n y\|_{F(\mathscr{M})} \\ &\leqslant \sum_{n=1}^{\infty} \|x_n\|_{\mathscr{M}} < \infty\end{aligned}$$

hold for each $y \in E(\mathscr{M})$ with $\|y\|_{E(\mathscr{M})} \leqslant 1$. Thus $\sum_{n=1}^{\infty} x_n \in M(E(\mathscr{M}),F(\mathscr{M}))$ and $\|\sum_{n=1}^{\infty} x_n\|_{\mathscr{M}} \leqslant \sum_{n=1}^{\infty} \|x_n\|_{\mathscr{M}} < \infty$. $\square$

PROPOSITION 5. *Let $E(\mathscr{M}) \neq \{0\}$ be a noncommutative symmetric function space, then*

$$M(E(\mathscr{M}),E(\mathscr{M})) = \mathscr{M}, \qquad M(\mathscr{M},E(\mathscr{M})) = E(\mathscr{M}).$$

Proof. It is clear that $\mathscr{M} \subseteq M(E(\mathscr{M}),E(\mathscr{M}))$ and $\|xy\|_{E(\mathscr{M})} \leqslant \|y\|_{E(\mathscr{M})}\|x\|$ hold for $x \in \mathscr{M}$ and $y \in E(\mathscr{M})$. Moreover, $\|x\|_{\mathscr{M}} \leqslant \|x\|, x \in \mathscr{M}$. Conversely, let $x \in M(E(\mathscr{M}),E(\mathscr{M}))$ and $x \notin \mathscr{M}$, then the projection $e_n = e_{(n^3,(n+1)^3]}(|x|)$ has positive trace for infinitely many $n \in \mathbb{N}^+$. Without loss of generality, we suppose that $0 < \tau(e_n) < \infty$ for all $n \in \mathbb{N}^+$. We put $a_n = \|e_n\|_{E(\mathscr{M})}$, $n \in \mathbb{N}^+$ and $y = \sum_{n=1}^{\infty} \frac{1}{n^2 a_n} e_n$. Then $y \in E(\mathscr{M})$. Since

$$\begin{aligned}\|xy\|_{E(\mathscr{M})} &\geqslant \|xye_n\|_{E(\mathscr{M})} \\ &\geqslant n^3\|ye_n\|_{E(\mathscr{M})} = n\end{aligned}$$

holds for all $n \in \mathbb{N}^+$, we have $x \notin M(E(\mathscr{M}),E(\mathscr{M}))$. This implies that

$$\mathscr{M} \supseteq M(E(\mathscr{M}),E(\mathscr{M})).$$

On the other hand, for $x \in M(E(\mathscr{M}),E(\mathscr{M})) \subseteq \mathscr{M}$ and $0<\lambda_0<\|x\|$, we put $y_{\lambda_0} = \frac{e_{(\lambda_0,\|x\|)}(|x|)}{\|e_{(\lambda_0,\|x\|)}(|x|)\|_{E(\mathscr{M})}} \in E(\mathscr{M})$, then $\|y_{\lambda_0}\|_{E(\mathscr{M})}=1$ and

$$\|xy_{\lambda_0}\|_{E(\mathscr{M})} = \Big\|x\frac{e_{(\lambda_0,\|x\|)}(|x|)}{\|e_{(\lambda_0,\|x\|)}(|x|)\|_{E(\mathscr{M})}}\Big\|_{E(\mathscr{M})} \geqslant \lambda_0,\ \ 0<\lambda_0<\|x\|,$$

which implies that $\|x\|_{\mathscr{M}} \geqslant \|x\|$. Therefore, $M(E(\mathscr{M}),E(\mathscr{M})) = \mathscr{M}$.

Let $x \in M(\mathscr{M},E(\mathscr{M}))$ and $x=u|x|$ be the polar decomposition of x. Note that $x=xuu^*$ and $uu^* \in \mathscr{M}$, by a simple computation, we derive $M(\mathscr{M},E(\mathscr{M})) = E(\mathscr{M})$. □

We put $M(E(\mathscr{M}),F(\mathscr{M})) = E(\mathscr{M})^{F(\mathscr{M})}$. The spaces $(E(\mathscr{M})^{F(\mathscr{M})})^{F(\mathscr{M})}$ is denoted by $E(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})}$. Here we say that $E(\mathscr{M})$ is $F(\mathscr{M})$-perfect, if $E(\mathscr{M}) = E(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})}$.

PROPOSITION 6. *Let E, E_1, E_2, F, F_1, F_2 be symmetric function spaces.*

(i) $E(\mathscr{M}) \hookrightarrow^1 E(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})}$.

(ii) *If $E_1(\mathscr{M}) \hookrightarrow^1 E_2(\mathscr{M})$, then $M(E_2(\mathscr{M}),F(\mathscr{M})) \hookrightarrow^1 M(E_1(\mathscr{M}),F(\mathscr{M}))$.*

(iii) *If $F_1(\mathscr{M}) \hookrightarrow^1 F_2(\mathscr{M})$, then $M(E(\mathscr{M}),F_1(\mathscr{M})) \hookrightarrow^1 M(E(\mathscr{M}),F_2(\mathscr{M}))$.*

(iv) $E(\mathscr{M})^{F(\mathscr{M})} = E(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})F(\mathscr{M})}$.

(v) *If $F,E_1 \subseteq L_1$, then*

$$\begin{aligned} M(E_0(\mathscr{M}),E_1(\mathscr{M})) &\hookrightarrow^1 M(E_1(\mathscr{M})^{F(\mathscr{M})},E_0(\mathscr{M})^{F(\mathscr{M})}) \\ &= M(E_0(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})},E_1(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})}). \end{aligned}$$

(vi) $E(\mathscr{M}) \hookrightarrow^1 E_1(\mathscr{M})^{F(\mathscr{M})}$ *if and only if* $E_1(\mathscr{M}) \hookrightarrow^1 E(\mathscr{M})^{F(\mathscr{M})}$.

(vii) *If $F,F_1 \subseteq L_1$ and $F(\mathscr{M})$ is $F_1(\mathscr{M})$-perfect, then $E(\mathscr{M})^{F(\mathscr{M})}$ is $F_1(\mathscr{M})$-perfect.*

Proof. (i): Let $y \in M(E(\mathscr{M}),F(\mathscr{M}))$ and $y=u|y|$ be the polar decomposition of y. Then $yx \in F(\mathscr{M})$ holds for all $x \in E(\mathscr{M})$. Thus $\mu(|y|x) = \mu(u^*yx) \leqslant \mu(yx) \in F$ holds for all $x \in E(\mathscr{M})$. This implies that $|y| \in M(E(\mathscr{M}),F(\mathscr{M}))$. For every $x \in E(\mathscr{M})$, it is clear that $u^*x \in E(\mathscr{M})$. Thus $\mu(y^*x) = \mu(|y|u^*x) \in F$ holds for all $x \in E(\mathscr{M})$. That is $y^* \in M(E(\mathscr{M}),F(\mathscr{M}))$. Moreover, we have $\mathrm{Re}y$, $\mathrm{Im}y \in M(E(\mathscr{M}),F(\mathscr{M}))$. Since $x \in E(\mathscr{M})$, it is clear that $\mathrm{Re}x$, $\mathrm{Im}x \in E(\mathscr{M})$. Hence, $\mathrm{Re}y\,\mathrm{Re}x$, $\mathrm{Re}y\,\mathrm{Im}x$, $\mathrm{Im}y\,\mathrm{Re}x$, $\mathrm{Im}y\,\mathrm{Im}x \in F(\mathscr{M})$. By Corollary 3.6 of [3], we deduce $\mathrm{Re}x\,\mathrm{Re}y$, $\mathrm{Im}x\mathrm{Re}y$, $\mathrm{Re}x\,\mathrm{Im}y$, $\mathrm{Im}x\,\mathrm{Im}y \in F(\mathscr{M})$. Since

$$\mu_t(xy) \leqslant D_4(\mu_t(\mathrm{Re}x\,\mathrm{Re}y)+\mu_t(\mathrm{Im}x\,\mathrm{Re}y)+\mu_t(\mathrm{Re}x\,\mathrm{Im}y)+\mu_t(\mathrm{Im}x\,\mathrm{Im}y))$$

and D_4 is bounded on F, we get $xy \in F(\mathscr{M})$. Therefore,

$$E(\mathscr{M}) \subseteq M(M(E(\mathscr{M}),F(\mathscr{M})),F(\mathscr{M})).$$

Let $y \in E(\mathcal{M}) \subseteq M(M(E(\mathcal{M}),F(\mathcal{M})),F(\mathcal{M}))$. By Corollary 3.6 of [3] and Proposition 3, we have

$$\begin{aligned}
\|y\|_{E(\mathcal{M})^{F(\mathcal{M})F(\mathcal{M})}} &= \||y|\|_{E(\mathcal{M})^{F(\mathcal{M})F(\mathcal{M})}} \\
&= \sup\{\||y|x\|_{F(\mathcal{M})} : 0 \leqslant x \in M(E(\mathcal{M}),F(\mathcal{M})), \|x\|_{\mathcal{M}} = 1\} \\
&\leqslant \sup\left\{\||y|x\|_{F(\mathcal{M})} : \begin{array}{l} 0 \leqslant x \in M(E(\mathcal{M}),F(\mathcal{M})),\ \ 0 \leqslant z \in E(\mathcal{M}), \\ \|xz\|_{F(\mathcal{M})} = \|zx\|_{F(\mathcal{M})} \leqslant \|z\|_{E(\mathcal{M})} \end{array}\right\} \\
&\leqslant \|y\|_{E(\mathcal{M})}.
\end{aligned}$$

(ii) and (iii) follow from the definition of $\|\cdot\|_{\mathcal{M}}$. (iv) follows immediately from (i) and (ii).

(v): Let $x \in M(E_0(\mathcal{M}),E_1(\mathcal{M}))$. Then $xz \in E_1(\mathcal{M})$ holds for all $z \in E_0(\mathcal{M})$. Therefore, for every $y \in M(E_1(\mathcal{M}),F(\mathcal{M}))$, we have $yxz \in F(\mathcal{M})$. A similar discussion to the proof of (i) shows that $zx \in E_1(\mathcal{M})$ holds for all $z \in E_0(\mathcal{M})$, and so $yzx \in F(\mathcal{M})$. This implies that $|yz|x \in F(\mathcal{M})$. Moreover, $(yz)^*x \in F(\mathcal{M})$. A similar discussion to the proof of (i) shows that $xyz \in F(\mathcal{M})$. That is $xy \in E_0(\mathcal{M})^{F(\mathcal{M})}$, i.e., $x \in M(E_1(\mathcal{M})^{F(\mathcal{M})},E_0(\mathcal{M})^{F(\mathcal{M})})$. Since $F,E_1 \subseteq L_1$, we have $F(\mathcal{M})$, $E_1(\mathcal{M}) \subseteq L_1(\mathcal{M})$. Thus, by Lemma 2 of [2], we have

$$\begin{aligned}
\|x\|_{M(E_1(\mathcal{M})^{F(\mathcal{M})},E_0(\mathcal{M})^{F(\mathcal{M})})} &= \||x|\|_{M(E_1(\mathcal{M})^{F(\mathcal{M})},E_0(\mathcal{M})^{F(\mathcal{M})})} \\
&= \sup\{\||x|y\|_{E_0(\mathcal{M})^{F(\mathcal{M})}} : 0 \leqslant y \in E_1(\mathcal{M})^{F(\mathcal{M})}, \|y\|_{E_1(\mathcal{M})^{F(\mathcal{M})}} = 1\} \\
&\leqslant \sup\left\{\||x|yz\|_{F(\mathcal{M})} : \begin{array}{l} 0 \leqslant y \in E_1(\mathcal{M})^{F(\mathcal{M})}, \|y\|_{E_1(\mathcal{M})^{F(\mathcal{M})}} = 1, \\ 0 \leqslant z \in E_0(\mathcal{M}), \|z\|_{E_0(\mathcal{M})} = 1 \end{array}\right\} \\
&= \sup\left\{\|z|x|y\|_{F(\mathcal{M})} : \begin{array}{l} 0 \leqslant y \in E_1(\mathcal{M})^{F(\mathcal{M})}, \|y\|_{E_1(\mathcal{M})^{F(\mathcal{M})}} = 1, \\ 0 \leqslant z \in E_0(\mathcal{M}), \|z\|_{E_0(\mathcal{M})} = 1 \end{array}\right\} \\
&\leqslant \sup\{\||x|z\|_{E_1(\mathcal{M})} : 0 \leqslant z \in E_0(\mathcal{M}), \|z\|_{E_0(\mathcal{M})} = 1\} \\
&= \|x\|_{M(E_0(\mathcal{M}),E_1(\mathcal{M}))}.
\end{aligned}$$

This completes the proof.

(vi): A similar discussion to the proof of (i) shows that $E(\mathcal{M}) \subseteq E_1(\mathcal{M})^{F(\mathcal{M})}$ if and only if $E_1(\mathcal{M}) \subseteq E(\mathcal{M})^{F(\mathcal{M})}$. If $\|x\|_{E(\mathcal{M})} \geqslant \|x\|_{E_1(\mathcal{M})^{F(\mathcal{M})}}$, then we have

$$\begin{aligned}
\|y\|_{E(\mathcal{M})^{F(\mathcal{M})}} &= \||y|\|_{E(\mathcal{M})^{F(\mathcal{M})}} \\
&= \sup\{\||y|x\|_{F(\mathcal{M})} : 0 \leqslant x \in E(\mathcal{M}), \|x\|_{E(\mathcal{M})} \leqslant 1\} \\
&\leqslant \sup\{\||y|x\|_{F(\mathcal{M})} : 0 \leqslant x \in E(\mathcal{M}), 0 \leqslant z \in E_1(\mathcal{M}), \|xz\|_{F(\mathcal{M})} \leqslant \|z\|_{E_1(\mathcal{M})}\} \\
&\leqslant \||y|\|_{E_1(\mathcal{M})} = \|y\|_{E_1(\mathcal{M})}.
\end{aligned}$$

Similarly, if $\|y\|_{E_1(\mathcal{M})} \geqslant \|y\|_{E(\mathcal{M})^{F(\mathcal{M})}}$, we have $\|x\|_{E(\mathcal{M})} \geqslant \|x\|_{E_1(\mathcal{M})^{F(\mathcal{M})}}$ and the proof is complete.

(vii): By (i) we have

$$E(\mathcal{M})^{F(\mathcal{M})} \hookrightarrow^1 E(\mathcal{M})^{F(\mathcal{M})F_1(\mathcal{M})F_1(\mathcal{M})}.$$

Let $F(\mathscr{M}) = F(\mathscr{M})^{F_1(\mathscr{M})F_1(\mathscr{M})}$. By (i) and (v), we obtain

$$\begin{aligned} E(\mathscr{M}) \hookrightarrow^1 E(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})} &\hookrightarrow^1 M(E(\mathscr{M})^{F(\mathscr{M})F_1(\mathscr{M})F_1(\mathscr{M})}, F(\mathscr{M})^{F_1(\mathscr{M})F_1(\mathscr{M})}) \\ &= M(E(\mathscr{M})^{F(\mathscr{M})F_1(\mathscr{M})F_1(\mathscr{M})}, F(\mathscr{M})). \end{aligned}$$

Thus, by (vi), we obtain $E(\mathscr{M})^{F(\mathscr{M})F_1(\mathscr{M})F_1(\mathscr{M})} \hookrightarrow^1 E(\mathscr{M})^{F(\mathscr{M})}$. It follows that $E(\mathscr{M})^{F(\mathscr{M})F_1(\mathscr{M})F_1(\mathscr{M})} = E(\mathscr{M})^{F(\mathscr{M})}$ and the proof is complete. □

By Proposition 5 and Proposition 6, we obtain the following corollary.

COROLLARY 1. *Let E, F be symmetric function spaces.*

(i) *$E(\mathscr{M})$ and $\mathscr{M}$ are $E(\mathscr{M})$-perfect spaces.*

(ii) *$E(\mathscr{M})^{F(\mathscr{M})} \hookrightarrow^1 F(\mathscr{M})$ if and only if $\mathscr{M} \hookrightarrow^1 E(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})}$.*

(iii) *$F(\mathscr{M}) \hookrightarrow^1 E(\mathscr{M})^{F(\mathscr{M})}$ if and only if $E(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})} \hookrightarrow^1 \mathscr{M}$.*

(iv) *$E(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})} \hookrightarrow^1 F(\mathscr{M})$ if and only if $E(\mathscr{M}) \hookrightarrow^1 F(\mathscr{M})$ if and only if $\mathscr{M} \hookrightarrow^1 E(\mathscr{M})^{F(\mathscr{M})}$.*

(v) *$F(\mathscr{M}) \hookrightarrow^1 E(\mathscr{M})^{F(\mathscr{M})F(\mathscr{M})}$ if and only if $E(\mathscr{M})^{F(\mathscr{M})} \hookrightarrow^1 \mathscr{M}$.*

Proof. The same proof as of Corollary 1 of Maligranda and Persson [16] works. □

EXAMPLE 1.

(i) Let $1 \leqslant p < r \leqslant \infty$ and $\mathscr{M}$ be a semifinite von Neumann algebra. Then

$$M(L^p(\mathscr{M}), L^r(\mathscr{M})) = \{0\}.$$

Indeed, suppose that there exists $x \in M(L^p(\mathscr{M}), L^r(\mathscr{M}))$ and $x \neq 0$. Then there exists a projection $e \in P(\mathscr{M})$ such that $ex = xe$ and $0 < \tau(e) < \infty$. Let $e_n = e_{[\frac{1}{n},n]}(|exe|)$, $n = 1,2,3\cdots$. Hence, $e_n \uparrow e$ and $\tau(e_n) > 0$ for $n > n_0$. Moreover, if $y \in L^p(e_n\mathscr{M}e_n)$, then $|y|e_n \in L^p(\mathscr{M})$ and

$$\begin{aligned} \mu\left(\frac{1}{n}ye_n\right) \leqslant \mu\left(\frac{1}{n}|y|e_n\right) &= \mu\left((|y|e_n)^{\frac{1}{2}}\frac{1}{n}(|y|e_n)^{\frac{1}{2}}\right) \\ &\leqslant \mu((|y|e_n)^{\frac{1}{2}}|e_nxe_n|(|y|e_n)^{\frac{1}{2}}) \\ &\leqslant \mu(e_nxe_n(|y|e_n)) \leqslant \mu(x|y|e_n). \end{aligned}$$

It follows that $ye_n \in L^r(\mathscr{M})$ and so $y \in L^r(e_n\mathscr{M}e_n)$. Thus, $L^r(e_n\mathscr{M}e_n) \supseteq L^p(e_n\mathscr{M}e_n)$, $\tau(e_n) > 0$. But, the embedding cannot holds even in the classical case. This contradiction implies that $M(L^p(\mathscr{M}), L^r(\mathscr{M})) = \{0\}$.

(ii) Let $1 \leqslant p \leqslant r \leqslant \infty$ with $\frac{1}{q} = \frac{1}{r} - \frac{1}{p}$ and $\mathscr{M}$ be a semifinite von Neumann algebra. Then

$$M(L^p(\mathscr{M}), L^r(\mathscr{M})) = L^q(\mathscr{M}).$$

3. Multipliers of noncommutative Calderón-Lozanovskiĭ spaces

LEMMA 1. *Let* $\varphi \in \mathscr{Y}_1 \cup \mathscr{Y}_2$ *and* $x = \sum_{k=1}^{N} c_k e_k$, *where* $e_k \in P(\mathscr{M})$, $e_i \perp e_j$, $i \neq j$ *and* $\tau(e_k) < \infty$. *Then* $\rho_{\mathscr{M}}^{\varphi}(\frac{x}{\|x\|_\varphi}) = \rho_\varphi(\frac{\mu(x)}{\|\mu(x)\|_{E_\varphi}}) = 1$.

Proof. Let $x = \sum_{k=1}^{N} c_k e_k$. It is clear that $|x| = \sum_{k=1}^{N} |c_k| e_k$. Without loss of generality, we suppose $|c_1| > |c_2| > \cdots > |c_N|$. Let $d_j = \sum_{k=1}^{j} \tau(e_k)$, $1 \leqslant j \leqslant N$ and $d_0 = 0$. Then

$$\mu(x) = \mu(|x|) = |c_1|\chi_{(d_0,d_1)} + \sum_{j=2}^{N} |c_j|\chi_{[d_{j-1},d_j)}.$$

Hence, the result follows immediately from Lemma 5.1 of [9] and $\rho_{\mathscr{M}}^{\varphi}(x) = \rho_\varphi(\mu(x))$. □

THEOREM 1. *Let* E *be a symmetric function space with Fatou property and* φ, φ_1, φ_2 *be Young functions. Assume also that at least one of the following conditions holds:*

(i) $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ *for all arguments,*

(ii) $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ *for large arguments and* $\mathscr{M} \hookrightarrow E(\mathscr{M})$,

(iii) $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ *for small arguments and* $E(\mathscr{M}) \hookrightarrow \mathscr{M}$.

Then $E_{\varphi_2}(\mathscr{M}) \hookrightarrow M(E_{\varphi_1}(\mathscr{M}), E_\varphi(\mathscr{M}))$.

Proof. (i). Let $x \in E_{\varphi_1}(\mathscr{M})$ and $y \in E_{\varphi_2}(\mathscr{M})$, then $\mu(x) \in E_{\varphi_1}$ and $\mu(y) \in E_{\varphi_2}$. Since $D_2 : E_{\varphi_1} \to E_{\varphi_1}$ and $D_2 : E_{\varphi_2} \to E_{\varphi_2}$ are bounded, then $D_2(\mu_t(x)) \in E_{\varphi_1}$ and $D_2(\mu_t(y)) \in E_{\varphi_2}$. By Theorem 4.1 of [9], we have $D_2(\mu(x))D_2(\mu(y)) \in E_\varphi$. It follows from Lemma 2.5 of [6] that

$$\mu_t(yx) \leqslant \mu_{\frac{t}{2}}(x)\mu_{\frac{t}{2}}(y) = D_2(\mu_t(x))D_2(\mu_t(y)),$$

and so $\mu(yx) \in E_\varphi$, i.e., $yx \in E_\varphi(\mathscr{M})$. Therefore, by Theorem 4.1 of [9], we deduce

$$\|yx\|_{E_\varphi(\mathscr{M})} = \|\mu(yx)\|_{E_\varphi} \leqslant c\|D_2\|_{E_{\varphi_1}\to E_{\varphi_1}}\|D_2\|_{E_{\varphi_2}\to E_{\varphi_2}}\|x\|_{E_{\varphi_1}(\mathscr{M})}\|y\|_{E_{\varphi_2}(\mathscr{M})}, \quad (3.1)$$

where the constant c is taken from the proof of Theorem 4.1 in [9]. From inequality (3.1) we obtain

$$\|y\|_{\mathscr{M}} = \sup\{\|yx\|_\varphi : \|x\|_{\varphi_1} \leqslant 1\} \leqslant A\|y\|_{\varphi_2}, \quad (3.2)$$

where $A = c\|D_2\|_{E_{\varphi_1}\to E_{\varphi_1}}\|D_2\|_{E_{\varphi_2}\to E_{\varphi_2}}$.

On the other hand, the fact $\mathscr{M} \hookrightarrow E(\mathscr{M})$ and $E(\mathscr{M}) \hookrightarrow \mathscr{M}$ imply $L^\infty \hookrightarrow E$ and $E \hookrightarrow L^\infty$, respectively. Thus, a similar discussion to the proof of the case (i) shows that the inequality (3.2) holds for the case (ii) and (iii). This completes the proof. □

The idea of the proof of the following theorem is taken from Theorem 5.2 of [9] and Theorem 1 of [15].

THEOREM 2. *Let E be a symmetric function space with Fatou property and φ, φ_1, φ_2 be Young functions with $b_{\varphi_2} = b_{\varphi_1} = b_{\varphi}$. Assume also that at least one of the following conditions holds:*

(i) $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$ *for all arguments,*

(ii) $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$ *for large arguments and* $\mathscr{M} \hookrightarrow E(\mathscr{M})$,

(iii) $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$ *for small arguments and* $E(\mathscr{M}) \hookrightarrow \mathscr{M}$.

Then $M(E_{\varphi_1}(\mathscr{M}), E_{\varphi}(\mathscr{M})) \hookrightarrow E_{\varphi_2}(\mathscr{M})$.

Proof. (i): Let $\varphi, \varphi_2 \in \mathscr{Y}_1 \cup \mathscr{Y}_2$ and let

$$x \in K = \left\{ x : \begin{array}{l} x = \sum_{k=1}^{n} c_k e_k,\ c_k \in \mathbb{C}, \\ e_k \in P(\mathscr{M}), e_k \perp e_j,\ \text{if } k \neq j, \tau(e_k) < \infty,\ \ j,k = 1,2,\cdots,n \end{array} \right\}$$

with $x \neq 0$. Since $M(E_{\varphi_1}(\mathscr{M}), E_{\varphi}(\mathscr{M}))$ and $E_{\varphi_2}(\mathscr{M})$ are Banach spaces, then $x \in M(E_{\varphi_1}(\mathscr{M}), E_{\varphi}(\mathscr{M}))$ and $x \in E_{\varphi_2}(\mathscr{M})$. By Lemma 3.1, we have $\rho_{\varphi_2}(\frac{\mu(x)}{\|x\|_{\varphi_2}}) = 1$. Thus $\frac{\|x\|}{\|x\|_{\varphi_2}} \leqslant b_{\varphi_2}$. Indeed, let $b_{\varphi_2} < \infty$. If for some $t_0 > 0$ we had $\frac{\mu_{t_0}(x)}{\|x\|_{\varphi_2}} > b_{\varphi_2}$, then $\frac{\mu_t(x)}{\|x\|_{\varphi_2}} \geqslant \frac{\mu_{t_0}(x)}{\|x\|_{\varphi_2}} > b_{\varphi_2}$ for all $0 < t \leqslant t_0$, which would force

$$\rho_{\varphi_2}\left(\frac{\mu_t(x)}{\|x\|_{\varphi_2}}\right) \geqslant \rho_{\varphi_2}\left(\frac{\mu_t(x)}{\|x\|_{\varphi_2}}\chi_{(0,t_0]}\right) = \rho_{\varphi_2}(\infty\chi_{(0,t_0]}) = \infty.$$

Thus we have $\frac{\mu_t(x)}{\|x\|_{\varphi_2}} \leqslant b_{\varphi_2}$, $t > 0$. This means that $\frac{\|x\|}{\|x\|_{\varphi_2}} = \lim_{t\to 0} \frac{\mu_t(x)}{\|x\|_{\varphi_2}} \leqslant b_{\varphi_2}$. If $b_{\varphi_2} = \infty$, we clearly have that $\frac{\|x\|}{\|x\|_{\varphi_2}} \leqslant b_{\varphi_2}$. Therefore, $\frac{\|x\|}{\|x\|_{\varphi_2}} \leqslant b_{\varphi_2}$. Let $|x| = \int_0^{\|x\|} \lambda de_\lambda(|x|)$ be the spectral decomposition of $|x|$ and let

$$y = \varphi_2\left(\frac{|x|}{\|x\|_{\varphi_2}}\right) \in L_0(\mathscr{M})^+$$

and

$$z = \varphi_1^{-1}\left(\varphi_2\left(\frac{|x|}{\|x\|_{\varphi_2}}\right)\right) e_{[\|x\|_{\varphi_2} a_{\varphi_2}, \|x\|)}(|x|) \in L_0(\mathscr{M})^+.$$

Lemma 3.1 of [9] and the Borel functional calculus tell us that $\varphi_1(z) \leqslant y$ and $\rho_{\mathscr{M}}^{\varphi_1}(z) \leqslant \rho_{\mathscr{M}}^{\varphi_2}\left(\frac{|x|}{\|x\|_{\varphi_2}}\right) \leqslant 1$. This means that $\|z\|_{\varphi_1} \leqslant 1$ and $zx \in E_{\varphi}(\mathscr{M})$. The assumption (i) implies that there exists a constant $c > 0$ such that

$$\varphi^{-1}(t) \leqslant c\varphi_1^{-1}(t)\varphi_2^{-1}(t) \quad \text{for all } t > 0.$$

Combining this and the fact $\frac{\|x\|}{\|x\|_{\varphi_2}} \leqslant b_{\varphi_2}$ with Lemma 3.1 of [9], we obtain

$$\begin{aligned}
&\varphi\left(c\varphi_1^{-1}\left(\varphi_2\left(\frac{t}{\|x\|_{\varphi_2}}\right)\right)\chi_{[\|x\|_{\varphi_2}a_{\varphi_2},\|x\|)}(t)\frac{t}{\|x\|_{\varphi_2}}\right)\\
&=\varphi\left(c\varphi_1^{-1}\left(\varphi_2\left(\frac{t}{\|x\|_{\varphi_2}}\right)\right)\varphi_2^{-1}\left(\varphi_2\left(\frac{t}{\|x\|_{\varphi_2}}\right)\right)\right)\chi_{[\|x\|_{\varphi_2}a_{\varphi_2},\|x\|)}(t)\\
&\geqslant\varphi\left(\varphi^{-1}\left(\varphi_2\left(\frac{t}{\|x\|_{\varphi_2}}\right)\right)\right)\chi_{[\|x\|_{\varphi_2}a_{\varphi_2},\|x\|)}(t)\\
&=\varphi_2\left(\frac{t}{\|x\|_{\varphi_2}}\right)\chi_{[\|x\|_{\varphi_2}a_{\varphi_2},\|x\|)}(t).
\end{aligned}$$

On the other hand, if $0 \leqslant t < \|x\|_{\varphi_2}a_{\varphi_2}$, then $\varphi_2(\frac{t}{\|x\|_{\varphi_2}}) = 0$ and

$$\varphi\left(c\varphi_1^{-1}\left(\varphi_2\left(\frac{t}{\|x\|_{\varphi_2}}\right)\right)\chi_{[\|x\|_{\varphi_2}a_{\varphi_2},\|x\|)}(t)\frac{t}{\|x\|_{\varphi_2}}\right)=0.$$

Therefore,

$$\varphi\left(c\varphi_1^{-1}\left(\varphi_2\left(\frac{t}{\|x\|_{\varphi_2}}\right)\right)\chi_{[\|x\|_{\varphi_2}a_{\varphi_2},\|x\|)}(t)\frac{t}{\|x\|_{\varphi_2}}\right)\geqslant\varphi_2\left(\frac{t}{\|x\|_{\varphi_2}}\right)$$

holds for $0 \leqslant t < \|x\|$. Thus, by the Borel functional calculus, we deduce

$$\varphi\left(c\left|z\frac{|x|}{\|x\|_{\varphi_2}}\right|\right)=\varphi\left(cz\frac{|x|}{\|x\|_{\varphi_2}}\right)\geqslant\varphi_2\left(\frac{|x|}{\|x\|_{\varphi_2}}\right)=y.$$

Since

$$\begin{aligned}
\varphi\left(\mu\left(c|z\frac{x}{\|x\|_{\varphi_2}}|\right)\right)&=\varphi\left(c\mu\left(z\frac{x}{\|x\|_{\varphi_2}}\right)\right)=\varphi\left(c\mu\left(\frac{x}{\|x\|_{\varphi_2}}z\right)\right)\\
&=\varphi\left(c\mu\left(z\frac{x^*}{\|x\|_{\varphi_2}}\frac{x}{\|x\|_{\varphi_2}}z\right)^{\frac{1}{2}}\right)=\varphi\left(c\mu(z\frac{|x|^2}{\|x\|_{\varphi_2}^2}z)^{\frac{1}{2}}\right)\\
&=\varphi\left(\mu\left(cz\frac{|x|}{\|x\|_{\varphi_2}}\right)\right),
\end{aligned}$$

we have $\rho_{\mathscr{M}}^{\varphi}(cz\frac{x}{\|x\|_{\varphi_2}}) = \rho_{\mathscr{M}}^{\varphi}(cz\frac{|x|}{\|x\|_{\varphi_2}})$. From Lemma 1, we infer

$$\begin{aligned}
\rho_{\mathscr{M}}^{\varphi}\left(cz\frac{x}{\|x\|_{\varphi_2}}\right)&=\left\|\varphi\left(c\left|z\frac{|x|}{\|x\|_{\varphi_2}}\right|\right)\right\|_{E(\mathscr{M})}\geqslant\|y\|_{E(\mathscr{M})}\\
&=\left\|\varphi_2\left(\frac{x}{\|x\|_{\varphi_2}}\right)\right\|_{E(\mathscr{M})}=\rho_{\mathscr{M}}^{\varphi_2}\left(\frac{x}{\|x\|_{\varphi_2}}\right)=1.
\end{aligned}$$

Therefore, $\|x\|_{\mathscr{M}} \geqslant \|zx\|_{\varphi} \geqslant \frac{1}{c}\|x\|_{\varphi_2}$. Let $x \in M(E_{\varphi_1}(\mathscr{M}), E_{\varphi}(\mathscr{M}))$ and $|x| = \int_0^\infty \lambda de_\lambda(|x|)$ be the spectral decomposition of $|x|$. Let $f_k(t) = \sum_{j=1}^{n2^n} \frac{j-1}{2^n}\chi_{[\frac{j-1}{2^n},\frac{j}{2^n})}$, then

$0 \leqslant f_k(t) \leqslant f_{k+1}(t) \leqslant t$. It is clear that $0 \leqslant f_k(t)\chi_{[\frac{1}{k},\infty)} \leqslant f_{k+1}(t)\chi_{[\frac{1}{k+1},\infty)} \leqslant t$. We write

$$x_n = f_n(|x|)e_{[\frac{1}{n},\infty)}(|x|) = (\sum_{j=1}^{n2^n} \frac{j-1}{2^n} e_{[\frac{j-1}{2^n},\frac{j}{2^n})}(|x|))e_{[\frac{1}{n},\infty)}(|x|).$$

It follows that $x_n \leqslant x_{n+1} \leqslant |x|$, $n = 1,2,3\cdots$ and

$$\begin{aligned}\mu_{3t}(x-x_n) &\leqslant \mu_t(|x|e_{[0,\frac{1}{n})}) + \mu_t(|x|e_{[\frac{1}{n},n)}(|x|)-x_n) + \mu_t(|x|e_{[n,\infty)}(|x|))\\ &\leqslant \frac{1}{n} + \frac{1}{2^n} + \mu_t(|x|)\chi_{(0,\tau(e_{[n,\infty)}(|x|)))}.\end{aligned}$$

From Proposition 21 of [Chapter I, [21]], we have $\tau(e_{[n,\infty)}(|x|)) \to 0$ as $n \to \infty$, and so $\mu_{3t}(x-x_n) \to 0$ as $n \to \infty$. From [Lemma 3.1, Lemma 2.5 and Lemma 3.4 of [6]] we deduce $x_n \uparrow |x|$ in the measure topology and $\mu(x_n) \uparrow \mu(x)$. It is clear that $\|x\|_{\mathscr{M}} \geqslant \|x_n\|_{\mathscr{M}} \geqslant \frac{1}{c}\|x_n\|_{\varphi_2}$. The Fatou property of $E_\varphi(\mathscr{M})$ tells us that $\|x\|_{\mathscr{M}} \geqslant \frac{1}{c}\|x\|_{\varphi_2}$.

If φ or φ_2 is in $\mathscr{Y}_3$, we consider only the case that both φ and φ_2 are in $\mathscr{Y}_3$, since other cases are similar. For $0<\delta<1$, there exist $\psi,\psi_2 \in \mathscr{Y}_2$ such that

$$\psi(\delta t) \leqslant \varphi(t) \leqslant \psi(t),\ \psi_2(\delta t) \leqslant \varphi_2(t) \leqslant \psi_2(t)$$

for all $t>0$ (cf. property (iv) of [p. 254, [15]]). This ensures that

$$\delta\varphi^{-1}(t) \leqslant \psi^{-1}(t) \leqslant \varphi^{-1}(t),\ \ \delta\varphi_2^{-1}(t) \leqslant \psi_2^{-1}(t) \leqslant \varphi_2^{-1}(t),\ \ 0<\delta<1$$

and

$$\delta\|x\|_\psi \leqslant \|x\|_\varphi \leqslant \|x\|_\psi,\ \ \delta\|x\|_{\psi_2} \leqslant \|x\|_{\varphi_2} \leqslant \|x\|_{\psi_2},\ \ 0<\delta<1.$$

Moreover,

$$\delta\|x\|_{M(E_{\varphi_1}(\mathscr{M}),E_\psi(\mathscr{M}))} \leqslant \|x\|_{M(E_{\varphi_1}(\mathscr{M}),E_\varphi(\mathscr{M}))} \leqslant \|x\|_{M(E_{\varphi_1}(\mathscr{M}),E_\psi(\mathscr{M}))},\ \ 0<\delta<1.$$

Therefore, the fact $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ implies that

$$\psi^{-1}(t) \leqslant \frac{c}{\delta}\varphi_1^{-1}(t)\psi_2^{-1}(t),\ \ 0<\delta<1.$$

It follows from the above case that $\|x\|_{M(E_{\varphi_1}(\mathscr{M}),E_\psi(\mathscr{M}))} \geqslant \frac{\delta}{c}\|x\|_{\psi_2}$, $0<\delta<1$. Thus,

$$\|x\|_{M(E_{\varphi_1}(\mathscr{M}),E_\varphi(\mathscr{M}))} \geqslant \frac{\delta^2}{c}\|x\|_{\varphi_2}$$

holds for all $0<\delta<1$. This indicates that

$$\|x\|_{M(E_{\varphi_1}(\mathscr{M}),E_\varphi(\mathscr{M}))} \geqslant \frac{1}{c}\|x\|_{\varphi_2}.$$

(ii): If $\mathscr{M} \hookrightarrow E(\mathscr{M})$, put $\alpha > a_{\varphi_2}$ with $\varphi_2(\alpha)\|1\|_{E(\mathscr{M})} < \frac{1}{2}$. For this $\varphi_2(\alpha)>0$, Lemma 3.2 of [9] indicates that there exists $c_1 \geqslant c$ such that

$$\varphi^{-1}(t) \leqslant c_1\varphi_1^{-1}(t)\varphi_2^{-1}(t) \tag{3.3}$$

for any $t \geqslant \varphi_2(\alpha)$, where the constant c is taken from the definition of $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$ for large arguments. Thus

$$\rho_{\varphi_2}(ze_{[\alpha,\infty}(|z|)) \geqslant \frac{1}{2} \tag{3.4}$$

for each z with $\rho_{\varphi_2}(z) = 1$. Otherwise, the inequality $\mu_{\lambda_t(z)}(z) \leqslant t$ ensures that

$$1 = \rho_{\mathcal{M}}^{\varphi_2}(z) \leqslant \rho_{\mathcal{M}}^{\varphi_2}(ze_{[\alpha,\infty)}(|z|)) + \rho_{\mathcal{M}}^{\varphi_2}(ze_{[0,\alpha)}(|z|)) < \frac{1}{2} + \varphi_2(\alpha)\|1\|_{E(\mathcal{M})} < 1,$$

and we get a contradiction. Let $\varphi, \varphi_2 \in \mathscr{Y}_1 \cup \mathscr{Y}_2$ and $x \in K$. Then we have

$$x \in M(E_{\varphi_1}(\mathcal{M}), E_{\varphi}(\mathcal{M})) \text{ and } x \in E_{\varphi_2}(\mathcal{M}).$$

A similar discussion to the proof of the case (i) shows that $\frac{\||x|\|}{\|x\|_{\varphi_2}} \leqslant b_{\varphi_2}$. Moreover, we have $\varphi_2(\frac{|x|}{\|x\|_{\varphi_2}}) \in L_0(\mathcal{M})$. We write $y = \varphi_2(\frac{|x|}{\|x\|_{\varphi_2}})$ and $z = \varphi_1^{-1}(y)$. It follows from Lemma 3.1 of [9] and Borel functional calculus that $\varphi_1(z) \leqslant y$, and so

$$\rho_{\mathcal{M}}^{\varphi_1}(z) \leqslant \rho_{\mathcal{M}}^{\varphi_2}\left(\frac{|x|}{\|x\|_{\varphi_2}}\right) = 1.$$

Then $\|z\|_{\varphi_1} \leqslant 1$ and $zx \in E_{\varphi}(\mathcal{M})$. Applying Lemma 3.1 of [9], we infer

$$\frac{t}{\|x\|_{\varphi_2}}\chi_{[\alpha\|x\|_{\varphi_2},\infty)\cap\sigma(|x|)}(t) \geqslant \varphi_2^{-1}\left(\varphi_2\left(\frac{t}{\|x\|_{\varphi_2}}\right)\right)\chi_{[\alpha\|x\|_{\varphi_2},\infty)\cap\sigma(|x|)}(t).$$

Together this with Borel functional calculus, we deduce $\frac{|x|}{\|x\|_{\varphi_2}}e_1 \geqslant \varphi_2^{-1}(y)e_1$,

$$\begin{aligned}\varphi\left(2c_1|e_1z\frac{|x|}{\|x\|_{\varphi_2}}|\right) = \varphi\left(2c_1e_1z\frac{|x|}{\|x\|_{\varphi_2}}\right) &\geqslant \varphi(2c_1e_1\varphi_1^{-1}(y)\varphi_2^{-1}(y)e_1) \\ &\geqslant \varphi(2\varphi^{-1}(y))e_1 \geqslant 2\varphi(\varphi^{-1}(y))e_1 = 2ye_1,\end{aligned}$$

where $e_1 = e_{[\alpha\|x\|_{\varphi_2},\infty)}(|x|)$. Since

$$\varphi\left(\mu\left(c\left|e_1z\frac{x}{\|x\|_{\varphi_2}}\right|\right)\right) = \varphi\left(\mu\left(ce_1z\frac{|x|}{\|x\|_{\varphi_2}}\right)\right),$$

Applying Lemma 1, we have $\rho_{\mathcal{M}}^{\varphi_2}(\frac{x}{\|x\|_{\varphi_2}}) = 1$. Consequently, by (3.4), we obtain

$$\rho_{\mathcal{M}}^{\varphi}\left(2c_1e_1z\frac{|x|}{\|x\|_{\varphi_2}}e_1\right) \geqslant 2\rho_{\mathcal{M}}^{\varphi_2}\left(\frac{|x|}{\|x\|_{\varphi_2}}e_1\right) \geqslant 1.$$

Thus $\|zx\|_{\varphi} \geqslant \|e_1zx\|_{\varphi} \geqslant \frac{1}{2c_1}\|x\|_{\varphi_2}$, which means that

$$\|x\|_{\mathcal{M}} \geqslant \frac{1}{2c_1}\|x\|_{\varphi_2}.$$

For $x \in M(E_{\varphi_1}(\mathscr{M}), E_{\varphi}(\mathscr{M}))$, a similar discussion to the proof of the case (i) shows that there exists $\{x_n\} \subseteq K$ such that $0 \leqslant x_n \uparrow |x|$ in measure topology. Thus the Fatou property of $E_{\varphi_2}(\mathscr{M})$ implies that $\|x\|_{\mathscr{M}} \geqslant \frac{1}{2c_1}\|x\|_{\varphi_2}$ holds for all $x \in M(E_{\varphi_1}(\mathscr{M}), E_{\varphi}(\mathscr{M}))$.

If φ or φ_2 is in $\mathscr{Y}_3$, we consider only the case that both φ and φ_2 are in $\mathscr{Y}_3$, since other cases are similar. For $0 < \delta < 1$, there exist $\psi, \psi_2 \in \mathscr{Y}_2$ such that

$$\psi(\delta t) \leqslant \varphi(t) \leqslant \psi(t), \ \psi_2(\delta t) \leqslant \varphi_2(t) \leqslant \psi_2(t)$$

for all $t > 0$ (cf. property (iv) of [p. 254, [15]]). Using a similar discussion to the proof of the case (i), we have $\|x\|_{\mathscr{M}} \geqslant \frac{1}{2c_1}\|x\|_{\varphi_2}$.

(iii): If $E_{\varphi}(\mathscr{M}) \subseteq \mathscr{M}$, put $x \in E_{\varphi}(\mathscr{M})$ with $\|x\|_{\varphi} \leqslant 1$, then $\|x\| \leqslant b_{\varphi}$ and

$$\varphi(|x|) = \int_0^{\infty} \varphi(\lambda) de_{\lambda} \in \mathscr{M}.$$

Write $r = \max\{\|\varphi(|x|)\|, \|x\|\}$, by Lemma 3.2 of [9], there exists a constant $c_2 > c$ such that $\varphi^{-1}(t) \leqslant c_2 \varphi_1^{-1}(t) \varphi_2^{-1}(t)$ for any $0 < t \leqslant r$. The rest of the proof goes as in case (i). □

THEOREM 3. *Let E be a symmetric function space with Fatou property and φ, φ_1, φ_2 be Young functions with $b_{\varphi_2} = b_{\varphi_1} = b_{\varphi}$. Assume also that at least one of the following conditions holds:*

(i) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ *for all arguments,*

(ii) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ *for large arguments and* $\mathscr{M} \hookrightarrow E(\mathscr{M})$,

(iii) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ *for small arguments and* $E(\mathscr{M}) \hookrightarrow \mathscr{M}$.

Then $M(E_{\varphi_1}(\mathscr{M}), E_{\varphi}(\mathscr{M})) = E_{\varphi_2}(\mathscr{M})$.

Proof. It follows immediately from Theorem 1 and Theorem 2. □

4. The product of noncommutative Calderón-Lozanovskiĭ spaces

Given two noncommutative symmetric function spaces $E(\mathscr{M})$ and $F(\mathscr{M})$. We define the product space $E(\mathscr{M}) \cdot F(\mathscr{M})$ as

$$E(\mathscr{M}) \cdot F(\mathscr{M}) = \{xy : x \in E(\mathscr{M}) \text{ and } y \in F(\mathscr{M})\}$$

with a functional $\|\cdot\|_{E(\mathscr{M})\cdot F(\mathscr{M})}$ defined by

$$\|x\|_{E(\mathscr{M})\cdot F(\mathscr{M})} = \inf\{\|y\|_{E(\mathscr{M})}\|z\|_{F(\mathscr{M})} : x = yz, y \in E(\mathscr{M}), z \in F(\mathscr{M})\},$$

where the product xy being the closure of the algebraic product. Pointwise product of some Banach ideal spaces were investigated by several authors, see for example [10, 12, 15, 20].

REMARK 1.

(i) If $x \in E(\mathcal{M}) \cdot F(\mathcal{M})$, then $\|x\|_{E(\mathcal{M})\cdot F(\mathcal{M})} = \||x|\|_{E(\mathcal{M})\cdot F(\mathcal{M})}$.

(ii) If $0 \leqslant x \leqslant y$, then $\|x\|_{E(\mathcal{M})\cdot F(\mathcal{M})} \leqslant \|y\|_{E(\mathcal{M})\cdot F(\mathcal{M})}$.

THEOREM 4. *Let E be a symmetric function space with Fatou property and φ, φ_1, φ_2 be Young functions. Assume also that at least one of the following conditions holds:*

(i) $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ *for all arguments,*

(ii) $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ *for large arguments and* $\mathcal{M} \hookrightarrow E(\mathcal{M})$,

(iii) $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ *for small arguments and* $E(\mathcal{M}) \hookrightarrow \mathcal{M}$.

Then $E_{\varphi_2}(\mathcal{M}) \cdot E_{\varphi_1}(\mathcal{M}) \hookrightarrow E_{\varphi}(\mathcal{M})$.

Proof. Let $y \in E_{\varphi_1}(\mathcal{M})$ and $z \in E_{\varphi_2}(\mathcal{M})$. Using Theorem A of [10], we get

$$\begin{aligned}
\|zy\|_{\varphi} &\leqslant \|D_2\|_{E_\varphi \to E_\varphi} \|\mu(y)\mu(z)\|_{E_\varphi} \\
&\leqslant c\|D_2\|_{E_\varphi \to E_\varphi} \|\mu(y)\mu(z)\|_{E_{\varphi_1}\cdot E_{\varphi_2}} \\
&\leqslant c\|D_2\|_{E_\varphi \to E_\varphi} \|\mu(y)\|_{E_{\varphi_1}} \|\mu(z)\|_{F_{\varphi_1}} \\
&= c\|D_2\|_{E_\varphi \to E_\varphi} \|y\|_{E_{\varphi_1}(\mathcal{M})} \|z\|_{F_{\varphi_1}(\mathcal{M})},
\end{aligned}$$

where the constant c is taken from the Theorem A of [10]. This ensures that

$$\|zy\|_{\varphi} \leqslant c\|D_2\|_{E_\varphi \to E_\varphi} \|zy\|_{E_{\varphi_2}(\mathcal{M})\cdot E_{\varphi_1}(\mathcal{M})}. \qquad \square$$

The idea of the proof of the following theorem is derived from that of the Theorem 5 of [10].

THEOREM 5. *Let E be a symmetric function space with Fatou property and φ, φ_1, φ_2 be Young functions with $b_{\varphi_2} = b_{\varphi_1} = b_{\varphi}$. Assume also that at least one of the following conditions holds:*

(i) $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$ *for all arguments,*

(ii) $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$ *for large arguments and* $\mathcal{M} \hookrightarrow E(\mathcal{M})$,

(iii) $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$ *for small arguments and* $E(\mathcal{M}) \hookrightarrow \mathcal{M}$.

Then $E_{\varphi}(\mathcal{M}) \hookrightarrow E_{\varphi_2}(\mathcal{M}) \cdot E_{\varphi_1}(\mathcal{M})$.

Proof. (i): Let $0 \neq x \in E_{\varphi}(\mathcal{M})$. Then $\|\frac{x}{\|x\|_{\varphi}}\|_{\varphi} = 1$. It follows from Lemma 1 that $\rho_{\mathcal{M}}^{\varphi}(\frac{|x|}{\|x\|_{\varphi}}) \leqslant 1$. A similar argument to the proof of Theorem 2 implies that $\frac{\|x\|}{\|x\|_{\varphi}} \leqslant b_{\varphi}$ and so $\varphi(\frac{|x|}{\|x\|_{\varphi}}) \in L_0(\mathcal{M})$. We write $y = \varphi(\frac{|x|}{\|x\|_{\varphi}})$. Put

$$g_i(t) = \begin{cases} \left(\frac{t}{\varphi_1^{-1}\left(\varphi\left(\frac{t}{\|x\|_{\varphi}}\right)\right)\varphi_2^{-1}\left(\varphi\left(\frac{t}{\|x\|_{\varphi}}\right)\right)} \right)^{\frac{1}{2}} \varphi_i^{-1}\left(\varphi\left(\frac{t}{\|x\|_{\varphi}}\right)\right), & \text{if } t \in \sigma(|x|), \\ 0, & \text{otherwise}. \end{cases} \tag{4.1}$$

The proof of Theorem 5 of [10] shows that

$$\frac{g_i(t)}{c^{\frac{1}{2}}\|x\|_{\varphi}^{\frac{1}{2}}} \leqslant \varphi_i^{-1}\left(\varphi\left(\frac{t}{\|x\|_{\varphi}}\right)\right),$$

it follows from Lemma 3.1 of [9] that

$$\varphi_i\left(\frac{g_i(t)}{c^{\frac{1}{2}}\|x\|_{\varphi}^{\frac{1}{2}}}\right) \leqslant \varphi\left(\frac{t}{\|x\|_{\varphi}}\right),$$

where the constant c is taken from the proof of Theorem 5 of [10]. Then, the Borel functional calculus indicates that $\varphi_i(\frac{x_i}{c^{\frac{1}{2}}\|x\|_{\varphi}^{\frac{1}{2}}}) \leqslant y$, $i = 1,2$, where $x_i = g_i(|x|)$. This implies that

$$\rho_{\mathcal{M}}^{\varphi_i}\left(\frac{x_i}{c^{\frac{1}{2}}\|x\|_{\varphi}^{\frac{1}{2}}}\right) \leqslant \|y\|_{E(\mathcal{M})} = \rho_{\mathcal{M}}^{\varphi}\left(\frac{z}{\|x\|_{\varphi}}\right) \leqslant 1, \quad i = 1,2.$$

Therefore, $\|x_i\|_{\varphi_i} \leqslant (c\|x\|_{\varphi})^{\frac{1}{2}}$, $i = 1,2$. Consequently,

$$|x| = x_2 x_1 \quad \text{and} \quad \|x\|_{E_{\varphi_2}(\mathcal{M}) \cdot E_{\varphi_1}(\mathcal{M})} \leqslant c\|x\|_{\varphi}.$$

That is $x \in E_{\varphi_2}(\mathcal{M}) \cdot E_{\varphi_1}(\mathcal{M})$.

(ii): First we assume that $b_{\varphi} < \infty$. Since $\mathcal{M} \hookrightarrow E(\mathcal{M})$, then $L^{\infty} \hookrightarrow E$, which means $E_{\varphi} = L^{\infty}$. It is clear that $E_{\varphi}(\mathcal{M}) = \mathcal{M}$. For $x \in \mathcal{M}$, by the fact $\mathcal{M} \hookrightarrow E(\mathcal{M})$ and Borel functional calculus, there exists $\lambda_i > 0$ such that

$$\varphi_i(\lambda_i|x|) \in \mathcal{M} \hookrightarrow E(\mathcal{M}), \ i = 1,2.$$

This tells us that $\mathcal{M} \hookrightarrow E_{\varphi_i}(\mathcal{M})$, $i = 1,2$. Thus, $E_{\varphi}(\mathcal{M}) = \mathcal{M} \hookrightarrow E_{\varphi_2}(\mathcal{M}) \cdot E_{\varphi_1}(\mathcal{M})$.

If $b_{\varphi} = \infty$, set $s_0 = \varphi^{-1}(t_0)$, where t_0 and the constant c are taken from the definition of $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$ for large arguments. Take $s > 0$ with

$$\max\{\varphi_1(s), \varphi_2(s)\}\|1\|_{E(\mathcal{M})} \leqslant \frac{1}{2}.$$

For $\|x\|_\varphi = 1$, we write $y = \varphi(|x|) \in L_0(\mathcal{M})$, $e_1 = e_{[s_0,\infty)}(|x|), e_2 = e_{[0,s_0)}(|x|)$ and define

$$g_i(t) = \begin{cases} \left(\frac{t}{\varphi_1^{-1}(\varphi(t))\varphi_2^{-1}(\varphi(t))} \right)^{\frac{1}{2}} \varphi_i^{-1}(\varphi(t)), & \text{if } t \in \sigma(|x|) \cap (s_0, \infty), \\ t^{\frac{1}{2}}, & t \in \sigma(|x|) \cap [0, s_0], \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

for $i = 1, 2$. Since $\varphi(s_0) > 0$, the functions $g_i(t)$ are well defined. From $\varphi_1^{-1}\varphi_2^{-1} \succ \varphi^{-1}$ for large arguments and Lemma 3.1 of [9], we obtain

$$g_i(t) \leqslant \left(\frac{ct}{\varphi^{-1}(\varphi(t))} \right)^{\frac{1}{2}} \varphi_i^{-1}(\varphi(t)) \leqslant c^{\frac{1}{2}} \varphi_i^{-1}(\varphi(t)), \;\; t \in \sigma(|x|) \cap (s_0, \infty).$$

Therefore,

$$\varphi_1 \left(\frac{g_1(t)}{2c^{\frac{1}{2}}} \chi_{[s_0,\infty)}(t) \right) \leqslant \frac{1}{2} \varphi_1(\varphi_1^{-1}(\varphi(t))\chi_{[s_0,\infty)}(t)) \leqslant \frac{1}{2}\varphi(t),$$

where the constant c is taken from the case (i). Put $x_i = g_i(|x|)$ and $y = \varphi(|x|)$. It follows from Borel functional calculus that

$$\rho_{\mathcal{M}}^{\varphi_1} \left(\frac{x_1}{2c^{\frac{1}{2}}} e_1 \right) \leqslant \frac{1}{2} \rho_{\mathcal{M}}^{\varphi_1} \left(\frac{x_1}{c^{\frac{1}{2}}} e_1 \right) \leqslant \frac{1}{2} \|y\|_\varphi \leqslant \frac{1}{2}.$$

Since $\mu_{\tau(e_1)}(x_1) \leqslant s_0$, we obtain

$$\begin{aligned} \rho_{\mathcal{M}}^{\varphi_1} \left(\frac{s x_1 e_2}{s_0} \right) &= \left\| \varphi_1 \left(\mu \left(\frac{s x_1 e_2}{s_0} \right) \right) \right\|_E \leqslant \left\| \varphi_1 \left(\frac{s}{s_0} \mu(x_1) \chi_{[\tau(e_1),\infty)} \right) \right\|_E \\ &\leqslant \left\| \varphi_1 \left(\frac{s}{s_0} \mu_{\tau(e_1)}(x_1) \chi_{[0,\infty)} \right) \right\|_E \leqslant \left\| \varphi_1 \left(\frac{s}{s_0} \mu_{\tau(e_1)}(x_1) \chi_{[0,\infty)} \right) \right\|_E \\ &\leqslant \|\varphi_1(s)\chi_{[0,\infty)}\|_E = \varphi_1(s)\|1\|_{E(\mathcal{M})} \leqslant \frac{1}{2}. \end{aligned}$$

Set $\lambda = \max\{\frac{s_0}{s}, 2c^{\frac{1}{2}}\}$ and we deduce

$$\begin{aligned} \rho_{\mathcal{M}}^{\varphi_1} \left(\frac{x_1}{\lambda} \right) &\leqslant \rho_{\mathcal{M}}^{\varphi_1} \left(\frac{x_1}{\lambda} e_1 \right) + \rho_{\mathcal{M}}^{\varphi_1} \left(\frac{x_1}{\lambda} e_2 \right) \\ &\leqslant \rho_{\mathcal{M}}^{\varphi_1} \left(\frac{x_1}{2c^{\frac{1}{2}}} e_1 \right) + \rho_{\mathcal{M}}^{\varphi_1} \left(\frac{s x_1}{s_0} e_2 \right) \leqslant 1. \end{aligned}$$

Hence, $\|x_1\|_{\varphi_1} \leqslant \lambda$ and similarly $\|x_2\|_{\varphi_2} \leqslant \lambda$. On the other hand, it is clear that $g_1(t)g_2(t) = t, t \in \sigma(|x|)$. This means that $|x| = x_2x_1$ and so $x \in E_{\varphi_2}(\mathcal{M}) \cdot E_{\varphi_1}(\mathcal{M})$. Moreover, $\|x\|_{E_{\varphi_2}(\mathcal{M}) \cdot E_{\varphi_1}(\mathcal{M})} \leqslant \lambda^2 \|x\|_\varphi$ holds for every $x \in E_\varphi(\mathcal{M})$.

(iii): The Lemma 3.2 of [9] and the fact $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for small arguments imply that for every $t_1 > t_0$, there exists a constant $c_1 \geqslant c$ such that $\varphi^{-1}(t) \leqslant c_1 \varphi_1^{-1}(t)\varphi_2^{-1}(t)$

for all $t \leqslant t_1$. By the fact $E(\mathscr{M}) \hookrightarrow \mathscr{M}$, we have $E_\varphi(\mathscr{M}) \subseteq \mathscr{M}$. For $x \in E_\varphi(\mathscr{M})$ with $\|x\|_\varphi = 1$, we get $\sigma(|x|) \subseteq [0, \|x\|]$. Put $t_1 = \|x\|$, the rest of the proof goes as in case (i). $\square$

Combining Theorem 3 and Theorem 4 with Theorem 5, we obtain the following result.

THEOREM 6. *Let E be a symmetric function space with Fatou property and φ, φ_1, φ_2 be Young functions with $b_{\varphi_2} = b_{\varphi_1} = b_\varphi$. Assume also that at least one of the following conditions holds:*

(i) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ *for all arguments,*

(ii) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ *for large arguments and* $\mathscr{M} \hookrightarrow E(\mathscr{M})$,

(iii) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ *for small arguments and* $E(\mathscr{M}) \hookrightarrow \mathscr{M}$.

Then $E_{\varphi_2}(\mathscr{M}) \cdot E_{\varphi_1}(\mathscr{M})$ is a quasi-Banach space and $E_{\varphi_2}(\mathscr{M}) \cdot E_{\varphi_1}(\mathscr{M}) = E_\varphi(\mathscr{M})$. Moreover,

$$E_\varphi(\mathscr{M}) = M(E_{\varphi_1}(\mathscr{M}), E_\varphi(\mathscr{M})) \cdot E_{\varphi_1}(\mathscr{M}) = (E_{\varphi_2} \cdot E_{\varphi_1})(\mathscr{M}).$$

5. Normability of the product spaces $E_{\varphi_1}(\mathscr{M}) \cdot E_{\varphi_2}(\mathscr{M})$

LEMMA 2. *Let E and F be two symmetric function spaces with the Fatou property. Let $x \in L_0(\mathscr{M})$ with $\mu_t(x) \in (E \cdot F)'$ and $\mu_t(x) \neq 0, a.e.$, we define*

$$L_x^1(\mathscr{M}) = \{y \in L_0(\mathscr{M}) : \mu(y) \in L^1((0,\infty), \mu_t(x)dt)\},$$

with the norm $\|y\|_{L_x^1(\mathscr{M})} = \|\mu(y)\|_{L^1((0,\infty),\mu_t(x)dt)}$. Then $L_x^1(\mathscr{M})$ is a Banach space and $L_x^1(\mathscr{M}) \hookrightarrow L_0(\mathscr{M})$.

Proof. For convenience, we denote $L^1((0,\infty), \mu_t(x)dt)$ by L_x^1. Given $z, y \in L_x^1(\mathscr{M})$. By Theorem 4.4 of [6] and Proposition 3.6 of [Chapter 2, [1]], we have

$$\int_0^\infty \mu_s(z+y)\mu_s(x)ds \leqslant \int_0^\infty (\mu_s(z) + \mu_s(y))\mu_s(x)ds, \ t > 0.$$

That is $\|y+z\|_{L_x^1(\mathscr{M})} \leqslant \|y\|_{L_x^1(\mathscr{M})} + \|z\|_{L_x^1(\mathscr{M})}$. If $\|z\|_{L_x^1(\mathscr{M})} = 0$, then we have

$$\|\mu(z)\mu(x)\|_{L^1(\mathbb{R}^+)} = \|z\|_{L_x^1(\mathscr{M})} = 0.$$

Combining this with the fact $\mu(x)$ is non-zero on $(0, t_x)$, we obtain $\mu(z) = 0$, a.e. on $(0, t_x)$ and so $z = 0$, where $t_x = \inf\{t > 0 : \mu_t(x) = 0\}$. Thus, $\|\cdot\|_{L_x^1(\mathscr{M})}$ is a norm. Applying a similar proof to that of Theorem 7.1 of [18], we obtain $L_x^1(\mathscr{M})$ is a Banach space and $L_x^1(\mathscr{M}) \hookrightarrow L_0(\mathscr{M})$. Indeed, let $\varepsilon, \delta > 0$, $\zeta = \varepsilon\|\chi_{[0,\delta)}\|_{L_x^1}$ and let $y \in L_x^1(\mathscr{M})$ with $\|y\|_{L_x^1(\mathscr{M})} \leqslant \zeta$. Since $\mu_t(y) \geqslant \mu_\delta \chi_{[0,\delta)} + \mu_t(y)\chi_{[\delta,\infty)}$, we conclude that

$\mu_\delta(y)\|\chi_{[0,\delta)}\|_{L^1_x} \leqslant \|y\|_{L^1_x(\mathscr{M})}$. This implies that $\mu_\delta(y) \leqslant \varepsilon$ and so $L^1_x(\mathscr{M}) \hookrightarrow L_0(\mathscr{M})$. Let $\{y_n\}_1^\infty$ be a Cauchy sequence in $L^1_x(\mathscr{M})$, then $\{y_n\}_1^\infty$ is a Cauchy sequence in the measure topology. Hence, there exists $y \in L_0(\mathscr{M})$ such that $y_n \to y$, $n \to \infty$ in the measure topology. Theorem 6.6 of [18] and Proposition 3.6 of [Chapter 2, [1]] tell us that $\{\mu(y_n)\}_1^\infty$ is a Cauchy sequence in L^1_x. Thus, there exists $f \in L^1_x$ such that $\|\mu(y_n) - f\|_{L^1_x} \to 0$, $n \to \infty$. Then it is clear that $\mu(y_n) \to f$ a.e. on $(0,t_x)$. In fact, for each $A \subset (0,t_x)$, we write $\nu_x(A) = \int_0^\infty \chi_A(t)\mu_t(x)dt$. Set $\nu_x(A) = 0$, since $\mu(x)$ is non-zero on $(0,t_x)$, we have $m(A) = 0$, where m denotes Lebesgue measure. On the other hand, It follows from Lemma 3.4 of [6] that $\mu(y_n) \to \mu(y)$ a.e. on $(0,\infty)$, which means that $f = \mu(y)$ a.e. and $y \in L^1_x(\mathscr{M})$. By Theorem 6.6 of [18] and Proposition 3.6 of [Chapter 2, [1]], we obtain

$$\begin{aligned}\|\mu(y-y_n) - \mu(y-y_m)\|_{L^1_x} &\leqslant \|\mu(y_m - y_n)\|_{L^1_x} \\ &= \|y_m - y_n\|_{L^1_x(\mathscr{M})}.\end{aligned}$$

Hence $\{\mu(y-y_n)\}_1^\infty$ is a Cauchy sequence in L^1_x. Thus, there exists $g \in L^1_x$ such that $\|\mu(y-y_n) - g\|_{L^1_x} \to 0$, $n \to \infty$, which implies that $\mu(y-y_n) \to g$ a.e. (since $\nu_x(A) = 0$ means $m(A) = 0$). Using Lemma 3.1 of [6] and the fact $y_n \to y$, $n \to \infty$ in the measure topology, we have $\mu(y-y_n) \to 0$, $n \to \infty$. Therefore $g = 0$ a.e. on $(0,t_x)$. Furthermore, $\|y - y_n\|_{L^1_x(\mathscr{M})} = \|\mu(y-y_n)\|_{L^1_x} \to 0$, $n \to \infty$. □

PROPOSITION 7. *Let $A = \{y \in L^1_x(\mathscr{M}) : \|y\|_{L^1_x(\mathscr{M})} \leqslant r\}$ and $r > 0$. Then A is bounded in the measure topology, i.e., for all $\varepsilon > 0$ there exists $0 < t_0 < \infty$ such that $\tau(e_{[t_0,\infty)}(|y|)) < \varepsilon$ for all $y \in A$.*

Proof. Without loss of generality, we suppose $A = \{y \in L^1_x(\mathscr{M}) : \|y\|_{L^1_x(\mathscr{M})} \leqslant 1\}$. If A is unbounded in the measure topology, then there exists $\varepsilon_0 > 0$ and $y_n \in A$ such that $\tau(e_{[n2^n,\infty)}(|y_n|)) \geqslant \varepsilon_0$ holds for all $n \in \mathbb{N}^+$. Let $y = \Sigma_{n=1}^\infty \frac{|x_n|}{2^n}$, we deduce

$$\|y\|_{L^1_x(\mathscr{M})} \leqslant \Sigma_{n=1}^\infty \frac{\|y_n\|_{L^1_x(\mathscr{M})})}{2^n} \leqslant 1,$$

and

$$\tau(e_{[n,\infty)}(|y|)) \geqslant \tau(e_{[n2^n,\infty)}(|y_n|)) \geqslant \varepsilon_0, \quad \text{for all} \quad n.$$

This implies that

$$\lim_{n\to\infty} \lambda_n(y) \geqslant \varepsilon_0. \tag{5.1}$$

On the other hand, from Proposition 21 of [Chapter I, [21]], we have $\lim_{n\to\infty} \lambda_n(y) = 0$, which contradicts (5.1). Consequently, A is bounded in the measure topology. □

PROPOSITION 8. *Let $E_{\varphi_1}(\mathscr{M}), E_{\varphi_2}(\mathscr{M})$ be two noncommutative Calderón-Lozanovskiĭ spaces, where E is a symmetric function space with Fatou property. If $E_{\varphi_1}(\mathscr{M}) \cdot E_{\varphi_2}(\mathscr{M})$ is a quasi-Banach spaces and there exists $x \in L_0(\mathscr{M})$ such that $\mu(x)E_{\varphi_2} \subseteq E'_{\varphi_1}$ and $\mu(x) \neq 0$ a.e., then $E_{\varphi_1}(\mathscr{M}) \cdot E_{\varphi_2}(\mathscr{M})$ is normable.*

Proof. Let $B_{E_{\varphi_1}(\mathscr{M})}$ and $B_{E_{\varphi_2}(\mathscr{M})}$ be the unit balls of $E_{\varphi_1}(\mathscr{M})$ and $E_{\varphi_2}(\mathscr{M})$, respectively. Set $y \in B_{E_{\varphi_1}(\mathscr{M})}$ and $z \in B_{E_{\varphi_2}(\mathscr{M})}$, then by Theorem 4.2 of [6] and Proposition 3.6 of [Chapter 2, [1]], we have

$$\int_0^\infty \mu_s(yz)\mu_s(x)ds \leqslant \int_0^\infty \mu_s(y)\mu_s(z)\mu_s(x)ds. \tag{5.2}$$

Combining $\mu(x)E_{\varphi_2} \subseteq E'_{\varphi_1}$ with the inequality (5.2) we obtain

$$\begin{aligned}\|yz\|_{L^1_x(\mathscr{M})} &= \int_0^\infty \mu_s(yz)\mu_s(x)ds \\ &\leqslant \|\mu(x)\|_{(E_{\varphi_1}\cdot E_{\varphi_2})'}\|\mu(y)\mu(z)\|_{E_{\varphi_1}\cdot E_{\varphi_2}} \\ &\leqslant \|\mu(x)\|_{(E_{\varphi_1}\cdot E_{\varphi_2})'} < \infty.\end{aligned}$$

Therefore, $B_{E_{\varphi_1}(\mathscr{M})}\cdot B_{E_{\varphi_2}(\mathscr{M})}$ is a norm bounded subset of $L^1_x(\mathscr{M})$. Let

$$\mathscr{B} = \{y \in L^1_x(\mathscr{M}) : \|y\|_{L^1_x(\mathscr{M})} \leqslant \|z\|_{L^1_x(\mathscr{M})} \text{ for some } z \in B_{E_{\varphi_1}(\mathscr{M})}\cdot B_{E_{\varphi_2}(\mathscr{M})}\}.$$

It is clear that $\mathscr{B}$ is a norm bounded convex subset of $L^1_x(\mathscr{M})$. Let $\rho(y) = \inf\{s > 0 : \frac{1}{s}y \in \mathscr{B}\}$, then

(i) $\rho(y) = 0$ if $y = 0$.

(ii) $\rho(\alpha y) = |\alpha|\rho(y)$.

(iii) $\rho(y_1 + y_2) \leqslant \rho(y_1) + \rho(y_2)$,

where $y, y_1, y_2 \in E_{\varphi_1}(\mathscr{M})\cdot E_{\varphi_2}(\mathscr{M})$. Given $\varepsilon > 0$, by Proposition 7, there exists $t_0 > 0$ such that

$$\tau(e_{(t_0,\infty)}(|y|)) < \varepsilon \text{ for all } y \in \mathscr{B}.$$

If $\rho(y) = 0$, then there exists N_0 such that $ny \in \mathscr{B}$ for all $n \geqslant N_0$. It follows that, $\tau(e_{(\frac{t_0}{n},\infty)}(|y|)) < \varepsilon$ for all $n \geqslant N_0$. Consequently, $\tau(e_{(0,\infty)}(|y|)) \leqslant \varepsilon$ for all $\varepsilon > 0$ and so $y = 0$. If $y = y_1y_2 \in E_{\varphi_1}(\mathscr{M})\cdot E_{\varphi_2}(\mathscr{M}), y_1 \in E_{\varphi_1}(\mathscr{M}), y_2 \in E_{\varphi_2}(\mathscr{M})$, then

$$\frac{y}{\|y_1\|_{E_{\varphi_1}(\mathscr{M})}\|y_2\|_{E_{\varphi_2}(\mathscr{M})}} = \frac{y_1}{\|y_1\|_{E_{\varphi_1}(\mathscr{M})}}\frac{y_2}{\|y_2\|_{E_{\varphi_2}(\mathscr{M})}} \in \mathscr{B},$$

which implies that $\rho(y) \leqslant \|y_1\|_{E_{\varphi_1}(\mathscr{M})}\|y_2\|_{E_{\varphi_2}(\mathscr{M})}$. Moreover, $\rho(y) \leqslant \|y\|_{E(\mathscr{M})\cdot F(\mathscr{M})}$. Conversely, for $\varepsilon > 0$, let $y = y_1y_2 \in E_{\varphi_1}(\mathscr{M})\cdot E_{\varphi_2}(\mathscr{M}), y_1 \in E_{\varphi_1}(\mathscr{M}), y_2 \in E_{\varphi_2}(\mathscr{M})$, we deduce $\frac{y}{\rho(y)+\varepsilon} \in \mathscr{B}$. It follows from the definition of $\mathscr{B}$ that

$$\left\|\frac{y}{\rho(y)+\varepsilon}\right\|_{E_{\varphi_1}(\mathscr{M})\cdot E_{\varphi_2}(\mathscr{M})} \leqslant c,$$

which implies that $\|y\|_{E_{\varphi_1}(\mathscr{M})\cdot E_{\varphi_2}(\mathscr{M})} \leqslant c\rho(y)$ holds for some $c > 0$. Then the functional $\rho(\cdot)$ is an equivalent norm to the original quasi-norm. □

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