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POLYNOMIALS WITH A SHARP CAUCHY BOUND AND THEIR ZEROS OF MAXIMAL MODULUS

HARALD K. WIMMER

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Abstract. The moduli of zeros of a complex polynomial are bounded by the positive zero of an associated auxiliary polynomial. The bound is due to Cauchy. This note describes polynomials with a sharp Cauchy bound and the location of peripheral zeros.

1. Introduction

Let

$$g(z) = z^m - (c_{m-1}z^{m-1} + \cdots + c_1 z + c_0) \tag{1.1}$$

be a complex polynomial.

Define

$$g_a(z) = z^m - \big(|c_{m-1}|z^{m-1} + \cdots + |c_1|z + |c_0|\big).$$

If $g(z) \neq z^m$ then (see e.g. [4, p. 122], [7, p. 3], [8, p. 243]) there exists a unique positive zero $R(g)$ of $g_a(z)$, and all zeros of $g(z)$ have modulus less or equal to $R(g)$. The number $R(g)$ is known (see [8]) as *Cauchy bound* of $g(z)$.

Set

$$\sigma(g) = \{\lambda \in \mathbb{C};\, g(\lambda) = 0\} \quad \text{and} \quad \rho(g) = \max\{|\lambda|;\, \lambda \in \sigma(g)\}.$$

Then $\rho(g) \leqslant R(g)$. In general, the numbers $\rho(g)$ and $R(g)$ do not coincide, that is, $\rho(g) < R(g)$. For example, the polynomials

$$g(z) = z^2 - (z-1) = \big(z - e^{2\pi i/6}\big)\big(z - e^{-2\pi i/6}\big)$$

and

$$g_a(z)) = z^2 - (z+1) = \big(z - \tfrac{1+\sqrt{5}}{2}\big)\big(z + \tfrac{1+\sqrt{5}}{2}\big)$$

satisfy $1 = \rho(g) < R(g) = (1+\sqrt{5})/2$. We say that the Cauchy bound is sharp, if $\rho(g) = R(g)$. Clearly, if $g(z) = g_a(z)$ then $\rho(g) = R(g)$, and $R(g) \in \sigma(g)$. But the Cauchy bound may be sharp, even if $g(z) \neq g_a(z)$. An example is the polynomial

$$g(z) = z^2 - (-z+2) = (z-1)(z+2)$$

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with

$$g_a(z) = z^2 - (z+2) = (z+1)(z-2) \quad \text{and} \quad R(g) = \rho(g) = 2.$$

In this note we are concerned with polynomials $g(z)$ which have the property that $R(g) = \rho(g)$ and we describe their zeros of maximal modulus. For the straightforward proof of the following result I am indebted to a referee.

THEOREM 1.1. *Let $g(z) \in \mathbb{C}[z]$ be given as in (1.1). Then $\rho(g) = R(g)$ if and only if*

$$g(z) = \lambda^m g_a(\lambda^{-1} z) \tag{1.2}$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Proof. Suppose $g(z) \neq z^m$. Let R be the Cauchy bound of $g(z)$, that is,

$$R^m = \sum_{j=0}^{m-1} |c_j| R^j. \tag{1.3}$$

Then $g(z)$ has a zero of modulus R if and only if

$$(\lambda R)^m = \sum_{j=0}^{m-1} c_j (\lambda R)^j \tag{1.4}$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Because of (1.3) the equation (1.4) is equivalent to

$$\sum_{j=0}^{m-1} |c_j| R^j = \sum_{j=0}^{m-1} c_j \lambda^{-m+j} R^j. \tag{1.5}$$

All terms on the left-hand side of (1.5) are nonnegative. Thus it is easy to see that (1.5) holds if and only if

$$|c_j| = c_j \lambda^{-m+j}, \quad j = 0, \ldots, m-1, \tag{1.6}$$

which is equivalent to (1.2). □

In Section 2 we apply Theorem 1.1 to obtain a result on rotational symmetry of zeros of maximal modulus and we consider polynomials with real coefficients. A different approach to deal with the Cauchy bound and its sharpness is described in Section 3. It is based on companion matrices and the Perron-Frobenius theory of nonnegative matrices.

2. Zeros of maximal modulus

Throughout this paper $g(z)$ will be a polynomial of the form (1.1) and we assume $g(z) \neq z^m$. The following notation will be used. With regard to (1.2) we define

$$(\kappa \cdot g)(z) = \kappa^m g(\kappa^{-1} z),$$

where $\kappa \in \mathbb{C}$, $\kappa \neq 0$. If $g(z) = \prod_{j=1}^m (z - \lambda_j)$ then

$$(\kappa \cdot g)(z) = \prod_{j=1}^m (z - \kappa \lambda_j),$$

and therefore $\sigma(\kappa \cdot g) = \kappa\sigma(g)$. Let $\partial\mathbb{D}$ denote the unit circle and let E_n be the group of n-th roots of unity,

$$E_n = \sigma(z^n - 1) = \{e^{2k\pi i/n};\, k = 0, \ldots, n-1\}.$$

The *support* $\Sigma(q)$ of a polynomial $q(z) = \sum_{j=0}^{k} q_j z^j$ is the set of indices j with nonzero coefficient q_j. Thus, for the polynomial $g(z)$ in (1.1) we have

$$\Sigma(g) = \{j;\, 0 \leqslant j \leqslant m-1, c_j \neq 0\} \cup \{m\}.$$

Define

$$d(g) = \gcd\{j \in \Sigma(g)\} \text{ and } \ell(g) = m/d(g).$$

If $d(g) = d$ and $\ell(g) = \ell$ then

$$g(z) = \left(z^d\right)^{\ell} - \left(c_{(\ell-1)d}\left(z^d\right)^{\ell-1} + \cdots + c_d z^d + c_0\right). \tag{2.1}$$

Set $\tilde{c}_k = c_{kd}$, $k = 0, 1, \ldots, \ell - 1$, and

$$\tilde{g}(z) = z^{\ell} - \left(\tilde{c}_{\ell-1} z^{\ell-1} + \cdots + \tilde{c}_1 z + \tilde{c}_0\right). \tag{2.2}$$

Then $g(z) = \tilde{g}(z^d)$. Moreover, $\Sigma(g) = d\Sigma(\tilde{g})$ implies $d(\tilde{g}) = 1$. In accordance with [1] we denote by π_+^{k-1} the set of real polynomials $p(z) = \sum_{i=0}^{k-1} a_i z^i$ satisfying

$$0 < a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{k-1} = 1.$$

Define $S(g) = \sum_{j=1}^{m-1} |c_j|$. Then $S(g) = 1$ is equivalent to $1 \in \sigma(g_a)$. On the other hand, $R(g) = 1$ means that $\lambda = 1$ is the (unique) positive zero of $g_a(z)$. Hence we have $R(g) = 1$ if and only if $S(g) = 1$.

In this section we deal with polynomials $g(z)$ with a sharp Cauchy bound and we focus on zeros of $g(z)$ of maximal modulus. For the sake of simplicity we shall assume $0 \notin \sigma(g)$. The following theorem can be traced back to Hurwitz [3]. We include a proof to make the note self-contained. The theorem has an interesting history, which is indicated in [1]. Only the special case with $d(g) = 1$ seems to be widely known [6, p. 92]), [7, p. 3].

THEOREM 2.1. (Hurwitz) *Assume $g(z) = g_a(z)$. Suppose $R(g) = 1$ and $g(0) \neq 0$. Let $d(g) = d$ and $\ell(g) = \ell$. Then $g(z) = (z^d - 1)\tilde{p}(z^d)$ with $\tilde{p}(z) \in \pi_+^{\ell-1}$ and $\rho(\tilde{p}) < 1$. The unimodular zeros of $g(z)$ are simple, and $\sigma(g) \cap \partial\mathbb{D} = E_d$.*

Proof. Suppose first that $d = 1$ such that $g(z) = \tilde{g}(z)$. The assumption $\gcd\{j \in \Sigma(g)\} = 1$ yields a Bezout identity $\sum_{j\in\Sigma(g)} \gamma_j j = 1$ with $\gamma_j \in \mathbb{Z}$. Let $\lambda \in \sigma(g) \cap \partial\mathbb{D}$. From the proof of Theorem 1.1 we know that λ satisifies (1.6). We have $c_j = |c_j|$ for all j. Then $g(0) = c_0 \neq 0$ and (1.6) imply $\lambda^m = 1$, and we obtain $j \in \Sigma(g)$ if and only if $\lambda^j = 1$. Hence $\lambda = \prod_{j\in\Sigma(g)} \lambda^{\gamma_j j} = 1$, that is, $\sigma(g) \cap \partial\mathbb{D} = \{1\}$. From

$$g'(1) = m - \sum_{j=1}^{m-1} j c_j > m - (m-1)\sum_{j=1}^{m-1} c_j > 1$$

we see that $\lambda = 1$ is a simple zero of $g(z)$. Hence $g(z) = (z-1)p(z)$ for some polynomial $p(z) = \sum_{k=0}^{m-1} a_k z^k$ with $a_{k-1} = 1$ and $\rho(p) < 1$. The coefficients of $g(z)$ and $p(z)$ satisfy $a_k = \sum_{i=0}^{k} c_j$, $k = 0, \dots, m-1$. Thus $p(z) \in \pi_+^{m-1}$. In the general case, if $d(g) = d$, it suffices to note that $g(z) = \tilde{g}(z^d)$ with $d(\tilde{g}) = 1$. $\square$

Combining Theorem 2.1 with Theorem 1.1 we obtain the following.

COROLLARY 2.2. *Let $d(g) = d$ and $g(0) \neq 0$. Suppose $\rho(g) = R(g) = R$. If $|\lambda| = R$ and $g(\lambda) = 0$ then $\sigma(g) \cap R\partial\mathbb{D} = \lambda E_d$. In other words, the zeros of maximal modulus are the vertices of a regular d-sided polygon in the complex plane.*

We now consider polynomials $g(z)$ with real coefficients.

THEOREM 2.3. *Let $g(z) \in \mathbb{R}[z]$ and $d = d(g)$. Suppose $g(0) \neq 0$. Then $\rho(g) = R(g)$ if and only if $g(z) = g_a(z)$ or*

$$g(z) = \eta \cdot g_a(z) = z^{\ell d} - (-1)^{\ell} \sum_{\nu=0}^{\ell-1} (-1)^{\nu} |c_{\nu d}| z^{\nu d} \tag{2.3}$$

where $\ell d = m$ and $\eta = \mathrm{e}^{\pi i/d}$.

Proof. Suppose $\rho(g) = R(g)$. Let $\tilde{g}(z)$ be the polynomial in (2.2). Then $\rho(\tilde{g}) = R(\tilde{g})$, and it follows from Theorem 1.1 that $\tilde{g}(z) = \lambda^{\ell} \tilde{g}_a(\lambda^{-1} z)$ for some $\lambda \in \partial\mathbb{D}$. Assuming $R(g) = 1$ we apply Theorem 2.1. Because of $d(\tilde{g}) = 1$ we obtain $\sigma(\tilde{g}_a) \cap \partial\mathbb{D} = \{1\}$. Therefore

$$\tilde{g}(z) = \lambda^{\ell} (\lambda^{-1} z - 1) \tilde{p}(\lambda^{-1} z) = (z - \lambda) \lambda^{\ell-1} \tilde{p}(\lambda^{-1} z) = (z - \lambda)\, \lambda \cdot \tilde{p}(z).$$

The real polynomial $\tilde{q}(z) = \lambda \cdot \tilde{p}(z)$ satisfies $\rho(\tilde{q}) < 1$. Thus $\tilde{g}(z) \in \mathbb{R}[z]$ implies $\lambda \in \{1, -1\}$. If $\lambda = 1$ then $\tilde{g}(z) = \tilde{g}_a(z)$, and therefore

$$g(z) = g_a(z).$$

If $\lambda = -1$ then

$$\tilde{g}(z) = (-1)^{\ell} \tilde{g}_a(-z) = z^{\ell} - (-1)^{\ell} \sum_{\nu=0}^{\ell-1} (-1)^{\nu} |\tilde{c}_{\nu}| z^{\nu}.$$

Hence

$$g(z) = \tilde{g}(z^d)$$

and $\eta \cdot g = \eta^d \cdot \tilde{g}$ imply (2.3). $\square$

The real polynomial $g(z) = z^2 + 1$ is an example with a sharp Cauchy bound and $g(z) \neq (\pm 1) \cdot g_a(z)$. Here we have $d = 2$, $\ell = 1$, $\eta = i$, $g_a(z) = z^2 - 1$, and

$$g(z) = i \cdot g_a(z).$$

3. Companion matrices

A different approach to study zeros of polynomials uses companion matrices and takes advantage of the theory of Perron-Frobenius-Wielandt (see e.g. [9], [1], [8]). We indicate how results of this note can be viewed in that context. Let

$$F = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{m-1} \end{pmatrix} \quad \text{and} \quad F_a = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ |c_0| & |c_1| & |c_2| & \dots & |c_{m-1}| \end{pmatrix} \tag{3.1}$$

be companion matrices associated with the polynomials $g(z)$ and $g_a(z)$, respectively. Thus, $g(z) = \det(zI - F)$ and $g_a(z) = \det(zI - F_a)$. If $\sigma(F)$ and $\rho(F)$ denote the spectrum and the spectral radius of F then $\sigma(F) = \sigma(g)$, $\rho(F) = \rho(g)$ and $\rho(F_a) = R(g)$. The matrix F_a is a nonnegative matrix, and F_a is irreducible if and only if $c_0 \neq 0$.

Let $A = (a_{ij}) \in \mathbb{R}^{m\times m}$ be a nonnegative matrix and let $B = (b_{ij}) \in \mathbb{C}^{m\times m}$. We write $|B| \leqslant A$ if $|b_{ij}| \leqslant a_{ij}$ for all i,j. The following theorem is due to Wielandt (see [2, Theorem 8.4.5] or [5, Chapter 8]).

THEOREM 3.1. *Let $A \in \mathbb{R}^{m\times m}$ be nonnegative and irreducible. Suppose $|B| \leqslant A$. Then*

$$\rho(B) \leqslant \rho(A). \tag{3.2}$$

We have $\rho(B) = \rho(A)$ if and only if

$$B = e^{i\phi} DAD^{-1} \quad \text{for some} \quad D = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_m}). \tag{3.3}$$

If $B = F$ and $A = F_a$ are given by (3.1) then (3.2) yields $\rho(g) \leqslant R(g)$. Moreover, if $\rho(g) = R(g)$ then (3.3) implies

$$\begin{aligned} F &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{m-1} \end{pmatrix} \\ &= e^{i\phi} \begin{pmatrix} 0 & e^{i(\theta_1-\theta_2)} & 0 & \dots & 0 \\ 0 & 0 & e^{i(\theta_2-\theta_3)} & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & e^{i(\theta_{m-1}-\theta_m)} \\ e^{i(\theta_m-\theta_1)}|c_0| & e^{i(\theta_m-\theta_2)}|c_1| & e^{i(\theta_m-\theta_3)}|c_2| & \dots & e^{i(\theta_m-\theta_m)}|c_{m-1}| \end{pmatrix}. \end{aligned} \tag{3.4}$$

It follows that $\theta_1 - \theta_2 + \phi = 0$, $\dots$, $\theta_{m-1} - \theta_m + \phi = 0$. Therefore $\theta_m - \theta_1 = (m-1)\phi$, $\dots$, $\theta_m - \theta_{m-1} = \phi$. Set $\lambda = e^{i\phi}$. Then (3.4) yields $F = F_\lambda$ where

$$F_\lambda = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ |c_0|\lambda^m & |c_1|\lambda^{m-1} & |c_2|\lambda^{m-2} & \dots & |c_{m-1}|\lambda| \end{pmatrix}.$$

The matrix F_λ is the companion matrix of

$$z^m - \left(|c_0|\lambda^m + |c_1|\lambda^{m-1}z + \cdots + |c_{m-1}|\lambda\right) = \lambda^m g_a\left(\frac{z}{\lambda}\right) = \lambda \cdot g_a(z).$$

Hence the polynomial $g(z)$ satisfies (1.2), in accordance with Theorem 1.1.

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Harald K. Wimmer
Mathematisches Institut
Universität Würzburg
D-97074 Würzburg, Germany
e-mail: wimmer@mathematik.uni-wuerzburg.de