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HARTLEY–FOURIER COSINE GENERALIZED CONVOLUTION INEQUALITIES

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Abstract. In this paper, we study some inequalities related to a certain generalized convolution for the Hartley-Fourier cosine integral transforms. Specially, we will apply these inequalities to estimate the solutions of some integral equations, differential equations and partial differential equations.

1. Introduction

Recall that the Hartley-Fourier cosine convolution $(f \underset{1}{*} g)$ of functions f and g is introduced in [13]

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty [g(x+u)+g(x-u)]\, f(u)\, du, \;\; x \in \mathbb{R}, \tag{1}$$

and the following factorization properties hold

$$H_{\left\{\substack{1\\2}\right\}}(f \underset{1}{*} g)(y) = (F_c f)(y)\cdot (H_{\left\{\substack{1\\2}\right\}} g)(y), \;\; \forall y \in \mathbb{R}. \tag{2}$$

Here, the Hartley integral transforms of $f \in L_1(\mathbb{R})$ are of the form (see [3, 16])

$$(H_{\left\{\substack{1\\2}\right\}} f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\operatorname{cas}(\pm xy) dx, \;\; y \in \mathbb{R}, \tag{3}$$

where $\operatorname{cas} u = \cos u + \sin u$ is the cosine-and-sine, or Hartley kernel.

The Fourier cosine transform is of the form (see [8])

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(xy)\, dx, \;\; y \in \mathbb{R}. \tag{4}$$

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In case $F_c f \in L_1(\mathbb{R}_+)$ its inverse formula has the form

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(yx) \cdot (F_c f)(y) dy. \tag{5}$$

The following Young's Theorem for the Fourier convolution is well-known (see [2]):

PROPOSITION 1. ([2]) *For $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R})$, $h \in L_r(\mathbb{R})$, here $p > 1$, $q > 1$, $r > 1$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, we have*

$$\left| \int_{-\infty}^\infty (f \underset{F}{*} g)(x) \cdot h(x) dx \right| \leqslant \|f\|_{L_p(\mathbb{R})} \cdot \|g\|_{L_q(\mathbb{R})} \cdot \|h\|_{L_r(\mathbb{R})}. \tag{6}$$

An important corollary of this theorem is the so-called Young's inequality [2] for the Fourier convolution

$$\|f \underset{F}{*} g\|_{L_r(\mathbb{R})} \leqslant \|f\|_{L_p(\mathbb{R})} \cdot \|g\|_{L_q(\mathbb{R})}, \ \ f \in L_p(\mathbb{R}), \ \ g \in L_q(\mathbb{R}), \ \ \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{7}$$

Here, $(f \underset{F}{*} g)(x)$ is the well-known convolution of two functions f and g for the Fourier integral transform (see [8])

$$(f \underset{F}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x-y) g(y) dy, \ \ x \in \mathbb{R}, \ \ f, g \in L_1(\mathbb{R}). \tag{8}$$

In this paper, the following inequality idea is basic:

PROPOSITION 2. ([11]) *For two non-negative functions $\rho_j \in L_1(\mathbb{R})$ $(j = 1, 2)$, the $L_p(\mathbb{R})$ $(p > 1)$ weighted convolution inequality*

$$\|((F_1\rho_1) \underset{F}{*} (F_2\rho_2))(\rho_1 \underset{F}{*} \rho_2)^{\frac{1}{p}-1}\|_{L_p(\mathbb{R})} \leqslant \|F_1\|_{L_p(\mathbb{R},\rho_1)} \cdot \|F_2\|_{L_p(\mathbb{R},\rho_2)} \tag{9}$$

holds for $F_j \in L_p(\mathbb{R}, \rho_j)$ $(j = 1, 2)$.

Here

$$\|F\|_{L_p(\mathbb{R},\rho)} = \left\{ \int_{-\infty}^\infty |F(x)|^p \rho(x) dx \right\}^{\frac{1}{p}}.$$

Unlike Young's inequality, inequality (9) holds also in case $p = 2$. Furthermore, in many cases of interest, the convolution is given in the form

$$\rho_2(x) \equiv 1, \quad F_2(x) \equiv G(x), \tag{10}$$

where $G(x-\xi)$ is some Green's function. Then the inequality (9) takes the form

$$\|(F\rho)\underset{F}{*}G\|_{L_p(\mathbb{R})} \leqslant \|\rho\|_{L_1(\mathbb{R}_+)}^{1-\frac{1}{p}} \cdot \|G\|_{L_p(\mathbb{R})} \cdot \|F\|_{L_p(\mathbb{R},\rho)}, \tag{11}$$

where ρ, F, and G are such that the right hand side of (11) is finite.

Inequality (11) enables us to estimate the output function

$$\int_{-\infty}^{\infty} (F\rho)(y)\cdot G(x-y)dy, \tag{12}$$

in terms of the input function F in the related differential equation. For various applications, we refer the reader to [9, 10, 11, 12] and references therein. An inequality of this type for the Fourier cosine transform has introduced in [5].

In this paper, we are interested in the following famous reverse Hölder's inequality (see [7]).

PROPOSITION 3. ([7]) *For two positive functions f and g satisfying*

$$0 < m \leqslant \frac{f}{g} \leqslant M < \infty \tag{13}$$

on the set X, and for $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$,

$$\left(\int_X f d\mu\right)^{\frac{1}{p}} \left(\int_X g d\mu\right)^{\frac{1}{q}} \leqslant A_{p,q}\left(\frac{m}{M}\right) \int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{14}$$

if the right hand side integral converges. Here

$$A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} \frac{t^{-\frac{1}{pq}}(1-t)}{\left(1-t^{\frac{1}{p}}\right)^{\frac{1}{p}} \left(1-t^{\frac{1}{q}}\right)^{\frac{1}{p}}}.$$

By using Proposition 3, in [12], S. Saitoh, V. K. Tuan, M. Yamamoto obtained the following reverse inequality.

PROPOSITION 4. ([12]) *Let F_1, F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leqslant F_1(x) \leqslant M_1^{\frac{1}{p}} < \infty; \quad 0 < m_2^{\frac{1}{p}} \leqslant F_2(x) \leqslant M_2^{\frac{1}{p}} < \infty, \quad p>1, \quad x \in \mathbb{R}. \tag{15}$$

Then for any positive continuous functions ρ_1, ρ_2 we have the reverse L_p-weighted convolution inequality

$$\|(F_1\rho_1 \underset{F}{*} F_2\rho_2)\cdot(\rho_1 \underset{F}{*} \rho_2)^{\frac{1}{p}-1}\|_{L_p(\mathbb{R})} \geqslant C^{-1}\, \|F_1\|_{L_p(\mathbb{R},\rho_1)} \cdot \|F_2\|_{L_p(\mathbb{R},\rho_2)}, \tag{16}$$

if the left hand side is finite. Here $C = A_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right)$.

In formula (16), replacing ρ_1 by 1 and $F_2(x-\xi)$ by $G(x-\xi)$, and integrating with respect to x from c to d, S. Saitoh and co-operations arrive at the following inequality (see [12])

$$\int_c^d \Big(\int_{-\infty}^{\infty} F(\xi)\rho(\xi)G(x-\xi)d\xi\Big)^p dx$$

$$\geqslant A_{p,q}^{-p}\Big(\frac{m}{M}\Big)\Big(\int_{-\infty}^{\infty}\rho(\xi)d\xi\Big)^{p-1}\int_{-\infty}^{\infty}F^p(\xi)\rho(\xi)d\xi\int_{c-\xi}^{d-\xi}G^p(x)dx, \tag{17}$$

if positive continuous functions ρ, F and G satisfy

$$0<m^{\frac{1}{p}}\leqslant F(\xi)\cdot G(x-\xi)\leqslant M^{\frac{1}{p}}<\infty,\ \ x\in[c,\,d],\ \ \xi\in\mathbb{R}. \tag{18}$$

Inequality (17) is specially important when $G(x-\xi)$ is a Green's function. An reverse inequality of this type for the Laplace transform has been studied in [12]. However, similar problems for other integral transforms and generalized convolutions have not been studied.

The integral equation with the Toeplitz plus Hankel kernel is of the form [6, 15]

$$f(x)+\int_0^{\infty}[k_1(x+y)+k_2(x-y)]f(y)dy=g(x),\ \ x>0, \tag{19}$$

here g, k_1, k_2 are given, and f is an unknown function. This equation has many useful applications [6, 15]. However, this integral equation can be solved in closed form only in some particular cases of the Hankel kernel k_1 and the Toeplitz kernel k_2. The solution of equation (19) in closed form in the general case is still open.

This paper is organized as follows. In Section 2, we study some inequalities for the generalized convolution (1) which related to the Hartley and the Fourier cosine transforms. In Section 3, we will apply the above inequalities in estimating the solutions of some integral equations, integral equation with the Toeplitz plus Hankel kernel, differential equations and partial differential equations.

2. Hartley-Fourier cosine generalized convolution inequalities

In this section, we will prove analogue Young inequalities and an analogue of inequality (9) for the Hartley-Fourier cosine generalized convolutions.

THEOREM 1. (Young's type theorem) *Let $p,q,r>1$ be such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$ and let $f\in L_p(\mathbb{R})$, $g\in L_q(\mathbb{R})$, $h\in L_r(\mathbb{R})$. Then, the following inequality holds true*

$$\left|\int_{-\infty}^{\infty}(f\underset{1}{*}g)(x)\cdot h(x)dx\right|\leqslant(2\pi)^{-\frac{1}{2}}2^{\frac{1}{p}}\,\|f\|_{L_p(\mathbb{R})}\cdot\|g\|_{L_q(\mathbb{R})}\cdot\|h\|_{L_r(\mathbb{R})}. \tag{20}$$

Proof. Let p_1, q_1, r_1 be the conjugate exponentials of p, q, r, respectively, it means

$$\frac{1}{p}+\frac{1}{p_1}=1, \quad \frac{1}{q}+\frac{1}{q_1}=1, \quad \frac{1}{r}+\frac{1}{r_1}=1.$$

Then it is obviously that $\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}=1$ and $\frac{q}{p_1}+\frac{q}{r_1}=1$, $\frac{r}{p_1}+\frac{r}{q_1}=1$, $\frac{p}{q_1}+\frac{p}{r_1}=1$.

Put

$$\begin{aligned} F(x,u) &= |g(x+u)+g(x-u)|^{\frac{q}{p_1}} \cdot |h(x)|^{\frac{r}{p_1}}, \quad (x\in\mathbb{R}, u\in\mathbb{R}_+); \\ G(x,u) &= |f(u)|^{\frac{p}{q_1}} \cdot |h(x)|^{\frac{r}{q_1}}, \quad (x\in\mathbb{R}, u\in\mathbb{R}_+); \\ H(x,u) &= |f(u)|^{\frac{p}{r_1}} \cdot |g(x+u)+g(x-u)|^{\frac{q}{r_1}}, \quad (x\in\mathbb{R}, u\in\mathbb{R}_+). \end{aligned}$$

We see that F, G, H are functions defined in $\Omega=\mathbb{R}\times\mathbb{R}_+$, moreover,

$$(F\cdot G\cdot H)(x,u) = |f(u)|\cdot|h(x)|\cdot|[g(x+u)+g(x-u)]|. \tag{21}$$

On the other hand, in the space $L_{p_1}(\Omega)$, we have

$$\begin{aligned} \|F\|^{p_1}_{L_{p_1}(\Omega)} &= \int_{\Omega} |g(x+u)+g(x-u)|^q |h(x)|^r du dx \\ &= \int_{-\infty}^{\infty} \left(\int_0^{\infty} |g(x+u)+g(x-u)|^q du \right) |h(x)|^r dx. \end{aligned}$$

Note that t^q $(q>1)$ is a convex function, therefore, by changing variables we have

$$\begin{aligned} \int_0^{\infty} |g(x+u)+g(x-u)|^q du &\leqslant 2^{q-1}\Big(\int_0^{\infty} |g(x+u)|^q du + \int_0^{\infty} |g(x-u)|^q du \Big) \\ &= 2^{q-1} \int_{-\infty}^{\infty} |g(t)|^q dt. \end{aligned}$$

It yields

$$\|F\|^{p_1}_{L_{p_1}(\Omega)} \leqslant 2^{q-1} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |g(t)|^q dt \right) |h(x)|^r dx = 2^{q-1} \|g\|^q_{L_q(\mathbb{R})} \cdot \|h\|^r_{L_r(\mathbb{R})}.$$

Therefore

$$\|F\|_{L_{p_1}(\Omega)} \leqslant 2^{\frac{q-1}{p_1}} \|g\|^{\frac{q}{p_1}}_{L_q(\mathbb{R})} \cdot \|h\|^{\frac{r}{p_1}}_{L_r(\mathbb{R})}. \tag{22}$$

Similarly,

$$\|H\|_{L_{r_1}(\Omega)} \leqslant 2^{\frac{q}{r_1}} \|f\|^{\frac{p}{r_1}}_{L_q(\mathbb{R})} \cdot \|g\|^{\frac{q}{r_1}}_{L_r(\mathbb{R})}. \tag{23}$$

It is obviously that $\|G\|_{L_{q_1}(\Omega)} = \|f\|^{\frac{p}{q_1}}_{L_p(\mathbb{R}_+)} \cdot \|h\|^{\frac{r}{q_1}}_{L_r(\mathbb{R}_+)}$. Hence, from (22) and (23), we have

$$\|F\|_{L_{p_1}(\Omega)} \cdot \|G\|_{L_{q_1}(\Omega)} \cdot \|H\|_{L_{r_1}(\Omega)} \leqslant 2^{\frac{1}{p}} \|f\|_{L_p(\mathbb{R}_+)} \cdot \|g\|_{L_q(\mathbb{R})} \cdot \|h\|_{L_r(\mathbb{R})}. \tag{24}$$

From (21) and (24), by the three-function form of Hölder's inequality (see [2]):

We have

$$\begin{aligned}\left|\int_0^\infty (f \underset{1}{*} g)(x) \cdot h(x)dx\right| &\leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_0^{\infty} |f(u)|\,|[g(x+u)+g(x-u)]|\,|h(x)|dudx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_0^{\infty} F(x,u) \cdot G(x,u) \cdot H(x,u)dudx \\ &\leqslant (2\pi)^{-\frac{1}{2}} \|F\|_{L_{p_1}(\Omega)} \cdot \|G\|_{L_{q_1}(\Omega)} \cdot \|H\|_{L_{r_1}(\Omega)} \\ &\leqslant (2\pi)^{-\frac{1}{2}} 2^{\frac{1}{p}} \|f\|_{L_p(\mathbb{R}_+)} \cdot \|g\|_{L_q(\mathbb{R})} \cdot \|h\|_{L_r(\mathbb{R})}.\end{aligned}$$

The theorem is proved. □

Like Young's inequality for the Fourier convolution, the following Young's type inequality for the Hartley, Fourier cosine convolution is a direct corollary of the above theorem.

COROLLARY 1. (Young's type Inequality) *Let $p,q,r > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Let $f \in L_p(\mathbb{R}_+)$, $g \in L_q(\mathbb{R})$, then $(f \underset{1}{*} g) \in L_r(\mathbb{R})$, moreover*

$$\|f \underset{1}{*} g\|_{L_r(\mathbb{R})} \leqslant (2\pi)^{-\frac{1}{2}} 2^{\frac{1}{p}}\ \|f\|_{L_p(\mathbb{R}_+)} \cdot \|g\|_{L_q(\mathbb{R})}. \tag{25}$$

This inequality, however, also does not hold for the typical case of $f \in L_2(\mathbb{R}_+)$, $g \in L_2(\mathbb{R})$.

Next, we are interested in analogue inequality of $L_p(\mathbb{R})$ weighted inequality (9) for the Hartley-Fourier cosine convolution, which also has the meaning in case $p = q = 2$.

Our main result is the following theorem.

THEOREM 2. (Saitoh's type Inequality) *For two non-vanishing positive functions ρ_j, $(j = 1,2)$, the following $L_p(\mathbb{R}_+)$-weighted inequality for the Hartley-Fourier cosine convolution holds for any $F_1 \in L_p(\mathbb{R}_+,\rho_1)$, $F_2 \in L_p(\mathbb{R},\rho_2)$, $p > 1$,*

$$\|((F_1\rho_1) \underset{1}{*} (F_2\rho_2)) \cdot (\rho_1 \underset{1}{*} \rho_2)^{\frac{1}{p}-1}\|_{L_p(\mathbb{R})} \leqslant \sqrt{\frac{2}{\pi}} \|F_1\|_{L_p(\mathbb{R}_+,\rho_1)} \cdot \|F_2\|_{L_p(\mathbb{R},\rho_2)}. \tag{26}$$

Proof. By raising the left hand side of (26) to power p we obtain

$$\begin{aligned}
&\|((F_1\rho_1) \underset{1}{*} (F_2\rho_2)) \cdot (\rho_1 \underset{1}{*} \rho_2)^{\frac{1}{p}-1}\|^p_{L_p(\mathbb{R})} \\
= &\int_{-\infty}^{\infty} \left|((F_1\rho_1) \underset{1}{*} (F_2\rho_2))(x) \cdot (\rho_1 \underset{1}{*} \rho_2)^{\frac{1}{p}-1}(x)\right|^p dx \\
= &\left(\frac{1}{\sqrt{2\pi}}\right)^p \left(\frac{1}{\sqrt{2\pi}}\right)^{1-p} \int_{-\infty}^{\infty} \left|\int_0^{\infty} (F_1\rho_1)(u)[(F_2\rho_2)(x+u) + (F_2\rho_2)(x-u)]du\right|^p \\
&\times \left|\int_0^{\infty} \rho_1(u)[\rho_2(x+u) + \rho_2(x-u)]du\right|^{1-p} dx.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|((F_1\rho_1) \underset{1}{*} (F_2\rho_2)) \cdot (\rho_1 \underset{1}{*} \rho_2)^{\frac{1}{p}-1}\|^p_{L_p(\mathbb{R})} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|\int_0^{\infty} (F_1\rho_1)(u)[(F_2\rho_2)(x+u) + (F_2\rho_2)(x-u)]du\right|^p A^{1-p}, \qquad (27)
\end{aligned}$$

here $A = \int_0^{\infty} \rho_1(u)[\rho_2(x+u) + \rho_2(x-u)]du$.

On the other hand,

$$\begin{aligned}
&\left|\int_0^{\infty} (F_1\rho_1)(u)[(F_2\rho_2)(x+u) + (F_2\rho_2)(x-u)]du\right| \\
&\leqslant \int_0^{\infty} |(F_1\rho_1)(u)||(F_2\rho_2)(x+u)|du + \int_0^{\infty} |(F_1\rho_1)(u)||(F_2\rho_2)(x-u)|dy. \qquad (28)
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^{\infty} |F_1(u)|\rho_1(u)|F_2(x+u)|\rho_2(x+u)du \\
= &\int_0^{\infty} \left[(|F_1^p(u)|(\rho_1(u))^{\frac{1}{p}} (|F_2^p(x+u)|(\rho_2(x+u))^{\frac{1}{p}}\right] \left[(\rho_1(u))^{\frac{1}{q}} (\rho_2(x+u))^{\frac{1}{q}}\right] du.
\end{aligned}$$

Using Hölder's inequality, for q is the exponential conjugate to p, we have

$$\begin{aligned}
&\int_0^{\infty} |F_1(u)|\rho_1(u)|F_2(x+u)|\rho_2(x+u)du \\
\leqslant &\left(\int_0^{\infty} |F_1^p(u)|\rho_1(u)|F_2^p(x+u)|\rho_2(x+u)du\right)^{\frac{1}{p}} \left(\int_0^{\infty} \rho_1(u)\rho_2(x+u)du\right)^{\frac{1}{q}}. \qquad (29)
\end{aligned}$$

Similarly,

$$\int_0^\infty |F_1(u)|\rho_1(y)|F_2(x-u)|\rho_2(x-u)du$$
$$\leqslant \left(\int_0^\infty |F_1^p(u)|\rho_1(u)|F_2^p(x-u)|\rho_2(x-u)du\right)^{\frac{1}{p}}\left(\int_0^\infty \rho_1(u)\rho_2(x-u)du\right)^{\frac{1}{q}}. \quad (30)$$

Therefore, recalling that $t^{\frac{1}{p}}$, $t^{\frac{1}{q}}$ are concave functions we have

$$\int_0^\infty |(F_1\rho_1)(u)||(F_2\rho_2)(x+u)|du+\int_0^\infty |(F_1\rho_1)(u)||(F_2\rho_2)(x-u)|du$$
$$\leqslant \left(\int_0^\infty |F_1^p(u)|\rho_1(u)|F_2^p(x+u)|\rho_2(x+u)du\right)^{\frac{1}{p}}\left(\int_0^\infty \rho_1(u)\rho_2(x+u)du\right)^{\frac{1}{q}}$$
$$+\left(\int_0^\infty |F_1^p(u)|\rho_1(u)|F_2^p(x-u)|\rho_2(x-u)du\right)^{\frac{1}{p}}\left(\int_0^\infty \rho_1(u)\rho_2(x-u)du\right)^{\frac{1}{q}}$$
$$\leqslant 2\left(\int_0^\infty |F_1^p(u)|\rho_1(u)\big(|F_2^p(x+u)|\rho_2(x+u)+|F_2^p(x-u)|\rho_2(x-u)\big)du\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty \rho_1(u)\big(\rho_2(x+u)+\rho_2(x-u)\big)du\right)^{\frac{1}{q}}.$$

Thus

$$\int_0^\infty |(F_1\rho_1)(u)||(F_2\rho_2)(x+u)|du+\int_0^\infty |(F_1\rho_1)(u)||(F_2\rho_2)(x-u)|du$$
$$\leqslant \left(\int_0^\infty |F_1^p(u)|\rho_1(u)\big(|F_2^p(x+u)|\rho_2(x+u)+|F_2^p(ux-u)|\rho_2(x-u)\big)du\right)^{\frac{1}{p}} A^{\frac{1}{q}}. \quad (31)$$

From formulas (27), (28) and (31), using the Fubini theorem to interchange the order of integrations, we have

$$\|((F_1\rho_1)\underset{1}{*}(F_2\rho_2))\cdot(\rho_1\underset{1}{*}\rho_2)^{\frac{1}{p}-1}\|^p_{L_p(\mathbb{R})}$$
$$\leqslant \frac{A^{\frac{p}{q}+1-p}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_0^\infty |F_1^p(u)|\rho_1(u)\big(|F_2^p(x+u)|\rho_2(x+u)+|F_2^p(x-u)|\rho_2(x-u)\big)du\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \left(\int_{-\infty}^\infty \left(|F_2^p(x+u)|\rho_2(x+u) + |F_2^p(x-u)|\rho_2(x-u)\right) dx \right) |F_1^p(u)|\rho_1(u) du$$
$$= \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty |F_2^p(v)|\rho_2(v) dv \cdot \int_0^\infty |F_1^p(u)|\rho_1(u) du$$
$$= \sqrt{\frac{2}{\pi}} \, \|F_1\|^p_{L_p(\mathbb{R}_+,\rho_1)} \cdot \|F_2\|^p_{L_p(\mathbb{R},\rho_2)}.$$

The proof is completed. □

In particular, for $\rho_1 \equiv 1$, $\rho_2 = \rho$, the inequality (26) takes the form

$$\|F_1 \underset{1}{*} (F_2\rho)\|_{L_p(\mathbb{R})} \leqslant 2\, \|\rho\|^{1-\frac{1}{p}}_{L_1(\mathbb{R}_+)} \|F_1\|_{L_p(\mathbb{R})} \cdot \|F_2\|_{L_p(\mathbb{R}_+,\rho)}. \tag{32}$$

The inequality (32) enables us to estimate the output function

$$y(x) = \int_0^\infty F(y)\rho(y) \cdot G(x,y) dy, \tag{33}$$

in term of the input function F in the related differential equation, where for some F_2, $G(x,y) = F_2(x+y) + F_2(x-y)$ is a Green function.

THEOREM 3. (Reverse Saitoh's type Inequality) *Let $F_1(u)$ and $F_2(x)$ be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leqslant F_1(u) \leqslant M_1^{\frac{1}{p}} < \infty; \quad 0 < m_2^{\frac{1}{p}} \leqslant F_2(x) \leqslant M_2^{\frac{1}{p}} < \infty, \;\; p > 1, \;\; u \in \mathbb{R}_+, \;\; x \in \mathbb{R}. \tag{34}$$

Then for any positive functions $\rho_1(u)$, $\rho_2(x)$, we have the reverse Hartley-Fourier cosine convolution inequality

$$\|(F_1\rho_1 \underset{1}{*} F_2\rho_2) \cdot (\rho_1 \underset{1}{*} \rho_2)^{\frac{1}{p}-1}\|_{L_p(\mathbb{R})} \geqslant 2C^{-1} \, \|F_1\|_{L_p(\mathbb{R}_+,\rho_1)} \cdot \|F_2\|_{L_p(\mathbb{R},\rho_2)}, \tag{35}$$

here $C = A_{p,q}\left(\dfrac{m_1 m_2}{M_1 M_2}\right)$.

This inequality has validity in case the left hand side of (35) *is finite.*

Proof. Set

$$f(\xi) = F_1^p(\xi)[F_2(x+\xi) + F_2(x-\xi)]^p \rho_1(\xi)[\rho_2(x+\xi) + \rho_2(x-\xi)],$$
$$g(\xi) = \rho_1(\xi)[\rho_2(x+\xi) + \rho_2(x-\xi)].$$

Then the condition (34) implies

$$0 < 2^p m_1 m_2 \leqslant \frac{f(\xi)}{g(\xi)} \leqslant 2^p M_1 M_2 < \infty, \;\; \xi \in \mathbb{R}.$$

Using the reverse Hölder's inequality (14), we get

$$\left(\int_0^\infty F_1^p(\xi)[F_1(x+\xi)+F_2(x-\xi)]^p\rho_1(\xi)[\rho_1(x+\xi)+\rho_2(x-\xi)]d\xi\right)^{\frac{1}{p}} \tag{36}$$

$$\times\left(\int_0^\infty \rho_1(\xi)[\rho_2(x+\xi)+\rho_2(x-\xi)]d\xi\right)^{\frac{1}{q}}$$

$$\leqslant C\int_0^\infty F_1(\xi)\rho_1(\xi)[F_2(x+\xi)+F_2(x-\xi)][\rho_2(x+\xi)+\rho_2(x-\xi)]d\xi.$$

Exponent p on either side of the inequality, we have

$$\int_0^\infty F_1^p(\xi)[F_2(x+\xi)+F_2(x-\xi)]^p\rho_1(\xi)[\rho_2(x+\xi)+\rho_2(x-\xi)]d\xi\times \tag{37}$$

$$\times\left(\int_0^\infty \rho_1(\xi)[\rho_2(x+\xi)+\rho_2(x-\xi)]d\xi\right)^{p-1}$$

$$\leqslant C\left(\int_0^\infty F_1(\xi)\rho_1(\xi)[F_2(x+\xi)+F_2(x-\xi)][\rho_2(x+\xi)+\rho_2(x-\xi)]d\xi\right)^p.$$

Therefore

$$C^{-p}\int_0^\infty F_1^p(\xi)[F_2(x+\xi)+F_2(x-\xi)]^p\rho_1(\xi)[\rho_2(x+\xi)+\rho_2(x-\xi)]d\xi$$

$$\leqslant\left(\int_0^\infty F_1(\xi)\rho_1(\xi)[F_2(x+\xi)+F_2(x-\xi)][\rho_2(x+\xi)+\rho_2(x-\xi)]d\xi\right)^p$$

$$\times\left(\int_0^\infty \rho_1(\xi)[\rho_2(x+\xi)+\rho_2(x-\xi)]d\xi\right)^{1-p}.$$

Taking integration to both side of (37) with respect to x form $-\infty$ to ∞, we obtain the inequality

$$C^{-p}\int_0^\infty F_1^p(\xi)\rho_1(\xi)d\xi\int_{-\infty}^\infty[F_2(x+\xi)+F_2(x-\xi)]^p[\rho_2(x+\xi)+\rho_2(x-\xi)]dx$$

$$\leqslant\int_{-\infty}^\infty\left(\int_0^\infty(F_1\rho_1)(\xi)[F_2\rho_1(x+\xi)+F_2\rho_2(x-\xi)]d\xi\right)^p$$

$$\times \left(\int_0^\infty \rho_1(\xi)[\rho_2(x+\xi)+\rho_2(x-\xi)]d\xi \right)^{1-p} dx.$$

Exponent $\frac{1}{p}$ on either side of the inequality, we have

$$C^{-1}\left(\int_0^\infty F_1^p(\xi)\rho_1(\xi)d\xi \int_{-\infty}^\infty [F_2(x+\xi)+F_2(x-\xi)]^p[\rho_2(x+\xi)+\rho_2(x-\xi)]dx\right)^{\frac{1}{p}}$$
$$\leqslant \left(\int_{-\infty}^\infty (F_1\rho_1 \underset{1}{*} F_2\rho_2)^p \cdot (\rho_1 \underset{1}{*} \rho_2)^{1-p} dx\right)^{\frac{1}{p}}. \tag{38}$$

Moreover, since the fact that $(a+b)^p > a^p + b^p$ for all $a,b,p>0$, we have

$$\int_{-\infty}^\infty [F_2(x+\xi)+F_2(x-\xi)]^p[\rho_2(x+\xi)+\rho_2(x-\xi)]dx$$
$$\geqslant \int_{-\infty}^\infty (F_2^p(x+\xi)+F_2^p(x-\xi))(\rho_2(x+\xi)+\rho_2(x-\xi))d\xi$$
$$\geqslant 2\int_{-\infty}^\infty F_2^p(\xi)\cdot\rho_2(\xi)d\xi.$$

Combined with (38) we obtain

$$\|(F_1\rho_1 \underset{1}{*} F_2\rho_2)\cdot(\rho_1 \underset{1}{*} \rho_2)^{\frac{1}{p}-1}\|_{L_p(\mathbb{R})} \geqslant 2C^{-1} \|F_1\|_{L_p(\mathbb{R}_+,\rho_1)} \cdot \|F_2\|_{L_p(\mathbb{R},\rho_2)}.$$

The proof is completed. □

3. Applications

In this section, we will use inequality (32) to estimate the solutions of several ordinary differential equations, integral equations, and partial differential equations.

REMARK 1. Suppose that f is a continuous and piecewise smooth of order $2n$ such that $f^{(k)} \in L_1(\mathbb{R})$, and $\lim_{|x|\to\infty} f^{(2k)} = 0$, $(k=\overline{1,n})$, then we have

$$\left(H_1 f^{(2k)}\right)(y) = (-1)^k y^{2k}(H_1 f)(y), \quad (k=\overline{0,n}). \tag{39}$$

a) Ordinary differential equations

Let $a_0, a_n > 0$ and $a_k \geqslant 0$, $(k=\overline{1,n-1})$ such that there exists $Q \in L_1(\mathbb{R}_+) \cap L_p(\mathbb{R}_+)$ defined by

$$(F_c Q)(y) = \frac{1}{\sum_{k=0}^n a_k y^{2k}}, \quad y>0.$$

Consider the $2n^{th}$ order linear ordinary differential equation with constant coefficients

$$\Big(\sum_{k=0}^{n}(-1)^k a_k \frac{d^{2k}}{dx^{2k}}\Big) f(x) = g(x)\cdot\rho(x),\ \ x\in\mathbb{R}, \tag{40}$$

here g,ρ are given such that $g\in L_1(\mathbb{R},\rho)\cap L_p(\mathbb{R},\rho)$, $p>1$, $\rho\in L_1(\mathbb{R}_+)$ and $f\in L_1(\mathbb{R}_+)$ is an unknown function. We suppose in addition that

$$\frac{d^k}{dx^k} f(x)\to 0 \quad \text{as} \quad |x|\to\infty,\ \ k=0,1,\dots,2n. \tag{41}$$

Applying the Hartley transform to both sides of (40) and using condition (41) we get

$$\left(\sum_{k=0}^{n} a_k y^{2k}\right)(H_1 f)(y) = H_1(g\rho)(y). \tag{42}$$

Therefore, from (42) and the factorization property (2) of generalized convolution $f\underset{1}{*}g$, we have

$$\begin{aligned}(H_1 f)(y) &= \frac{1}{\sum\limits_{k=0}^{n} a_k y^{2k}}(H_1(g\rho))(y)\\ &= (F_c Q)(y)\cdot(H_1(g\rho))(y) = H_1\big(Q\underset{1}{*}(g\rho)\big)(y).\end{aligned}$$

It yields

$$f(x) = (Q\underset{1}{*}(g\rho))(x),\ \ x\in\mathbb{R}.$$

Using inequality (32) we obtain

$$\|f\|_{L_p(\mathbb{R})}\leqslant \|\rho\|_{L_1(\mathbb{R})}^{1-\frac{1}{p}}\|g\|_{L_p(\mathbb{R},\rho)}\cdot\|Q\|_{L_p(\mathbb{R}_+)}. \tag{43}$$

b) Integral equations of Toeplitz plus Hankel type

Consider the integral equation with the Toeplitz plus Hankel kernel in cases $k_1 = k_2 = f$:

$$f(x)+\frac{1}{2\pi}\int_0^\infty k(y)[f(x+y)+f(x-y)]dy = h(x)\rho(x),\ \ x\in\mathbb{R}. \tag{44}$$

Here, $k\in L_1(\mathbb{R}_+)\cap L_p(\mathbb{R}_+)$, $h\in L_1(\mathbb{R},\rho)\cap L_p(\mathbb{R},\rho)$ are given; f is an unknown function.

By Theorem (3.2) in [13], we have

$$f(x) = h\rho(x) - (l\underset{1}{*}h\rho)(x),\ \ x\in\mathbb{R}. \tag{45}$$

Using inequality (32) we obtain

$$\|f(x)\|_{L_p(\mathbb{R})} = \|h\rho-(l\underset{1}{*}h\rho)\|_{L_p(\mathbb{R})}\leqslant\|h\rho\|_{L_p(\mathbb{R}_+)}+\|\rho\|_{L_1(\mathbb{R}_+)}^{1-\frac{1}{p}}\|h\|_{L_p(\mathbb{R},\rho)}\cdot\|l\|_{L_p(\mathbb{R}_+)}.$$

c) Dirichlet's problem on the half-plane

Let us consider the equation

$$u_{xx} + u_{tt} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{46}$$

with the boundary conditions

$$u(x,0) = f(x)\rho(x), \quad -\infty < x < \infty, \tag{47}$$

$$u_x(x,t) \to 0 \text{ as } |x| \to \infty, \quad t \to \infty, \tag{48}$$

here, f, ρ are given such that $f \in L_1(\mathbb{R},\rho) \cap L_p(\mathbb{R},\rho)$, $p > 1$.

We introduce the Hartley transform with respect to x of a function of two variables $u(x,t)$

$$(H_1 u)(y,t) \equiv U(y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\,\mathrm{cas}(xy)dx. \tag{49}$$

Applying the Hartley transform (49) to both sides of (46), using conditions (47)–(48) we have

$$\frac{d^2}{dt^2}U(y,t) - y^2 U(y,t) = 0, \tag{50}$$

with the boundary condition

$$U(y,0) = (H_1(f\rho))(y). \tag{51}$$

The solution of the equation (50) with condition (51) is of the form

$$U(y,t) = e^{-yt}(H_1(f\rho))(y).$$

Using formula $(1.4.1)$ in ([4], p. 23) and the factorization property of $f \underset{1}{*} g$ we have

$$U(y,t) = \sqrt{\frac{2}{\pi}} F_c\left(\frac{t}{t^2+\tau^2}\right)(y) \cdot (H_1(f\rho))(y) = H_1\left(\frac{t}{t^2+\tau^2} \underset{1}{*} (f\rho)(\tau)\right)(y). \tag{52}$$

Therefore,

$$u(x,t) = \left(\frac{t}{t^2+\tau^2} \underset{1}{*} (f\rho)(\tau)\right)(x). \tag{53}$$

For each $t > 0$, using inequality (32) we obtain the following estimation

$$\|u\|_{L_p(\mathbb{R})} \leqslant 2^{2-\frac{1}{p}} \left\|\frac{t}{t^2+\rho^2}\right\|_{L_p(\mathbb{R})} \|\rho\|_{L_1(\mathbb{R})}^{1-\frac{1}{p}} \cdot \|f\|_{L_p(\mathbb{R},\rho)},$$

or

$$\|u\|_{L_p(\mathbb{R})} \leqslant 2^{1-\frac{1}{p}} \frac{\Gamma(p-\frac{1}{2})}{\Gamma(p)} t^{1-p} \|\rho\|_{L_1(\mathbb{R})}^{1-\frac{1}{p}} \cdot \|f\|_{L_p(\mathbb{R},\rho)}. \tag{54}$$

Here, $\Gamma(\cdot)$ denotes the Gamma function [1, 4]

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

d) Cauchy problem for the diffusion equation

Finally, consider the initial value problem for the one-dimensional diffusion equation with no sources or sinks

$$ku_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0, \tag{55}$$

with the boundary conditions

$$u_x(x,t) \to 0 \text{ as } |x| \to \infty, \tag{56}$$

$$u(x,t) \to 0 \text{ as } |x| \to \infty, \tag{57}$$

and the initial condition

$$u(x,0) = f(x)\rho(x), \tag{58}$$

where $f \in L_1(\mathbb{R},\rho) \cap L_p(\mathbb{R},\rho)$, $p > 1$ is given, and $k > 0$ is a diffusivity constant.

Again, by applying the Hartley transform (49) with respect to x to both sides of equation (55) and the condition (58) with set $(H_1 u) = U$, we obtain

$$\frac{d}{dt} U(y,t) = -ky^2 U(y,t), \tag{59}$$

with the initial condition

$$U(y,0) = (H_1(f\rho))(y). \tag{60}$$

The solution of the problem (59)–(60) is of the form

$$U(y,t) = e^{-ky^2 t}(H_1(f\rho))(y).$$

Using formula $(1.4.11)$ in ([4], p. 24) we obtain

$$U(y,t) = \sqrt{\frac{2}{\pi}} F_c \left(\frac{e^{-\frac{\tau^2}{4kt}}}{\sqrt{kt}} \right)(y) \cdot (H_1(f\rho))(y) = H_1 \left(\frac{e^{-\frac{\tau^2}{4kt}}}{\sqrt{kt}} \underset{1}{*} (f\tau) \right)(y).$$

Thus

$$u(x,t) = \left(\frac{e^{-\frac{\tau^2}{4kt}}}{\sqrt{kt}} \underset{1}{*} (f\tau) \right)(x). \tag{61}$$

For each $t > 0$, using inequality (32) we get

$$\|u\|_{L_p(\mathbb{R}_+)} \leqslant 2^{2-\frac{1}{p}} \|\rho\|_{L_1(\mathbb{R})}^{1-\frac{1}{p}} \cdot \|f\|_{L_p(\mathbb{R},\rho)} \left\| \frac{e^{-\frac{\tau^2}{4kt}}}{\sqrt{kt}} \right\|_{L_p(\mathbb{R})}.$$

Therefore,

$$\|u\|_{L_p(\mathbb{R})} \leqslant 2^{2-\frac{1}{p}} \left(\frac{\pi}{\sqrt{p}(\sqrt{kt})^{p-1}} \right)^{\frac{1}{p}} \|\rho\|_{L_1(\mathbb{R})}^{1-\frac{1}{p}} \cdot \|f\|_{L_p(\mathbb{R},\rho)}. \tag{62}$$

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