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SOME RESULTS FOR HAUSDORFF OPERATORS

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Abstract. In this paper, we give the sufficient conditions for the boundedness of the (fractional) Hausdorff operators on the Lebesgue spaces with power weights. In some special cases, these conditions are the same and also necessary. As an application, we obtain a better lower bound of fractional Hardy operators on the Lebesgue spaces compared with a result of the paper [25].

1. Introduction

Hausdorff operators (Hausdorff summability methods) are an very useful tool for solving certain classical problems in analysis. They have a deep root in the study of the one dimensional Fourier analysis, particularly the summability of the classical Fourier series. Modern theory of Hausdorff operators started with the work of Siskakis [23] in complex analysis setting and with the work of Georgakis [10] and Liflyand and Móricz [19] in the Fourier transform setting. A brief overview of the study for Hausdorff operators can be found in [18]. One can see [1–6, 16–24] to find details of some recent developments for Hausdorff operators.

The one-dimensional Hausdorff operator is defined by

$$h_\Phi f(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt,$$

where Φ is a locally integrable function on $(0,\infty)$. Liflyand and Móricz [19] proved that h_Φ generated by a function $\Phi \in L^1(\mathbb{R})$ is a bounded linear operator on the real Hardy space $H^1(\mathbb{R})$ by the theory of Fourier transform and Hilbert transform. Following this, Hausdorff operators were considered in various spaces, for example, see [2, 13, 20, 22].

The one-dimensional Hausdorff operator contains the classical Hardy operator and its adjoint operator if we choose suitable functions Φ. For $x > 0$, when one chooses $\Phi(t)$ as $t^{-1}\chi_{(1,\infty)}(t)$ and $\chi_{(0,1]}(t)$, we obtain the classical Hardy operator h and the adjoint Hardy operator h^* respectively, where

$$hf(x) := \frac{1}{x}\int_0^x f(t)dt, \quad \text{and} \quad h^* f(x) := \int_x^\infty \frac{f(t)}{t} dt.$$

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It is well known that Hardy operators are important operators in Harmonic analysis, for instance, see [12, 14]. On the other hand, if we choose $\Phi(t) = \alpha(1-t)^{\alpha-1}\chi(0,1)(t)$ for $\alpha = 1,2,\ldots$, then $H_\Phi = C_\alpha$ is called the Cesàro operator of order α. A brief history of the study of the Cesäro operator can be found in [13].

For multidimensional Hausdorff operators, there are many kinds of definitions [1, 3, 4, 16–18, 21, 22]. One of the interesting definitions of the Hausdorff operators is

$$H_\Phi f(x) = \int_{\mathbb{R}^n} \frac{\Phi(x/|y|)}{|y|^n} f(y)dy.$$

Similar to h_Φ, H_Φ contains the high dimensional Hardy operator H and its adjoint operator H^* (see the below definitions). Recently, Chen, Fan and Li [4] obtained that if Φ is a radial function and $1 \leqslant p \leqslant \infty$, then

$$\|H_\Phi f\|_{L^p(\mathbb{R}^n)} \leqslant \omega_{n-1} \int_0^\infty |\Phi(t)| t^{-1+\frac{n}{p}} dt \cdot \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.1}$$

For a general function Φ, Wang [24] proved

$$\|H_\Phi f\|_{L^p(\mathbb{R}^n)} \leqslant \omega_{n-1}^{\frac{1}{p'}} \int_0^\infty \left(\int_{S^{n-1}} |\Phi(t\varphi)|^p d\varphi \right)^{\frac{1}{p}} t^{-1+\frac{n}{p}} dt \cdot \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.2}$$

In [22], Lin and Sun defined the n-dimensional fractional Hausdorff operator for a radial function Φ as follows

$$H_{\Phi,\beta} f(x) = \int_{\mathbb{R}^n} \frac{\Phi(|x|/|y|)}{|y|^{n-\beta}} f(y)dy, \qquad 0 \leqslant \beta < n.$$

They obtained that

THEOREM A. *Let* $1 \leqslant p,\ q < \infty$, $0 < \beta < n$, $\gamma > \beta p - n$ *and* $\frac{n+\gamma}{p} - \beta = \frac{n+\gamma}{q}$. *If* $\int_0^\infty |\Phi(t)|^{\frac{p}{p-1}} t^{\frac{\gamma-\beta p+n-p+1}{p-1}\pm\varepsilon} dt < \infty$ *for small* $\varepsilon > 0$, *then*

$$\|H_{\Phi,\beta} f\|_{L^q(\mathbb{R}^n,|x|^\gamma)} \leqslant C\|f\|_{L^p(\mathbb{R}^n,|x|^\gamma)}, \quad p > 1$$

and if $\||\cdot|^{n-\beta+\gamma\pm\varepsilon}\Phi(\cdot)\|_{L^\infty(\mathbb{R}^n)} < \infty$, *then*

$$\|H_{\Phi,\beta} f\|_{L^q(\mathbb{R}^n,|x|^\gamma)} \leqslant C\|f\|_{L^1(\mathbb{R}^n,|x|^\gamma)}.$$

If we choose Φ as the following radial functions $|t|^{\beta-n}\chi_{(1,\infty)}(|t|)$ and $\chi_{(0,1]}(|t|)$, $H_{\Phi,\beta} f$ becomes the high dimensional fractional Hardy operator H_β and its adjoint operator H_β^* respectively, where

$$H_\beta f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} f(y)dy, \quad x \in \mathbb{R}^n \setminus \{0\} \text{ and } H_\beta^* f(x) = \int_{|y|\geqslant|x|} \frac{f(y)}{|y|^{n-\beta}} dy.$$

For some results about H_β and H_β^*, please refer to [8] and [25]. Let $\mathbb{H}_\beta := \nu_n^{-1+\frac{\beta}{n}} H_\beta$, where ν_n is the volume of the unit ball in $\mathbb{R}^n$. Recently, Lu, Yan and Zhao [25] proved that the following result.

THEOREM B. *Suppose that* $0 \leqslant \beta < n$, $1 < p \leqslant \frac{n}{\beta}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$. *If* $f \in L^p(\mathbb{R}^n)$, *then*

$$\|\mathbb{H}_\beta f\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)},$$

where

$$\left(\frac{p}{q}\right)^{\frac{1}{q}}\left(\frac{p}{p-1}\right)^{\frac{1}{q}}\left(\frac{q}{q-1}\right)^{1-\frac{1}{q}}\left(1-\frac{p}{q}\right)^{\frac{1}{p}-\frac{1}{q}} \leqslant C \leqslant \left(\frac{p}{p-1}\right)^{\frac{p}{q}}.$$

In this paper, we consider the general function Φ for $H_{\Phi,\beta}$ for $0 \leqslant \beta < n$. In Section 2, we give sufficient conditions for the boundedness of the one-dimensional (fractional) Hausdorff operators $h_{\Phi,\beta}$ defined on $\mathbb{R}^+$ on the Lebesgue spaces with power weights. In Section 3, we give two sufficient conditions for the boundedness of the high (fractional) Hausdorff operators on the Lebesgue spaces with power weights. In Section 4, as applications, we obtain some explicit bounds of fractional Hardy operators on the Lebesgue spaces. In particular, we have a better lower bound of fractional Hardy operators on the Lebesgue spaces compared with a result of the paper by Lu, Yan and Zhao [25].

Throughout this paper, ω_{n-1} denotes the area of the unit sphere S^{n-1} and ν_n is the volume of the unit ball in $\mathbb{R}^n$. We use $B(0,R)$ to denote the ball of radius R in $\mathbb{R}^n$ centered at the origin and $B(0,R)^c = \mathbb{R}^n \setminus B(0,R)$. We denote $H_{\Phi,\beta} := H_\Phi$, $H_0 := H$ and $H_0^* := H^*$.

2. Estimates for one-dimensional Hausdorff operators

It is well known that using Minkowski inequality and scaling, we can show the operator h_Φ is bounded on $L^p(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$, if

$$K_{\Phi,p} = \int_0^\infty |\Phi(t)| t^{-1+\frac{1}{p}} dt < \infty$$

and

$$\|h_\Phi f\|_{L^p(\mathbb{R})} \leqslant K_{\Phi,p}\|f\|_{L^p(\mathbb{R})}.$$

Here we point out that h_Φ can be regarded as convolution in the multiplicative group $\mathbb{R}^+$ with Haar measure $\frac{dx}{x}$. So we can use Young's inequality (see Lemma 2.1) to obtain the boundedness of h_Φ on $L^p(\mathbb{R}^+)$. In this section, we can prove the boundedness of h_Φ on the Lesbesgue spaces with power weights. Moreover, we can extend these results to the fractional Hausdorff operator $h_{\Phi,\beta}$, where

$$h_{\Phi,\beta} f(x) = \int_0^\infty \frac{\Phi(xt^{-1})}{t^{1-\beta}} f(t) dt, \quad x > 0.$$

THEOREM 2.1. *Let* $1 \leqslant p,\ q \leqslant \infty$ *and* $0 \leqslant \beta < 1$ *satisfy* $\frac{1}{q} = \frac{1}{p} - \beta$. *If*

$$K_{\Phi,\beta,q} = \left(\int_0^\infty |\Phi(t)|^{\frac{1}{1-\beta}} t^{-1+\frac{1}{q(1-\beta)}} dt\right)^{1-\beta} < \infty,$$

then the operator $h_{\Phi,\beta}$ *from* $L^p(\mathbb{R}^+)$ *into* $L^q(\mathbb{R}^+)$ *is bounded, i.e.,*

$$\|h_{\Phi,\beta}f\|_{L^q(\mathbb{R}^+)} \leqslant K_{\Phi,\beta,q}\|f\|_{L^p(\mathbb{R}^+)}$$

for all $f \in L^p(\mathbb{R}^+)$.

More generally, we have the following estimate.

THEOREM 2.2. *Let* $1 \leqslant p,\ q \leqslant \infty,\ 0 \leqslant \beta < 1$ *and* $\alpha,\ \gamma \in \mathbb{R}$ *satisfy* $\frac{\gamma+1}{q} = \frac{\alpha+1}{p} - \beta$. *If*

$$K_{\Phi,s,q,\gamma} = \left(\int_0^\infty |\Phi(t)|^s t^{-1+\frac{s(\gamma+1)}{q}}\,dt\right)^{\frac{1}{s}} < \infty,$$

where s *satisfies* $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$, *then the operator* $h_{\Phi,\beta}$ *from* $L^p(\mathbb{R}^+, x^\alpha)$ *into* $L^q(\mathbb{R}^+, x^\gamma)$ *is bounded, i.e.,*

$$\|h_{\Phi,\beta}f\|_{L^q(\mathbb{R}^+,x^\gamma)} \leqslant K_{\Phi,s,q,\gamma}\|f\|_{L^p(\mathbb{R}^+,x^\alpha)}$$

for all $f \in L^p(\mathbb{R}^+, x^\alpha)$.

REMARK 2.1. Obviously, let $\alpha = \gamma = 0$ in Theorem 2.2, then we can obtain $s = \frac{1}{1-\beta}$. Furthermore, we know that Theorem 2.1 holds. If $\beta = 0$ in Theorem 2.1, then we obtain the boundedness of h_Φ on $L^p(\mathbb{R}^+)$.

REMARK 2.2. Let $\beta = 0$, $\gamma = \alpha$ and $\Phi \geqslant 0$, then $s = 1$ and

$$K_{\Phi,1,q,\gamma} = \int_0^\infty \Phi(t)t^{-1+\frac{\gamma+1}{q}}\,dt < \infty$$

is the sufficient and necessary condition for the operator h_Φ on $L^q(\mathbb{R}^+, x^\gamma)$. See [27] for the detail.

Before proving the main results, we first recall the following Young's inequality for convolution.

LEMMA 2.1. *([11]) Let* $1 \leqslant p,\ q,\ r \leqslant \infty$ *satisfy* $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, *and* μ *be a Haar measure on a locally compact group* G, *then*

$$\|f * g\|_{L^q(G,\ \mu)} \leqslant \|g\|_{L^r(G,\mu)}\|f\|_{L^p(G,\ \mu)}$$

for all f *in* $L^p(G,\ \mu)$ *and for all* g *in* $L^r(G,\ \mu)$ *satisfying* $\|g\|_{L^r(G,\ \mu)} = \|\widetilde{g}\|_{L^r(G,\ \mu)}$, *where* $\widetilde{g}(x) = g(x^{-1})$.

Proof of Theorem 2.2. The proof is based on an idea used in [7] for proving the boundedness of Hardy operator on $L^p(\mathbb{R})$. It is well known that the multiplicative group $\mathbb{R}^+$ is a locally compact group with Haar measure $\frac{dx}{x}$. Note that $\frac{\gamma+1}{q} = \frac{\alpha+1}{p} - \beta$, we have

$$\begin{aligned}\|h_{\Phi,\beta}f\|_{L^q(\mathbb{R}^+,x^\gamma)} &= \left(\int_0^\infty \left|\int_0^\infty \Phi(xt^{-1})x^{\frac{\gamma+1}{q}} f(t)t^\beta \frac{dt}{t}\right|^q \frac{dx}{x}\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left|\int_0^\infty \Phi(xt^{-1})(xt^{-1})^{\frac{\gamma+1}{q}} f(t)t^{\frac{\alpha+1}{p}} \frac{dt}{t}\right|^q \frac{dx}{x}\right)^{\frac{1}{q}} \\ &\leqslant \left(\int_0^\infty \left(\int_0^\infty |\Phi(xt^{-1})|(xt^{-1})^{\frac{\gamma+1}{q}} |f(t)t^{\frac{\alpha+1}{p}}| \frac{dt}{t}\right)^q \frac{dx}{x}\right)^{\frac{1}{q}}.\end{aligned}$$

Because the right side of the above inequality is a convolution inequality on the multiplicative group $\mathbb{R}^+$ with Haar measure $\frac{dx}{x}$, so by Lemma 2.1 for $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$, we get

$$\left(\int_0^\infty \left(\int_0^\infty \left| \Phi(xt^{-1}) \right| (xt^{-1})^{\frac{\gamma+1}{q}} \left| f(t) t^{\frac{\alpha+1}{p}} \right| \frac{dt}{t} \right)^q \frac{dx}{x} \right)^{\frac{1}{q}} \leqslant K_{\Phi,s,q,\gamma} \|f\|_{L^p(\mathbb{R}^+, x^\alpha)},$$

where

$$K_{\Phi,s,q,p,\gamma} = \left(\int_0^\infty |\Phi(t)|^s t^{-1+\frac{s(\gamma+1)}{q}} dt \right)^{\frac{1}{s}}.$$

Therefore, we finish the proof. $\square$

3. High-dimensional case: $n \geqslant 2$

In this section, we consider the n-dimensional fractional Hausdorff operator for a general function Φ as follows

$$H_{\Phi,\beta} f(x) = \int_{\mathbb{R}^n} \frac{\Phi(x/|y|)}{|y|^{n-\beta}} f(y) dy, \qquad 0 \leqslant \beta < n.$$

First of all, we formulate our main results.

THEOREM 3.1. *Let* $1 \leqslant p,\ q \leqslant \infty,\ 0 \leqslant \beta < n$ *and* $\alpha,\ \gamma \in \mathbb{R}$ *satisfy* $\frac{\gamma+n}{q} = \frac{\alpha+n}{p} - \beta$. *If*

$$K_{\Phi,s,n,q,\gamma} = \left(\int_0^\infty \left(\int_{S^{n-1}} |\Phi(t\varphi)|^q d\varphi \right)^{\frac{s}{q}} t^{-1+\frac{(n+\gamma)s}{q}} dt \right)^{\frac{1}{s}} < \infty, \tag{3.1}$$

where s *satisfies* $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$, *then the operator* $H_{\Phi,\beta}$ *from* $L^p(\mathbb{R}^n, |x|^\alpha)$ *into* $L^q(\mathbb{R}^n, |x|^\gamma)$ *is bounded, i.e.,*

$$\|H_{\Phi,\beta} f\|_{L^q(\mathbb{R}^n, |x|^\gamma)} \leqslant \omega_{n-1}^{\frac{1}{p'}} K_{\Phi,s,n,q,\gamma} \|f\|_{L^p(\mathbb{R}^n, |x|^\alpha)} \tag{3.2}$$

for all $f \in L^p(\mathbb{R}^n, |x|^\alpha)$.

COROLLARY 3.1. *Let* $1 \leqslant p,\ q \leqslant \infty$ *and* $0 \leqslant \beta < n$ *satisfy* $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. *If*

$$K_{\Phi,s,n,q} = \left(\int_0^\infty \left(\int_{S^{n-1}} |\Phi(t\varphi)|^q d\varphi \right)^{\frac{s}{q}} t^{-1+\frac{ns}{q}} dt \right)^{\frac{1}{s}} < \infty,$$

where $s = \frac{n}{n-\beta}$, *then the operator* $H_{\Phi,\beta}$ *from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$ *is bounded, i.e.,*

$$\|H_{\Phi,\beta} f\|_{L^q(\mathbb{R}^n)} \leqslant \omega_{n-1}^{\frac{1}{p'}} K_{\Phi,s,n,q} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$.

THEOREM 3.2. *Let* $1 \leqslant p,\ q \leqslant \infty,\ 0 \leqslant \beta < n$ *and* $\alpha,\ \gamma \in \mathbb{R}$ *satisfy* $\frac{\gamma+n}{q} = \frac{\alpha+n}{p} - \beta$.

(i) For any radial function $\Phi(x)$, *if*

$$\widetilde{K}_{\Phi,s,n,q,\gamma} = \left(\int_{\mathbb{R}^n} |\Phi(x)|^s |x|^{-n+\frac{(\gamma+n)s}{q}} dx \right)^{\frac{1}{s}} < \infty, \tag{3.3}$$

where s *satisfies* $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$, *then the operator* $H_{\Phi,\beta}$ *from* $L^p(\mathbb{R}^n, |x|^\alpha)$ *into* $L^q(\mathbb{R}^n, |x|^\gamma)$ *is bounded, i.e.,*

$$\|H_{\Phi,\beta} f\|_{L^q(\mathbb{R}^n,|x|^\gamma)} \leqslant \widetilde{K}_{\Phi,s,n,q,\gamma} \|f\|_{L^p(\mathbb{R}^n,|x|^\alpha)} \tag{3.4}$$

for all $f \in L^p(\mathbb{R}^n, |x|^\alpha)$;

(ii) For any general function $\Phi(x)$, *if*

$$\widetilde{\widetilde{K}}_{\Phi,s,n,q,\gamma} = \omega_{n-1}^{\frac{1}{p'}} \left(\int_{S^{n-1}} \left(\int_0^\infty |\Phi(\rho\varphi)|^s \rho^{-1+\frac{(\gamma+n)s}{q}} d\rho \right)^{\frac{q}{s}} d\varphi \right)^{\frac{1}{q}} < \infty,$$

where s *satisfies* $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$, *then the operator* $H_{\Phi,\beta}$ *from* $L^p(\mathbb{R}^n, |x|^\alpha)$ *into* $L^q(\mathbb{R}^n, |x|^\gamma)$ *is bounded, i.e.,*

$$\|H_{\Phi,\beta} f\|_{L^q(\mathbb{R}^n,|x|^\gamma)} \leqslant \widetilde{\widetilde{K}}_{\Phi,s,n,q,\gamma} \|f\|_{L^p(\mathbb{R}^n,|x|^\alpha)} \tag{3.5}$$

for all $f \in L^p(\mathbb{R}^n, |x|^\alpha)$.

REMARK 3.1. When $\Phi(x)$ is a radial function, then by (3.1), we have

$$K_{\Phi,s,n,q,\gamma} = \omega_{n-1}^{\frac{1}{q}} \left(\int_0^\infty |\Phi(t)|^s t^{-1+\frac{(n+\gamma)s}{q}} dt \right)^{\frac{1}{s}},$$

and by (3.3) we obtain

$$\widetilde{K}_{\Phi,s,n,q,\gamma} = \widetilde{\widetilde{K}}_{\Phi,s,n,q,\gamma} = \omega_{n-1}^{\frac{1}{p'}+\frac{1}{q}} \left(\int_0^\infty |\Phi(t)|^s t^{-1+\frac{(n+\gamma)s}{q}} dt \right)^{\frac{1}{s}}.$$

Therefore, compared (3.2) and (3.4), we obtain that Theorem 3.1 coincides with Theorem 3.2 if Φ is a radial function. For any general function $\Phi(x)$, by Minkowski's inequality, we get $\widetilde{\widetilde{K}}_{\Phi,s,n,q,\gamma} \leqslant \omega_{n-1}^{\frac{1}{p'}} K_{\Phi,s,n,q,\gamma}$, see the proof of Theorem 3.1 for the details. But we point out that the proofs of Theorem 3.1 and Theorem 3.2 are very different.

REMARK 3.2. Let $\beta = 0,\ \gamma = \alpha$ and Φ be a non-negative radial function, then

$$\widetilde{K}_{\Phi,s,n,q,\gamma} = \omega_{n-1} \int_0^\infty \Phi(t) t^{-1+\frac{(n+\gamma)}{q}} dt < \infty$$

is the sufficient and necessary condition for the operator h_Φ on $L^q(\mathbb{R}^n, |x|^\gamma)$. See [27] for the detail.

REMARK 3.3. Obviously, if $\beta = 0$, then $s = 1$ and $p = q$. Therefore using Corollary 3.1, we obtain the inequality (1.2). Furthermore, if we choose Φ is a radial function, then we have (1.1).

REMARK 3.4. Unluckily, our results are not comparable to Theorem A. Our idea and method are different from [22].

Proof of Theorem 3.1. By polar coordinates, we know

$$H_{\Phi,\beta}f(x)=\int_0^\infty\int_{S^{n-1}}\Phi\Big(\frac{x}{t}\Big)t^\beta f(t\theta)d\theta\frac{dt}{t}$$

and

$$\|H_{\Phi,\beta}f\|^q_{L^q(\mathbb{R}^n,|x|^\gamma)}=\int_0^\infty\int_{S^{n-1}}\Big|\int_0^\infty\int_{S^{n-1}}\Phi(\rho\varphi t^{-1})\rho^{\frac{\gamma+n}{q}}f(t\theta)t^\beta d\theta\frac{dt}{t}\Big|^q d\varphi\frac{d\rho}{\rho}.$$

We apply $\frac{\gamma+n}{q}=\frac{\alpha+n}{p}-\beta$ and then to Fubini's theorem to interchange the integrals in ρ and φ. Then

$$\begin{aligned}
&\|H_{\Phi,\beta}f\|^q_{L^q(\mathbb{R}^n,|x|^\gamma)}\\
&=\int_0^\infty\int_{S^{n-1}}\Big|\int_0^\infty\int_{S^{n-1}}\Phi(\rho\varphi t^{-1})(\rho t^{-1})^{\frac{\gamma+n}{q}}f(t\theta)t^{\frac{n+\alpha}{p}}d\theta\frac{dt}{t}\Big|^q d\varphi\frac{d\rho}{\rho}\\
&\leqslant\int_0^\infty\int_{S^{n-1}}\Big(\int_0^\infty\int_{S^{n-1}}|\Phi(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{\gamma+n}{q}}|f(t\theta)|t^{\frac{n+\alpha}{p}}d\theta\frac{dt}{t}\Big)^q d\varphi\frac{d\rho}{\rho}\\
&=\int_{S^{n-1}}\int_0^\infty\Big(\int_{S^{n-1}}\int_0^\infty|\Phi(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{\gamma+n}{q}}|f(t\theta)|t^{\frac{n+\alpha}{p}}\frac{dt}{t}d\theta\Big)^q\frac{d\rho}{\rho}d\varphi.
\end{aligned}$$

By Minkowski's inequality, we obtain

$$\begin{aligned}
&\Big(\int_0^\infty\Big(\int_{S^{n-1}}\int_0^\infty|\Phi(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{\gamma+n}{q}}|f(t\theta)|t^{\frac{n+\alpha}{p}}\frac{dt}{t}d\theta\Big)^q\frac{d\rho}{\rho}\Big)^{\frac{1}{q}}\\
&\leqslant\int_{S^{n-1}}\Big(\int_0^\infty\Big|\int_0^\infty|\Phi(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{\gamma+n}{q}}|f(t\theta)|t^{\frac{n+\alpha}{p}}\frac{dt}{t}\Big|^q\frac{d\rho}{\rho}\Big)^{\frac{1}{q}}d\theta.
\end{aligned}$$

For $\int_0^\infty|\Phi(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{\gamma+n}{q}}|f(t\theta)|t^{\frac{n+\alpha}{p}}\frac{dt}{t}$, we can regard it as a convolution inequality on the multiplicative group $\mathbb{R}^+$ with Haar measure $\frac{dx}{x}$. Applying Lemma 2.1 for $\frac{1}{q}=\frac{1}{s}+\frac{1}{p}-1$, we have

$$\begin{aligned}
&\Big(\int_0^\infty\Big(\int_0^\infty|\Phi(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{\gamma+n}{q}}|f(t\theta)|t^{\frac{n+\alpha}{p}}\frac{dt}{t}\Big)^q\frac{d\rho}{\rho}\Big)^{\frac{1}{q}}\\
&\leqslant\Big(\int_0^\infty|\Phi(\rho\varphi)|^s\rho^{\frac{(\gamma+n)s}{q}}\frac{d\rho}{\rho}\Big)^{\frac{1}{s}}\Big(\int_0^\infty|f(\rho\theta)|^p\rho^{n+\alpha}\frac{d\rho}{\rho}\Big)^{\frac{1}{p}}\\
&=\Big(\int_0^\infty|\Phi(t\varphi)|^s t^{-1+\frac{(\gamma+n)s}{q}}dt\Big)^{\frac{1}{s}}\Big(\int_0^\infty|f(t\theta)|^p t^{\alpha+n-1}dt\Big)^{\frac{1}{p}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|H_{\Phi,\beta}f\|^q_{L^q(\mathbb{R}^n,|x|^\gamma)}\\
&\leqslant \int_{S^{n-1}}\left(\int_{S^{n-1}}\left(\int_0^\infty |\Phi(t\varphi)|^s t^{-1+\frac{(\gamma+n)s}{q}}dt\right)^{\frac{1}{s}}\left(\int_0^\infty |f(t\theta)|^p t^{\alpha+n-1}dt\right)^{\frac{1}{p}}d\theta\right)^q d\varphi\\
&=\left(\int_{S^{n-1}}\left(\int_0^\infty |\Phi(t\varphi)|^s t^{-1+\frac{(\gamma+n)s}{q}}dt\right)^{\frac{q}{s}}d\varphi\right)\left(\int_{S^{n-1}}\left(\int_0^\infty |f(t\theta)|^p t^{\alpha+n-1}dt\right)^{\frac{1}{p}}d\theta\right)^q.
\end{aligned}$$

Applying Hölder's inequality, we deduce that

$$\begin{aligned}
\int_{S^{n-1}}\left(\int_0^\infty |f(t\theta)|^p t^{\alpha+n-1}dt\right)^{\frac{1}{p}}d\theta &\leqslant |S^{n-1}|^{\frac{1}{p'}}\left(\int_{S^{n-1}}\int_0^\infty |f(t\theta)|^p t^{\alpha+n-1}dt\,d\theta\right)^{\frac{1}{p}}\\
&=\omega_{n-1}^{\frac{1}{p'}}\cdot\|f\|_{L^p(\mathbb{R}^n,|x|^\alpha)}.
\end{aligned}$$

Hence, we have

$$\|H_{\Phi,\beta}f\|_{L^q(\mathbb{R}^n,|x|^\gamma)}\leqslant \omega_{n-1}^{\frac{1}{p'}}\left(\int_{S^{n-1}}\left(\int_0^\infty |\Phi(t\varphi)|^s t^{-1+\frac{(\gamma+n)s}{q}}dt\right)^{\frac{q}{s}}d\varphi\right)^{\frac{1}{q}}\|f\|_{L^p(\mathbb{R}^n,|x|^\alpha)}.$$

Note that $s\leqslant q$ and then by Minkowski's inequality, so we obtain

$$\begin{aligned}
&\left(\int_{S^{n-1}}\left(\int_0^\infty |\Phi(t\varphi)|^s t^{-1+\frac{(\gamma+n)s}{q}}dt\right)^{\frac{q}{s}}d\varphi\right)^{\frac{1}{q}}\\
&\leqslant\left(\int_0^\infty\left(\int_{S^{n-1}}|\Phi(t\varphi)|^q d\varphi\right)^{\frac{s}{q}}t^{-1+\frac{(\gamma+n)s}{q}}dt\right)^{\frac{1}{s}}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|H_{\Phi,\beta}f\|_{L^q(\mathbb{R}^n,|x|^\gamma)}&\leqslant \omega_{n-1}^{\frac{1}{p'}}\left(\int_0^\infty\left(\int_{S^{n-1}}|\Phi(t\varphi)|^q d\varphi\right)^{\frac{s}{q}}t^{-1+\frac{(\gamma+n)s}{q}}dt\right)^{\frac{1}{s}}\|f\|_{L^p(\mathbb{R}^n,|x|^\alpha)}\\
&=\omega_{n-1}^{\frac{1}{p'}}K_{\Phi,s,n,q,\gamma}\|f\|_{L^p(\mathbb{R}^n,|x|^\alpha)}.\quad\square
\end{aligned}$$

Proof of Theorem 3.2. We adapt some ideas and methods used in [9]. Let

$$g_f(x)=\frac{1}{\omega_{n-1}}\int_{S^{n-1}}f(|x|\varphi)d\varphi.$$

Obviously, g_f is a radial function. We can obtain $\|g_f\|_{L^p(\mathbb{R}^n,|x|^\gamma)}\leqslant\|f\|_{L^p(\mathbb{R}^n,|x|^\gamma)}$, because

$$\begin{aligned}
\|g_f\|_{L^p(\mathbb{R}^n,|x|^\gamma)} &\leqslant \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left| f(|x|\varphi) \right|^p |x|^\gamma dx \right)^{\frac{1}{p}} d\varphi \\
&= \frac{1}{\omega_{n-1}^{1/p'}} \int_{S^{n-1}} \left(\int_0^\infty \left| f(r\varphi) \right|^p r^{\gamma+n-1} dr \right)^{\frac{1}{p}} d\varphi \\
&\leqslant \left(\int_{S^{n-1}} \int_0^\infty \left| f(r\varphi) \right|^p r^{\gamma+n-1} dr d\varphi \right)^{\frac{1}{p}} \\
&= \|f\|_{L^p(\mathbb{R}^n,|x|^\gamma)}.
\end{aligned}$$

Note that we have used Minkowski's inequality, polar coordinates and Hölder's inequality. On the other hand, by Fubini's Theorem, we have

$$\begin{aligned}
H_{\Phi,\beta}(g_f)(x) &= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \int_{\mathbb{R}^n} \frac{\Phi(x|y|^{-1})}{|y|^{n-\beta}} f(|y|\varphi) dy d\varphi \\
&= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \int_0^\infty \int_{S^{n-1}} \frac{\Phi(xr^{-1})}{r^{n-\beta}} f(r\varphi) r^{n-1} d\sigma dr d\varphi \\
&= \int_{S^{n-1}} \int_0^\infty \frac{\Phi(xr^{-1})}{r^{n-\beta}} f(r\varphi) r^{n-1} dr d\varphi \\
&= H_{\Phi,\beta}(f)(x).
\end{aligned}$$

In consequence,

$$\frac{\|H_{\Phi,\beta}(f)\|_{L^q(\mathbb{R}^n,|x|^\gamma)}}{\|f\|_{L^p(\mathbb{R}^n,|x|^\gamma)}} \leqslant \frac{\|H_{\Phi,\beta}(g_f)\|_{L^q(\mathbb{R}^n,|x|^\gamma)}}{\|g_f\|_{L^p(\mathbb{R}^n,|x|^\gamma)}}.$$

Therefore, this implies that if we want to obtain the operator norm of $H_{\Phi,\beta}$ from $L^p(\mathbb{R}^n,|x|^\gamma)$ to $L^q(\mathbb{R}^n,|x|^\gamma)$, we can restrict to f radial functions. So we can assume that $f(x)$ is a radial function in the following proof.

$$\begin{aligned}
&\|H_{\Phi,\beta}(f)\|_{L^q(\mathbb{R}^n,|x|^\gamma)} \\
&= \left(\int_0^\infty \int_{S^{n-1}} \left| \int_0^\infty \int_{S^{n-1}} \frac{\Phi(\rho\varphi r^{-1})}{r^{1-\beta}} f(r) d\sigma dr \right|^q \rho^{\gamma+n-1} d\varphi d\rho \right)^{\frac{1}{q}} \qquad (3.6) \\
&= \omega_{n-1} \left(\int_{S^{n-1}} \int_0^\infty \left| \int_0^\infty \frac{\Phi(\rho\varphi r^{-1})}{r^{1-\beta}} f(r) dr \right|^q \rho^{\gamma+n-1} d\rho d\varphi \right)^{\frac{1}{q}}.
\end{aligned}$$

Now we prove the inequalities (3.4) and (3.5), respectively.

(i) If Φ is a radial function, then we have

$$\begin{aligned}
\|H_{\Phi,\beta}(f)\|_{L^q(\mathbb{R}^n,|x|^\gamma)} &= \omega_{n-1}^{1+\frac{1}{q}} \left(\int_0^\infty \left| \int_0^\infty \frac{\Phi(\rho r^{-1})}{r^{1-\beta}} f(r) dr \right|^q \rho^{\gamma+n-1} d\rho \right)^{\frac{1}{q}} \\
&= \omega_{n-1}^{1+\frac{1}{q}} \|h_{\Phi,\beta} f\|_{L^q(\mathbb{R}^+,\rho^{\gamma+n-1})}.
\end{aligned}$$

By Theorem 2.2, note that $\frac{\gamma+n}{q}=\frac{\alpha+n}{p}-\beta$ and $\frac{1}{q}=\frac{1}{p}+\frac{1}{s}-1$, therefore we deduce that

$$\begin{aligned}
&\|H_{\Phi,\beta}(f)\|_{L^q(\mathbb{R}^n,|x|^\gamma)}\\
&\leqslant \omega_{n-1}^{1+\frac{1}{q}}\left(\int_0^\infty |\Phi(t)|^s t^{-1+\frac{(\gamma+n)s}{q}}dt\right)^{\frac{1}{s}}\|f\|_{L^p(\mathbb{R}^+,\rho^{\alpha+n-1})}\\
&=\left(\int_0^\infty\int_{S^{n-1}}|\Phi(t)|^s t^{-1+\frac{(\gamma+n)s}{q}}d\sigma dt\right)^{\frac{1}{s}}\left(\int_0^\infty\int_{S^{n-1}}|f(\rho)|^p\rho^{\alpha+n-1}d\sigma d\rho\right)^{\frac{1}{p}}\\
&=\widetilde{K}_{\Phi,s,n,q,\gamma}\|f\|_{L^p(\mathbb{R}^n,|x|^\alpha)},
\end{aligned}$$

where

$$\widetilde{K}_{\Phi,s,n,q,\gamma}=\left(\int_{\mathbb{R}^n}|\Phi(x)|^s|x|^{-n+\frac{(\gamma+n)s}{q}}dx\right)^{\frac{1}{s}}.$$

(ii) For a general function Φ, let

$$I_{\Phi,\beta,q,n}(\varphi)=\left(\int_0^\infty\left|\int_0^\infty\frac{\Phi(\rho\varphi r^{-1})}{r^{1-\beta}}f(r)dr\right|^q\rho^{\gamma+n-1}d\rho\right)^{\frac{1}{q}}. \tag{3.7}$$

Noticing that $\frac{\gamma+n}{q}=\frac{\alpha+n}{p}-\beta$, so

$$I_{\Phi,\beta,q,n}(\varphi)=\left(\int_0^\infty\left|\int_0^\infty\Phi(\rho\varphi r^{-1})(\rho r^{-1})^{\frac{\gamma+n}{q}}f(r)r^{\frac{\alpha+n}{p}}\frac{dr}{r}\right|^q\frac{d\rho}{\rho}\right)^{\frac{1}{q}}.$$

Using Lemma 2.1 for $\frac{1}{q}=\frac{1}{p}+\frac{1}{s}-1$, we obtain that

$$I_{\Phi,\beta,q,n}(\varphi)\leqslant\left(\int_0^\infty|\Phi(\rho\varphi)|^s\rho^{\frac{(\gamma+n)s}{q}-1}d\rho\right)^{\frac{1}{s}}\left(\int_0^\infty|f(\rho)|^p\rho^{\alpha+n-1}d\rho\right)^{\frac{1}{p}}.$$

Therefore, by (3.6), (3.7) and the above inequality, we have

$$\begin{aligned}
&\|H_{\Phi,\beta}(f)\|_{L^q(\mathbb{R}^n,|x|^\gamma)}\\
&\leqslant\omega_{n-1}\left(\int_{S^{n-1}}\left(\int_0^\infty|\Phi(\rho\varphi)|^s\rho^{-1+\frac{(\gamma+n)s}{q}}d\rho\right)^{\frac{q}{s}}d\varphi\right)^{\frac{1}{q}}\left(\int_0^\infty|f(\rho)|^p\rho^{\alpha+n-1}d\rho\right)^{\frac{1}{p}}\\
&\leqslant\omega_{n-1}^{\frac{1}{p'}}\left(\int_{S^{n-1}}\left(\int_0^\infty|\Phi(\rho\varphi)|^s\rho^{-1+\frac{(\gamma+n)s}{q}}d\rho\right)^{\frac{q}{s}}d\varphi\right)^{\frac{1}{q}}\left(\int_0^\infty\int_{S^{n-1}}|f(\rho)|^p\rho^{\alpha+n-1}d\varphi d\rho\right)^{\frac{1}{p}}\\
&=\widetilde{\widetilde{K}}_{\Phi,s,n,q,\gamma}\|f\|_{L^p(\mathbb{R}^n,|x|^\alpha)}.
\end{aligned}$$

This completes the proof of Theorem 3.2. □

4. Applications

APPLICATION 1. Let $\Phi(t) = t^{\beta-1}\chi_{(1,\infty)}(t)$, then $h_{\Phi,\beta} = h_\beta$, where h_β is the fractional Hardy operator. If $\alpha < p-1$, then by Theorem 2.2 we get

$$\|h_\beta f\|_{L^q(\mathbb{R}^+,x^\gamma)} \leqslant \left(\frac{p}{(p-\alpha-1)s}\right)^{\frac{1}{s}} \|f\|_{L^p(\mathbb{R}^+,x^\alpha)} \tag{4.1}$$

for all $f \in L^p(\mathbb{R}^+,x^\alpha)$, where $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$. Similarly, when $\Phi(t) = \chi_{(0,1]}(t)$ and $\alpha > p\beta - 1$, we have

$$\|h^*_\beta f\|_{L^q(\mathbb{R}^+,x^\gamma)} \leqslant \left(\frac{q}{(\gamma+1)s}\right)^{\frac{1}{s}} \|f\|_{L^p(\mathbb{R}^+,x^\alpha)} \tag{4.2}$$

for all $f \in L^p(\mathbb{R}^+,x^\alpha)$, where $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$. In particular, we have

COROLLARY 4.1. *Let $1 < p,\ q \leqslant \infty$ and $0 \leqslant \beta < 1$ satisfy $\frac{1}{q} = \frac{1}{p} - \beta$. Then the operators h_β and h^*_β from $L^p(\mathbb{R}^+)$ into $L^q(\mathbb{R}^+)$ are bounded, i.e.,*

$$\|h_\beta f\|_{L^q(\mathbb{R}^+)} \leqslant \left(\frac{p'}{q} + 1\right)^{\frac{1}{p'}+\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^+)},$$

$$\|h^*_\beta f\|_{L^q(\mathbb{R}^+)} \leqslant \left(\frac{q}{p'} + 1\right)^{\frac{1}{p'}+\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^+)}$$

for all $f \in L^p(\mathbb{R}^+)$. When $\beta = 0$, Wang, Lu and Yan [26] *obtain the inequality (4.1). In fact they obtain $\alpha < p-1$ and $\frac{\gamma+1}{q} = \frac{\alpha+1}{p}$ is a necessary condition for (4.1). For (4.2), they have similar results. Furthermore, if $s = 1$, we can obtain $\left(\frac{q}{(\gamma+1)s}\right)^{\frac{1}{s}} = \frac{p}{p-1}$, which is the best possible constant for the boundedness of Hardy operator on $L^p(\mathbb{R}^+)$. See* [12].

For high dimensional Hausdorff operators, choosing Φ as the radial functions $|t|^{\beta-n}\chi_{(1,\infty)}(|t|)$ and $\chi_{(0,1]}(|t|)$ respectively, according to Theorem 3.1 or Theorem 3.2 in the third section, we have

APPLICATION 2. Let $1 < p,\ q \leqslant \infty$ and $0 \leqslant \beta < n$ satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. Then the operators H_β and H^*_β from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ are bounded, i.e.,

$$\|H_\beta f\|_{L^q(\mathbb{R}^n)} \leqslant \left(\frac{p'}{q}\nu_n + \nu_n\right)^{\frac{1}{p'}+\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^n)},$$

$$\|H^*_\beta f\|_{L^q(\mathbb{R}^n)} \leqslant \left(\frac{q}{p'}\nu_n + \nu_n\right)^{\frac{1}{p'}+\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$, where ν_n is the volume of the unit ball in $\mathbb{R}^n$.

REMARK 4.1. In fact, we have the following inequalities:

$$\left(\int_{\mathbb{R}^n}\left(\frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}}\int_{B(0,|x|)}|f(y)|dy\right)^q dx\right)^{\frac{1}{q}} \leqslant \left(\frac{p'}{q}+1\right)^{\frac{1}{p'}+\frac{1}{q}}\left(\int_{\mathbb{R}^n}|f(y)|^p dy\right)^{\frac{1}{p}}, \tag{4.3}$$

$$\left(\int_{\mathbb{R}^n}\left(\int_{B(0,|x|)^c}\frac{|f(y)|}{|B(0,|y|)|^{1-\frac{\beta}{n}}}dy\right)^q dx\right)^{\frac{1}{q}} \leqslant \left(\frac{q}{p'}+1\right)^{\frac{1}{p'}+\frac{1}{q}}\left(\int_{\mathbb{R}^n}|f(y)|^p dy\right)^{\frac{1}{p}}.$$

When $\beta = 0$, a simple calculation shows that

$$\left(\int_{\mathbb{R}^n}\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}|f(y)|dy\right)^p dx\right)^{\frac{1}{p}} \leqslant \frac{p}{p-1}\left(\int_{\mathbb{R}^n}|f(y)|^p dy\right)^{\frac{1}{p}}, \qquad (4.4)$$

$$\left(\int_{\mathbb{R}^n}\left(\int_{B(0,|x|)^c}\frac{|f(y)|}{|B(0,|y|)|}dy\right)^p dx\right)^{\frac{1}{p}} \leqslant p\left(\int_{\mathbb{R}^n}|f(y)|^p dy\right)^{\frac{1}{p}}.$$

Christ and Grafakos [7] proved that the inequality (4.4) holds and the constant $\frac{p}{p-1}$ is the best possible for $1 < p < \infty$. In [25], the authors obtained

$$\left(\int_{\mathbb{R}^n}\left(\frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}}\int_{B(0,|x|)}|f(y)|dy\right)^q dx\right)^{\frac{1}{q}} \leqslant \left(\frac{p}{p-1}\right)^{\frac{p}{q}}\left(\int_{\mathbb{R}^n}|f(y)|^p dy\right)^{\frac{1}{p}} \qquad (4.5)$$

by the inequality (4.4), see Theorem B in the Introduction. However, in the following we will prove $\left(\frac{p'}{q}+1\right)^{\frac{1}{p'}+\frac{1}{q}} \leqslant \left(\frac{p}{p-1}\right)^{\frac{p}{q}}$ for $1 < p \leqslant q \leqslant \infty$. Therefore we obtain the better lower bound from (4.3) for the boundedness of the fractional Hardy operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.

PROPOSITION 4.1. *If $1 < p \leqslant q \leqslant \infty$, then*

$$\left(\frac{p'}{q}+1\right)^{\frac{1}{p'}+\frac{1}{q}} \leqslant \left(\frac{p}{p-1}\right)^{\frac{p}{q}}.$$

Equality holds if and only if $p = q$.

Proof. It is obvious that $\left(\frac{p'}{q}+1\right)^{\frac{1}{p'}+\frac{1}{q}} \leqslant \left(\frac{p}{p-1}\right)^{\frac{p}{q}}$ is equivalent to

$$\left(1+\frac{p}{(p-1)q}\right)^{\frac{(p-1)q}{p}+1} \leqslant \left(1+\frac{p}{(p-1)p}\right)^{\frac{(p-1)p}{p}+1}.$$

Let $g(t) = (1+t)^{1+\frac{1}{t}}$, $t > 0$, then we can obtain $g'(t) > 0$, where g' is the derivative of the function g. So the function $g(t)$ is strictly increasing. For $1 < p \leqslant q \leqslant \infty$, we have $\frac{p}{(p-1)q} \leqslant \frac{p}{p-1}\cdot\frac{1}{p}$. Hence, $g(\frac{p}{(p-1)q}) \leqslant g(\frac{p}{(p-1)p})$, equality holds if and only if $p = q$. □

REFERENCES

[1] K. F. ANDERSEN, *Boundedness of Hausdorff operators on* $L^p(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$, *and BMO*$(\mathbb{R}^n)$, Acta Sci. Math.(Szeged) **69** (2003), 409–418.

[2] G. BROWN AND F. MÓRICZ, *The Hausdorff operator and the quasi Hausdorff operators on the space* L^p, $1 \leqslant p < \infty$, Math. Inequal. Appl. **3** (2000), 105–115.

[3] G. BROWN AND F. MÓRICZ, *Multivariate Hausdorff operators on the spaces* $L^p(\mathbb{R}^n)$, J. Math. Anal. Appl. **271** (2002), 443–454.

[4] J. C. CHEN, D. S. FAN AND J. LI, *Hausdorff operators on function spaces*, Chin. Annal. Math. Ser. B **33** (2012), 537–556.

[5] J. C. CHEN, D. S. FAN AND C. J. ZHANG, *Multilinear Hausdorf operators and their best constants*, Acta Math. Sinica (English Ser.) **28** (2012), 1521–1530.

[6] J. C. CHEN, D. S. FAN AND C. J. ZHANG, *Boundedness of Hausdorff operators on some product Hardy type spaces*, Appl. Math. J. Chinese Univ. Ser. B. **27** (2012), 114–126.

[7] M. CHRIST AND L. GRAFAKOS, *Best constants for two nonconvolution inequalities*, Proc. Amer. Math. Soc. **123** (1995), 1687–1693.

[8] Z. W. FU, Z. G. LIU, S. Z. LU AND H. B. WANG, *Characterization for commutators of* n*-dimensional fractional Hardy operators*, Sci. China Math. Ser. A **50** (2007), 1418–1426.

[9] Z. W. FU, L. GRAFAKOS, S. Z. LU AND F. Y. ZHAO, *Sharp bounds for* m*-linear Hardy and Hilbert operators*, Houst. J. Math. **38** (2012), 225–244.

[10] C. GEORGAKIS, *The Hausdorff mean of a Fourier-Stieltjes transform*, Proc. Amer. Math. Soc. **116** (1992), 465–471.

[11] L. GRAFAKOS, *Classical Fourier Analysis, 2nd edition*, New York: Springer Science, Business Media LLC, 2008.

[12] G. H. HARDY, *Note on a theorem of Hilbert*, Math. Z. **6** (1920), 314–317.

[13] Y. KANJIN, *The Hausdorff operators on the real Hardy spaces* $H^p(\mathbb{R})$, Studia Math. **148** (2001), 37–45.

[14] A. KUFNER AND L.-E. PERSSON, *Weighted Inequalities of Hardy type*, Singapore: World Scientific Publishing Co Pte Ltd, 2003.

[15] A. K. LERNER AND E. LIFLYAND, *Multidimensional Hausdorff operators on the real Hardy spaces*, J. Aust. Math. Soc. **83** (2007), 79–86.

[16] E. LIFLYAND, *Open problems on Hausdorff operators, Complex Analysis and potential Theorey*, Proceedings of the Conference, Istanbul, Tukley, Sep. 8–14, 2006.

[17] E. LIFLYAND, *Boundedness of multidimensional Hausdorff operators on* $H^1(\mathbb{R}^n)$, Acta Sci. Math. (Szeged) **74** (2008), 845–851.

[18] E. LIFLYAND, *Complex and real Hausdorff operators*, `http://www.crm.es/Publications/11/Pr1046.pdf`.

[19] E. LIFLYAND AND F. MÓRICZ, *The Hausdorff operators is bounded on the real Hardy* $H^1(\mathbb{R})$, Proc. Amer. Math. Soc. **128** (2000), 1391–1396.

[20] E. LIFLYAND AND A. MIYACHI, *Boundedness of the Hausdorff operators in* H^p *spaces,* $0 < p < 1$, Studia Math. **194** (2009), 279–292.

[21] F. MÓRICZ, *Multivariate Hausdorff operators on the spaces* $H^1(\mathbb{R}^n)$ *and BMO*$(\mathbb{R}^n)$, Analysis Math. **31** (2005), 31–41.

[22] X. Y. LIN AND L. J. SUN, *Some estimates on the Hausdorff operator*, Acta Sci. Math. (Szeged) **78** (2012), 669–681.

[23] A. G. SISKAKIS, *The Cesàro operator is bounded on* H^1, Proc. Amer. Math. Soc. **110** (1990), 461–462.

[24] S. L. WANG, *Hausdorff operators*, Reports in Zhejiang Normal University, 2013.

[25] S. Z. LU, D. Y. YAN AND F. Y. ZHAO, *Sharp bounds for Hardy type operators on higher-dimensional product spaces*, J. Ineq. Appl. **2013** (2013), 148.

[26] S. M. WANG, S. Z. LU AND D. Y. YAN, *Explicit constants for Hardy's inequality with power weight on* n*-dimensional product spaces*, Sci. China Math. **55** (2012), 2469–2480.

[27] X. M. WU AND J. C. CHEN, *Best constants for Hausdorff operators on n-dimensional product spaces*, Sci. China Math. **57** (2014) 57, doi: 10.1007/s11425-013-4725-7.

[28] X. R. ZHU AND J. C. CHEN, *Integrability of the general product Hardy operators on the product Hardy spaces*, Appl. Math. J. Chinese Univ. Ser. B **27** (2012), 225–233.

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