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A GENERALIZATION OF THE HILBERT'S TYPE INEQUALITY

GAOWEN XI

(*Communicated by J. Pečarić*)

Abstract. In this paper, by introducing two parameters A, B and using the Euler-Maclaurin expansion for the Riemann zeta function, we establish an inequality of a weight coefficient. Using this inequality, we derive generalizations of a Hilbert's type inequality.

1. Introduction

If $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n \geqslant 0$, $b_n \geqslant 0$, for $n \geqslant 1$, $n \in N$ and $0<\sum\limits_{n=1}^{\infty} a_n^p<\infty$, $0<\sum\limits_{n=1}^{\infty} b_n^q<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}}<pq\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [1].

In [4], [7] and [6], M. Krnić, J. Pečarić and B. Yang gave some generalization and reinforcement of inequality (1.1). In [2], J. Kuang and L. Debnath gave a reinforcement of inequality (1.2):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}}<\left\{\sum_{n=1}^{\infty}[pq-G(p,n)]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}[pq-G(q,n)]b_n^q\right\}^{\frac{1}{q}} \tag{1.3}$$

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where $G(r,n)=\frac{r+\frac{1}{3r}-\frac{4}{3}}{(2n+1)^{\frac{1}{r}}}>0\ (r=p,q)$.

In [5], G. Xi gave a generalization and reinforcement of inequalities (1.2) and (1.3):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^{\lambda},n^{\lambda}\}}<\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3qn^{\frac{q+\lambda-2}{q}}}\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3pn^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{1.4}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0,\ 2-\min\{p,q\}<\lambda\leqslant 2$.

In this paper, by introducing two parameters A,B and using the Euler-Maclaurin expansion for the Riemann zeta function, we establish an inequality for a weight coefficient. Using this inequality, we derive a generalization of inequalities (1.4).

2. A Lemma

First, we need the following formula of the Riemann-ζ function (see [3], [9] and [8]):

$$\zeta(\sigma)=\sum_{k=1}^{n}\frac{1}{k^{\sigma}}-\frac{n^{1-\sigma}}{1-\sigma}-\frac{1}{2n^{\sigma}}-\sum_{k=1}^{l-1}\frac{B_{2k}}{2k}\binom{-\sigma}{2k-1}\frac{1}{n^{\sigma+2k-1}} -\frac{B_{2l}}{2l}\binom{-\sigma}{2l-1}\frac{\varepsilon}{n^{\sigma+2l-1}}, \tag{2.1}$$

where $\sigma>0,\ \sigma\neq 1,\ n,\ l\geqslant 1,\ n,\ l\in N,\ 0<\varepsilon=\varepsilon(\sigma,l,n)<1$. The numbers $B_1=-1/2,\ B_2=1/6,\ B_3=0,\ B_4=-1/30,\ \cdots$ are Bernoulli numbers. In particular, $\zeta(\sigma)=\sum_{k=1}^{\infty}\frac{1}{k^{\sigma}}\ (\sigma>1)$.

Since $\zeta(0)=-1/2$, then the formula of the Riemann-ζ function (2.1) is also true for $\sigma=0$.

LEMMA 1. *If* $p,q>1,\ \frac{1}{p}+\frac{1}{q}=1,\ 2-\min\{p,q\}<\lambda\leqslant 2,\ n\geqslant 1$ *and* $n\in N$, $0\leqslant A\leqslant B\leqslant\min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$, *then*

$$\omega(n,\lambda,p,A,B)=\sum_{k=1}^{\infty}\frac{1}{\max\{k^{\lambda}+A,n^{\lambda}+B\}}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} <n^{1-\lambda}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A}{1+A}\right)\right], \tag{2.2}$$

and

$$\omega(n,\lambda,q,A,B) = \sum_{k=1}^{\infty} \frac{1}{\max\left\{k^{\lambda}+B, n^{\lambda}+A\right\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} < n^{1-\lambda}\left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q} - \frac{B}{1+B}\right)\right], \tag{2.3}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$. *When* $\lambda = 1$, *we have following the stronger inequality:*

$$\omega(n,1,p,A,B) = \sum_{k=1}^{\infty} \frac{1}{\max\{k+A, n+B\}} \left(\frac{n}{k}\right)^{\frac{1}{p}} < \left[pq - \frac{1}{n^{\frac{1}{q}}}\left(\frac{12q^2+3q+5p}{12pq} - \frac{A}{1+A}\right)\right], \tag{2.4}$$

and

$$\omega(n,1,q,A,B) = \sum_{k=1}^{\infty} \frac{1}{\max\{k+A, n+B\}} \left(\frac{n}{k}\right)^{\frac{1}{q}} < \left[pq - \frac{1}{n^{\frac{1}{p}}}\left(\frac{12p^2+3p+5q}{12pq} - \frac{B}{1+B}\right)\right]. \tag{2.5}$$

Proof. Equalities (2.2) and (2.3) define the weight coefficient. When $2-\min\{p,q\} < \lambda \leqslant 2$, taking $\sigma = \frac{2-\lambda}{p} \geqslant 0$, $l = 1$, in (2.1), we obtain

$$\zeta\left(\frac{2-\lambda}{p}\right) = \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} - \frac{1}{2n^{\frac{2-\lambda}{p}}} + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}}\varepsilon_1, \tag{2.6}$$

where $0 < \varepsilon_1 < 1$.

Taking $\sigma = \frac{2}{p} + \frac{\lambda}{q}$, $l = 1$, we obtain

$$\zeta\left(\frac{2}{p} + \frac{\lambda}{q}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda+2q}{12pqn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\varepsilon_2, \tag{2.7}$$

where $0 < \varepsilon_2 < 1$.

In addition,

$$\begin{aligned}\omega(n,\lambda,p,A,B) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^{\lambda}+A, n^{\lambda}+B\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\ &= \sum_{k=1}^{n} \frac{1}{\max\{k^{\lambda}+A, n^{\lambda}+B\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^{\lambda}+A} \\ &\quad + \sum_{k=n}^{\infty} \frac{1}{\max\{k^{\lambda}+A, n^{\lambda}+B\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \frac{1}{n^{\lambda}+B}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^{\lambda}+A} + \sum_{k=n}^{\infty} \frac{1}{k^{\lambda}+A}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&\leqslant \sum_{k=1}^{n} \frac{1}{n^{\lambda}}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^{\lambda}+A} + \sum_{k=n}^{\infty} \frac{1}{k^{\lambda}}\left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{1}{n^{\lambda}+A} + n^{\frac{2-\lambda}{p}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}}.
\end{aligned}
$$

By (2.6) and (2.7)

$$
\begin{aligned}
&\omega(n,\lambda,p,A,B) \\
&< \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\left[\zeta\left(\frac{2-\lambda}{p}\right) + \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2-\lambda}{p}}}\right] - \frac{1}{n^{\lambda}+A} \\
&\quad + n^{\frac{2-\lambda}{p}}\left[\frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda+2q}{12pqn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\right] \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right) + \frac{pn^{1-\lambda}}{p+\lambda-2} + \frac{1}{2n^{\lambda}} - \frac{1}{n^{\lambda}+A} + \frac{qn^{1-\lambda}}{q+\lambda-2} \\
&\quad + \frac{1}{2n^{\lambda}} + \frac{p\lambda+2q}{12pqn^{1+\lambda}} \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right) + \frac{pq\lambda n^{1-\lambda}}{(p+\lambda-2)(q+\lambda-2)} + \frac{p\lambda+2q}{12pqn^{1+\lambda}} + \frac{A}{n^{\lambda}(n^{\lambda}+A)} \\
&= n^{1-\lambda}\left\{\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}}\left[-\zeta\left(\frac{2-\lambda}{p}\right) - \frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}} - \frac{A}{n^{\frac{2-\lambda}{p}}(n^{\lambda}+A)}\right]\right\}.
\end{aligned}
$$

In (2.6), taking $n=1$, by $2-\min\{p,q\}<\lambda\leqslant 2$, we obtain

$$
\begin{aligned}
\zeta\left(\frac{2-\lambda}{p}\right) &= 1 - \frac{p}{p+\lambda-2} - \frac{1}{2} + \frac{(2-\lambda)\varepsilon_1}{12p} \\
&< \frac{1}{2} - \frac{p}{p+\lambda-2} + \frac{2-\lambda}{12p} \\
&= -\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)} \\
&< 0.
\end{aligned}
$$

So for $n\geqslant 1$, $n\in N$, $2-\min\{p,q\}<\lambda\leqslant 2$, $0\leqslant A\leqslant B\leqslant \min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$, we have

$$
\begin{aligned}
&-\zeta\left(\frac{2-\lambda}{p}\right)-\frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}}-\frac{A}{n^{\frac{2-\lambda}{p}}(n^{\lambda}+A)}\\
&>\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)}-\frac{p\lambda+2q}{12pq}-\frac{\alpha}{1+\alpha}\\
&=\frac{q(\lambda-2-3p)(\lambda-2-2p)-(p\lambda+2q)(p+\lambda-2)}{12pq(p+\lambda-2)}-\frac{A}{1+A}\\
&>\frac{-p(p\lambda+2q)+6p^2q}{12pq(p+\lambda-2)}-\frac{A}{1+A}\\
&\geqslant\frac{-(2p+2q)+6pq}{12q(p+\lambda-2)}-\frac{A}{1+A}\\
&>\frac{1}{3p}-\frac{A}{1+A}\\
&\geqslant 0.
\end{aligned}
$$

Using the last result and the inequality for $\omega(n,\lambda,p,A,B)$ above, we obtain (2.2).

When $\lambda=1$, we have

$$
\begin{aligned}
&-\zeta\left(\frac{2-\lambda}{p}\right)-\frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}}-\frac{A}{n^{\frac{2-\lambda}{p}}(n^{\lambda}+A)}\\
&>\frac{q(\lambda-2)^2+(p\lambda+5pq+2q)(2-\lambda)-p(p\lambda+2q)+6p^2q}{12pq(p+\lambda-2)}-\frac{A}{1+A}\\
&=\frac{p+3q-p^2+3pq+6p^2q}{12pq(p-1)}-\frac{A}{1+A}\\
&=\frac{5p^2+10p+12q}{12pq(p-1)}-\frac{A}{1+A}\\
&=\frac{(5p^2+10p+12q)(q-1)}{12pq}-\frac{A}{1+A}\\
&=\frac{12q^2+3q+5p}{12pq}-\frac{A}{1+A}.
\end{aligned}
$$

Using the last result and the inequality for $\omega(n,\lambda,p,A,B)$ above, we obtain (2.4).

In a similar way, we have

$$
\begin{aligned}
&\omega(m,\lambda,q,A,B)\\
&=\sum_{k=1}^{\infty}\frac{1}{\max\{m^{\lambda}+A,k^{\lambda}+B\}}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}}\\
&=\sum_{k=1}^{n}\frac{1}{\max\{m^{\lambda}+A,k^{\lambda}+B\}}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}}-\frac{1}{m^{\lambda}+B}\\
&\quad+\sum_{k=m}^{\infty}\frac{1}{\max\{m^{\lambda}+A,k^{\lambda}+B\}}\left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}}
\end{aligned}
$$

$$= \sum_{k=1}^{m} \frac{1}{m^\lambda + A} \left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{m^\lambda + B} + \sum_{k=m}^{\infty} \frac{1}{k^\lambda + B} \left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}}$$
$$\leqslant \sum_{k=1}^{m} \frac{1}{m^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{m^\lambda + B} + \sum_{k=m}^{\infty} \frac{1}{k^\lambda} \left(\frac{m}{k}\right)^{\frac{2-\lambda}{q}}$$
$$= \frac{1}{m^{\frac{(q+1)\lambda-2}{q}}} \sum_{k=1}^{m} \frac{1}{k^{\frac{2-\lambda}{q}}} - \frac{1}{m^\lambda + B} + m^{\frac{2-\lambda}{q}} \sum_{k=m}^{\infty} \frac{1}{k^{\frac{2}{q}+\frac{\lambda}{p}}}.$$

By (2.6) and (2.7)

$$\omega(m,\lambda,q,A,B)$$
$$< \frac{1}{m^{\frac{(q+1)\lambda-2}{q}}} \left[\zeta\left(\frac{2-\lambda}{q}\right) + \frac{qm^{\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2m^{\frac{2-\lambda}{q}}}\right] - \frac{1}{m^\lambda + B}$$
$$+ m^{\frac{2-\lambda}{q}} \left[\frac{pm^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2m^{\frac{2}{q}+\frac{\lambda}{p}}} + \frac{q\lambda+2p}{12qpm^{1+\frac{2}{q}+\frac{\lambda}{p}}}\right]$$
$$= m^{1-\lambda} \left\{\kappa(\lambda) - \frac{1}{m^{\frac{q+\lambda-2}{q}}} \left[-\zeta\left(\frac{2-\lambda}{q}\right) - \frac{q\lambda+2p}{12qpm^{\frac{q-\lambda+2}{q}}} - \frac{B}{m^{\frac{2-\lambda}{q}}(m^\lambda+B)}\right]\right\}.$$

Since for $m \geqslant 1$, $m \in N$, $2-\min\{p,q\} < \lambda \leqslant 2$, $0 \leqslant A \leqslant B \leqslant \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$, we have

$$-\zeta\left(\frac{2-\lambda}{q}\right) - \frac{q\lambda+2p}{12qpm^{\frac{q-\lambda+2}{q}}} - \frac{B}{m^{\frac{2-\lambda}{q}}(m^\lambda+B)}$$
$$> \frac{(\lambda-2-3q)(\lambda-2-2q)}{12q(q+\lambda-2)} - \frac{q\lambda+2p}{12pq} - \frac{B}{1+B}$$
$$> \frac{1}{3q} - \frac{B}{1+B}$$
$$\geqslant 0.$$

Using the last result and the inequality for $\omega(m,\lambda,q,A,B)$ above, we obtain (2.3).

When $\lambda = 1$, we have

$$-\zeta\left(\frac{2-\lambda}{q}\right) - \frac{q\lambda+2p}{12qpm^{\frac{q-\lambda+2}{q}}} - \frac{B}{m^{\frac{2-\lambda}{q}}(m^\lambda+B)}$$
$$> \frac{p(\lambda-2)^2 + (q\lambda+5pq+2p)(2-\lambda) - q(q\lambda+2p) + 6q^2p}{12pq(q+\lambda-2)} - \frac{B}{1+B}$$
$$= \frac{12p^2+3p+5q}{12pq} - \frac{B}{1+B}.$$

Using the last result and the inequality for $\omega(m,\lambda,q,A,B)$ above, we obtain (2.5). □

3. Main results

THEOREM 1. *If* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-\min\{p,q\}<\lambda\leqslant 2$, $0\leqslant A\leqslant B\leqslant \min\{\frac{1}{3p-1},\frac{1}{3q-1}\}$, $a_n\geqslant 0$, $b_n\geqslant 0$, *for* $n\geqslant 1, n\in N$ *and* $0<\sum\limits_{n=1}^{\infty}a_n^p<\infty$, $0<\sum\limits_{n=1}^{\infty}b_n^q<\infty$, *then*

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m^\lambda+A,n^\lambda+B\}}\\&<\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{B}{1+B}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\\&\quad\times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A}{1+A}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}},\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&\sum_{m=1}^{\infty}m^{(p-1)(\lambda-1)}\left(\sum_{n=1}^{\infty}\frac{a_n}{\max\{m^\lambda+A,n^\lambda+B\}}\right)^p\\&<\kappa(\lambda)^{p-1}\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{A}{1+A}\right)\right]n^{1-\lambda}a_n^p,\end{aligned}\tag{3.2}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0$. *When* $\lambda=1$, *we have*

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m+A,n+B\}}\\&<\left\{\sum_{n=1}^{\infty}\left[pq-\frac{1}{n^{\frac{1}{p}}}\left(\frac{12p^2+3p+5q}{12pq}-\frac{B}{1+B}\right)\right]a_n^p\right\}^{\frac{1}{p}}\\&\quad\times\left\{\sum_{n=1}^{\infty}\left[pq-\frac{1}{n^{\frac{1}{q}}}\left(\frac{12q^2+3q+5p}{12pq}-\frac{A}{1+A}\right)\right]b_n^q\right\}^{\frac{1}{q}}.\end{aligned}\tag{3.3}$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m^\lambda+A,n^\lambda+B\}}\\&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m}{\max\{m^\lambda+A,n^\lambda+B\}^{\frac{1}{p}}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right]\\&\quad\times\left[\frac{b_n}{\max\{m^\lambda+A,n^\lambda+B\}^{\frac{1}{q}}}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}}\right]\end{aligned}$$

$$\leqslant \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{a_m^p}{\max\{m^\lambda + A, n^\lambda + B\}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}} \right] \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{b_n^q}{\max\{m^\lambda + A, n^\lambda + B\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{p}} \right] \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{m=1}^{\infty} \omega(m,\lambda,q,A,B) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \omega(n,\lambda,p,A,B) b_n^q \right\}^{\frac{1}{q}}.$$

By (2.2), (2.3), (2.4) and (2.5), we obtain (3.1) and (3.3).

By Hölder's inequality and Lemma 1, we have

$$\sum_{n=1}^{\infty} \frac{a_n}{\max\{m^\lambda + A, n^\lambda + B\}}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda + A, n^\lambda + B\}^{\frac{1}{p}}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}} a_n \frac{1}{\max\{m^\lambda, n^\lambda\}^{\frac{1}{q}}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}} \right]$$

$$\leqslant \left\{ \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda + A, n^\lambda + B\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}} a_n^p \right] \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda + A, n^\lambda + B\}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{p}} \right] \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda + A, n^\lambda + B\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}} a_n^p \right] \right\}^{\frac{1}{p}} [\omega(m,\lambda,p,A,B)]^{\frac{1}{q}}$$

$$< \left\{ \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda + A, n^\lambda + B\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}} a_n^p \right] \right\}^{\frac{1}{p}} [m^{1-\lambda} \kappa(\lambda)]^{\frac{1}{q}}.$$

So

$$\sum_{m=1}^{\infty} m^{(p-1)(\lambda-1)} \left(\sum_{n=1}^{\infty} \frac{a_n}{\max\{m^\lambda + A, n^\lambda + B\}} \right)^p$$

$$< \kappa(\lambda)^{p-1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{1}{\max\{m^\lambda + A, n^\lambda + B\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{q}} a_n^p \right]$$

$$< \kappa(\lambda)^{p-1} \sum_{n=1}^{\infty} \omega(n,\lambda,q,A,B) a_n^p.$$

By Lemma 1, the proof of the theorem is completed. □

In inequality (3.3), taking $p = q = 2$, we have:

COROLLARY 1. *Let* $a_n \geqslant 0,\ b_n \geqslant 0,\ 0 \leqslant A \leqslant B \leqslant \frac{1}{5}$, *and* $0 < \sum_{n=1}^{\infty} a_n^2 < \infty,\ 0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m+A,n+B\}} < 4\left\{\sum_{n=1}^{\infty}\left[1-\frac{1}{3\sqrt{n}}\left(1-\frac{3B}{4+4B}\right)\right]a_n^2\right\}^{\frac{1}{2}} \times\left\{\sum_{n=1}^{\infty}\left[1-\frac{1}{3\sqrt{n}}\left(1-\frac{3A}{4+4A}\right)\right]b_n^2\right\}^{\frac{1}{2}}. \quad (3.4)$$

In inequality (3.1), taking $A=0, B=0$, we obtain:

COROLLARY 2. *If* $p,q>1,\ \frac{1}{p}+\frac{1}{q}=1,\ a_n \geqslant 0,\ b_n \geqslant 0,\ 2-\min\{p,q\}<\lambda\leqslant 2$, *for* $n\geqslant 1, n\in N$ *and* $0<\sum_{n=1}^{\infty} a_n^p<\infty,\ 0<\sum_{n=1}^{\infty} b_n^q<\infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3qn^{\frac{q+\lambda-2}{q}}}\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3pn^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}. \quad (3.5)$$

Apparently, inequality (3.1) is a generalization of inequality (1.4).

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Gaowen Xi
College of Mathematics and Physics
Chongqing University of Science and Technology
Chongqing, 401331, P. R. China
e-mail: xigaowen@163.com