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ON CRITICAL CONDITION FOR A WEIGHTED INTEGRAL SYSTEM WITH NEGATIVE EXPONENTS

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Abstract. This paper is concerned with the integral system

$$\begin{cases} u(x) = \int_{R^n} |x|^{\alpha}|y|^{\beta}|x-y|^s v^q(y)dy, & u>0 \text{ in } R^n, \\ v(x) = \int_{R^n} |x|^{\beta}|y|^{\alpha}|x-y|^s u^p(y)dy, & v>0 \text{ in } R^n, \end{cases}$$

where $n \geqslant 1$, $\alpha,\beta,s>0$ and $p,q<0$. Such an integral system appears in the study of the conformal geometry and the weighted Hardy-Littlewood-Sobolev inequality. We obtain that

$$\frac{1}{p+1}+\frac{1}{q+1} = -\frac{\alpha+\beta+s}{n},$$

is a necessary condition for the existence of the C^1 positive entire solutions, which is also the necessary and sufficient condition for the invariant of the system and some energy functionals under the scaling transformation.

1. Introduction

In 1958, Stein and Weiss [15] proved the weighted Hardy-Littlewood-Sobolev (WHLS) inequality

$$\left| \int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}dxdy \right| \leqslant C_{\alpha,\beta,s,\lambda,n}\|f\|_r\|g\|_s, \tag{1.1}$$

where $1<r,s<\infty$, $0<\lambda<n$, $\alpha+\beta \geqslant 0$, $\alpha+\beta+\lambda \leqslant n$, and

$$1-\frac{1}{r}-\frac{\lambda}{n} < \frac{\alpha}{n} < 1-\frac{1}{r} \quad \text{and} \quad \frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n} = 2. \tag{1.2}$$

In order to obtain the best constant in the WHLS inequality (1.1), Lieb [13] maximized the functional

$$J(f,g) = \int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}dxdy$$

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under the constraints $\|f\|_r = \|g\|_s = 1$. The corresponding Euler-Lagrange equation is the following integral system:

$$\begin{cases} \lambda_1 r f(x)^{r-1} = \dfrac{1}{|x|^\alpha} \displaystyle\int_{R^n} \dfrac{g(y)}{|y|^\beta |x-y|^\lambda} dy \\ \lambda_2 s g(x)^{s-1} = \dfrac{1}{|x|^\beta} \displaystyle\int_{R^n} \dfrac{f(y)}{|y|^\alpha |x-y|^\lambda} dy \end{cases} \tag{1.3}$$

where $f, g \geqslant 0$, $x \in R^n$ and $\lambda_1 r = \lambda_2 s = J(f,g)$. Set $u = c_1 f^{r-1}$, $v = c_2 g^{s-1}$, $\frac{1}{p+1} = 1 - \frac{1}{r}$, $\frac{1}{q+1} = 1 - \frac{1}{s}$ with $pq \neq 1$. By a proper choice of constants c_1 and c_2, (1.3) becomes

$$\begin{cases} u(x) = \dfrac{1}{|x|^\alpha} \displaystyle\int_{R^n} \dfrac{v(y)^q}{|y|^\beta |x-y|^\lambda} dy \\ v(x) = \dfrac{1}{|x|^\beta} \displaystyle\int_{R^n} \dfrac{u(y)^p}{|y|^\alpha |x-y|^\lambda} dy \end{cases} \tag{1.4}$$

where $\alpha + \beta + \lambda \leqslant n$, and

$$\begin{cases} u, v \geqslant 0, \ 0 < p, q < \infty, \ 0 < \lambda < n, \ \alpha + \beta \geqslant 0, \\ \frac{\alpha}{n} < \frac{1}{p+1} < \frac{\lambda+\alpha}{n}, \ \frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda+\alpha+\beta}{n}. \end{cases} \tag{1.5}$$

When $\alpha = \beta = 0$ in (1.4), we get the Hardy-Littlewood-Sobolev (HLS) type integral system

$$\begin{cases} u(x) = \displaystyle\int_{R^n} \dfrac{v(y)^q}{|x-y|^\lambda} dy \\ v(x) = \displaystyle\int_{R^n} \dfrac{u(y)^p}{|x-y|^\lambda} dy \end{cases} \tag{1.6}$$

which is related to the study of extremal functions of the HLS inequality. When $p = q = (2n-\lambda)/\lambda$, it was proved $u = v$ in [2]. Then (1.6) is reduced to a single equation

$$u(x) = \int_{R^n} \frac{u^p(y)}{|x-y|^\lambda} dy. \tag{1.7}$$

Lieb [13] pointed out that (1.7) is a typical example with conformal property and obtained the explicit extremal function of the HLS inequality with $p = q$ and assumed the form

$$u(x) = c \left(\frac{t}{t^2 + |x - x_0|^2} \right)^{\lambda/2}$$

with $c, t > 0$ and $x_0 \in R^n$. Meanwhile, he also posed how to classify the positive solutions of the Euler-Lagrange equation (1.7). Twenty years later, Chen, Li, Ou [3] and Li [12] solved this open problem respectively. They proved that every positive solution of (1.7) is radially symmetric and decreasing about some point x_0, and hence obtained the classification result.

For systems (1.4) and (1.6), it seems difficult to obtain the explicit solutions. We can only find the qualitative properties which imply the shape of the positive solutions.

Jin and Li ([4] and [5]) proved the radial symmetry and obtained the optimal integrability of the positive solutions. Based on these properties, the fast decay rates were given in [8], [10] and [11]. Other related results with $p,q>0$ can be seen in [1], [9], [14] and [17].

In 2004, Li [12] studied integral equation (1.7) with $p<0$, and posed whether or not does (1.7) admit any positive (regular) solutions for all $n\geqslant 1$, $\lambda<0$ and $p<(2n-\lambda)/\lambda$. Xu [16] studied this problem and obtained the following results.

(R1) Let $\lambda<0$. Eq. (1.7) has a C^1 positive solution if and only if $2n-\lambda=p\lambda$. Now, u is given by

$$u(x)=a(b^2+|x-x_0|^2)^{-\lambda/2} \tag{1.8}$$

with $a,b>0$ and $x_0\in R^n$.

(R2) If $\lambda\in(0,n)$, then (1.7) has no C^1 positive solution.

Afterwards, Lei [6] studied system (1.6) with negative exponents and obtained some existence results. In particular, the critical condition $\frac{1}{1+p}+\frac{1}{1+q}=\frac{\lambda}{n}$ is a necessary condition for existence of C^1-solutions. In this paper, we study (1.4) with negative exponents and also give an analogous conclusion by using the ideas in [7].

For convenience, we write (1.4) with negative exponents as the following

$$\begin{cases} u(x)=\displaystyle\int_{R^n}|x|^\alpha|y|^\beta|x-y|^s v^q(y)dy, & u>0 \ in\ R^n,\\ v(x)=\displaystyle\int_{R^n}|x|^\beta|y|^\alpha|x-y|^s u^p(y)dy, & v>0 \ in\ R^n,\end{cases} \tag{1.9}$$

where $\alpha,\beta,s>0$, and $p<0,q<0$.

For $\lambda,\theta_1,\theta_2\neq 0$, set the scaling of u,v

$$u_\lambda(x)=\lambda^{\theta_1}u(\lambda x),\quad v_\lambda(x)=\lambda^{\theta_2}v(\lambda x).$$

THEOREM 1. *The scaling functions u_λ,v_λ still solve (1.9) and satisfy*

$$\begin{cases} \displaystyle\int_{R^n}u_\lambda^{1+p}(x)dx=\int_{R^n}u^{1+p}(x)dx\\ \displaystyle\int_{R^n}v_\lambda^{1+q}(x)dx=\int_{R^n}v^{1+q}(x)dx\end{cases} \tag{1.10}$$

if and only if the following critical condition holds

$$\frac{1}{1+p}+\frac{1}{1+q}=-\frac{\alpha+\beta+s}{n}. \tag{1.11}$$

THEOREM 2. *If $u,v\in C^1(R^n)$ are positive solutions of (1.9), then u^{p+1}, $v^{q+1}\in L^1(R^n)$. Furthermore, the critical condition (1.11) must be true.*

2. Proofs of Theorems

Proof of Theorem 1. By (1.9), we have

$$
\begin{aligned}
u_\lambda(x) &= \lambda^{\theta_1} u(\lambda x) = \lambda^{\theta_1} \int_{R^n} |\lambda x|^\alpha |y|^\beta |\lambda x - y|^s v^q(y)dy \\
&= \lambda^{\theta_1+\alpha+\beta+s+n} \int_{R^n} |x|^\alpha |z|^\beta |x-z|^s v^q(\lambda z)dz \\
&= \lambda^{\theta_1+\alpha+\beta+s+n-\theta_2 q} \int_{R^n} |x|^\alpha |z|^\beta |x-z|^s v_\lambda^q(z)dz.
\end{aligned}
$$

Similarly,

$$
v_\lambda(x) = \lambda^{\theta_2+\alpha+\beta+s+n-\theta_1 p} \int_{R^n} |x|^\beta |z|^\alpha |x-z|^s u_\lambda^p(z)dz.
$$

Since u_λ, v_λ still solve (1.9), we get that

$$
\theta_1 + \alpha + \beta + s + n - \theta_2 q = 0, \quad \theta_2 + \alpha + \beta + s + n - \theta_1 p = 0.
$$

Therefore,

$$
\theta_1 = \frac{(\alpha+\beta+s+n)(1+q)}{pq-1}, \quad \theta_2 = \frac{(\alpha+\beta+s+n)(1+p)}{pq-1}. \tag{2.1}
$$

On the other hand,

$$
\begin{aligned}
\int_{R^n} u_\lambda^{1+p}(x)dx &= \int_{R^n} (\lambda_1^\theta u(\lambda x))^{1+p} dx = \lambda^{\theta_1(1+p)} \int_{R^n} u^{1+p}(\lambda x)dx \\
&= \lambda^{\theta_1(1+p)-n} \int_{R^n} u^{1+p}(y)dy.
\end{aligned}
$$

This result, together with (1.10), implies $\theta_1(1+p) - n = 0$. By a same argument, we also get $\theta_2(1+q) - n = 0$. Combining with (2.1), we get that

$$
\frac{(\alpha+\beta+s+n)(1+q)}{pq-1} = \frac{n}{1+p}.
$$

Noting $pq - 1 = (p+1)(q+1) - (p+1) - (q+1)$, we obtain (1.11) finally.

On the contrary, the calculation above also shows that if (1.11) holds, then u_λ, v_λ still satisfy (1.9) and (1.10). Thus, we complete the proof. □

Proof of Theorem 2. Assume $u > 0$, $v > 0$ are C^1-solutions of (1.9).

Step 1. We claim that u^{p+1}, $v^{q+1} \in L^1(R^n)$.

To see this, first we observe the fact that $|y-e| \geqslant |y| - 1 \geqslant |y|/2$ when $|y| > R \gg 1$. Thus,

$$
\begin{aligned}
\infty > u(e) &= \int_{R^n} |y|^\beta |y-e|^s v^q(y)dy \geqslant C \int_{R^n \setminus B_R(0)} |y|^{\beta+s} v^q(y)dy \\
&\geqslant C \int_{R^n \setminus B_R(0)} |y|^\beta v^q(y)dy.
\end{aligned}
$$

Clearly, $|y|^{\beta} v^q(y),\ |y|^{\beta+s} v^q(y) \in L^1(B_1(0))$. Therefore,

$$|y|^{\beta} v^q(y),\ |y|^{\beta+s} v^q(y) \in L^1(R^n).$$

It follows from system (1.9) that

$$\begin{aligned} |x|^{s+\alpha} u\left(\frac{x}{|x|^2}\right) &= \int_{R^n} |x|^{s+\alpha} \left|\frac{x}{|x|^2} - y\right|^s \left|\frac{x}{|x|^2}\right|^{\alpha} |y|^{\beta} v^q(y) dy \\ &= \int_{R^n} |y|^{\beta} |x|^s \left|\frac{x}{|x|^2} - y\right|^s v^q(y) dy = \int_{R^n} |y|^{\beta} |y|^s \left|x - \frac{y}{|y|^2}\right|^s v^q(y) dy. \end{aligned} \tag{2.2}$$

Letting $|x| \to 0$ in (2.2), we have

$$\lim_{|x| \to 0} \left[|x|^{s+\alpha} u\left(\frac{x}{|x|^2}\right) \right] = \int_{R^n} |y|^{\beta} v^q(y) dy < \infty.$$

We should point out that here we can take the limit by using the dominated convergence theorem. To justify this, we only need to notice that when $s > 0$ and $|x| \leqslant 1$,

$$\left|x - \frac{y}{|y|^2}\right|^s \leqslant (|x| + 1/|y|)^s \leqslant (1 + 1/|y|)^s,$$

and notice that $|y|^{\beta+s}(1 + 1/|y|)^s v^q(y) \in L^1(R^n)$.

By doing variable change, we can see that there are constants $R > 0$ large and $C > 1$ such that

$$C^{-1}|x|^{s+\alpha} \leqslant u(x) \leqslant C|x|^{s+\alpha} \quad \text{for} \quad |x| \geqslant R. \tag{2.3}$$

Thus, for $R \gg 1$,

$$\begin{aligned} \int_{R^n \setminus B_R(0)} u^{p+1}(x) dx &= \int_{R^n \setminus B_R(0)} u^p(x) u(x) dx \\ &\leqslant C \int_{R^n \setminus B_R(0)} u^p(x) |x|^{s+\alpha} dx \leqslant C v(e). \end{aligned}$$

By a same argument, we also have

$$\int_{R^n \setminus B_R(0)} v^{q+1}(x) dx < +\infty.$$

This should be enough to conclude that $u^{p+1},\ v^{q+1} \in L^1(R^n)$.

Step 2. We claim that if system (1.9) has $C^1(R^n)$ positive solutions for $\alpha, \beta, s > 0$, and $p < 0,\ q < 0,$ then $\frac{1}{1+p} + \frac{1}{1+q} = -\frac{\alpha+\beta+s}{n}$.

First, by (1.9) and Fubini's theorem, we see easily that

$$\begin{aligned} \int_{R^n} u^{p+1}(x) dx &= \int_{R^n} u^p(x) \int_{R^n} |x|^{\alpha} |y|^{\beta} |x-y|^s v^q(y) dy dx \\ &= \int_{R^n} v^q(y) \int_{R^n} u^p(x) |x|^{\alpha} |y|^{\beta} |x-y|^s dx dy = \int_{R^n} v^{q+1}(y) dy. \end{aligned} \tag{2.4}$$

For $\lambda \neq 0$, there holds

$$u(\lambda x) = \int_{R^n} |\lambda x|^{\alpha}|y|^{\beta}|\lambda x - y|^{s} v^{q}(y)dy$$
$$= \lambda^{\alpha+\beta+s+n}\int_{R^n} |x|^{\alpha}|z|^{\beta}|x-z|^{s} v^{q}(\lambda z)dz.$$

Differentiating with respect to λ yields

$$\begin{aligned} x\cdot\nabla u(\lambda x) &= (\alpha+\beta+s+n)\lambda^{\alpha+\beta+s+n-1}\int_{R^n} |x|^{\alpha}|z|^{\beta}|x-z|^{s}v^{q}(\lambda z)dz \\ &\quad +\lambda^{\alpha+\beta+s+n}\int_{R^n} |x|^{\alpha}|z|^{\beta}|x-z|^{s} z\cdot\nabla v^{q}(\lambda z)dz. \end{aligned} \tag{2.5}$$

Let $\lambda = 1$, then

$$\nabla u(x)x = (\alpha+\beta+s+n)u(x) + \int_{R^n} |x|^{\alpha}|z|^{\beta}|x-z|^{s} z\cdot\nabla v^{q}(z)dz. \tag{2.6}$$

Multiplying by $u^p(x)$ on both sides of (2.6) and integrating the resulting equation over R^n, we obtain

$$\begin{aligned} &\frac{1}{p+1}\int_{R^n} x\nabla u^{p+1}(x)dx - (\alpha+\beta+s+n)\int_{R^n} u^{p+1}(x)dx \\ &= \int_{R^n}\int_{R^n} |x|^{\alpha}|z|^{\beta}|x-z|^{s}u^{p}(x)z\cdot\nabla v^{q}(z)dzdx \\ &= \int_{R^n} z\cdot\nabla v^{q}(z)dz\int_{R^n} |x|^{\alpha}|z|^{\beta}|x-z|^{s}u^{p}(x)dx \\ &= \int_{R^n} z\cdot\nabla v^{q}(z)v(z)dz = \frac{q}{q+1}\int_{R^n} z\cdot\nabla v^{q+1}(z)dz. \end{aligned} \tag{2.7}$$

Clearly, (2.7) is equivalent to

$$\begin{aligned} &\lim_{R\to\infty}\left[\frac{1}{p+1}\int_{B_R} x\cdot\nabla u^{p+1}(x)dx - (\alpha+\beta+s+n)\int_{B_R} u^{p+1}(x)dx\right] \\ &= \lim_{R\to\infty}\frac{q}{q+1}\int_{B_R} z\cdot\nabla v^{q+1}(z)dz. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} &\lim_{R\to\infty}\left[\frac{1}{p+1}\int_{\partial B_R} Ru^{p+1}(x)ds - \frac{n}{p+1}\int_{B_R} u^{p+1}(x)dx - (\alpha+\beta+s+n)\int_{R^n} u^{p+1}(x)dx\right] \\ &= \lim_{R\to\infty}\left[\frac{q}{q+1}\int_{\partial B_R} Rv^{q+1}(x)ds - \frac{nq}{q+1}\int_{B_R} v^{q+1}(x)dx\right]. \end{aligned} \tag{2.8}$$

Due to the fact that u^{p+1}, $v^{q+1} \in L^1(R^n)$, we can find $R_j \to \infty$ such that

$$\lim_{R_j\to\infty} R_j\int_{\partial B_{R_j}} u^{p+1}(x)ds = 0, \qquad \lim_{R_j\to\infty} R_j\int_{\partial B_{R_j}} v^{q+1}(x)ds = 0.$$

Thus, from (2.4) and (2.8) with $R = R_j$, we deduce that

$$\left[(\alpha+\beta+s+n)-n\left(\frac{q}{q+1}-\frac{1}{p+1}\right)\right]\int_{R^n} u^{p+1}(x)dx = 0,$$

which implies

$$\frac{1}{1+p}+\frac{1}{1+q} = -\frac{\alpha+\beta+s}{n}. \quad \square$$

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