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JOINTLY SUBADDITIVE MAPPINGS INDUCED BY OPERATOR CONVEX FUNCTIONS

MAREK NIEZGODA

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Abstract. In this paper, we study jointly subadditive mappings induced by operator convex functions and generalized inverses of positive linear maps. We formulate conditions under which the inequalities $TfT^{-}\left(\sum_{k=1}^{n} T_kA_k\right) \leqslant \sum_{k=1}^{n} T_k f(A_k)$ and $TfT^{-}\Phi(A) \leqslant \Phi(f(A))$ hold, where f is an operator convex function, $A, A_k \in \mathbb{B}(H)$ with Hilbert space H, and T, T_k and Φ are positive linear maps (not necessarily unital) on $\mathbb{B}(H)$, with a (reflexive) generalized inverse T^{-} of T. We also show that the transformation $TfT^{-}(B)$ is jointly subadditive in (T,B) and antimonotone in $T(I)$.

1. Introduction

Throughout the paper the symbol $\mathbb{B}(H)$ stands for the C^*-algebra of all bounded linear operators on Hilbert space H with inner product $\langle \cdot, \cdot \rangle$.

A self-adjoint operator A in $\mathbb{B}(H)$ is said to be *positive*, written $0 \leqslant A$, if $\langle Ax, x \rangle \geqslant 0$ for $x \in H$. If moreover A is invertible then A is said to be *strictly positive*, written $0 < A$.

For self-adjoint operators A and B in $\mathbb{B}(H)$, we write $A \leqslant B$ (resp. $A < B$) if $B - A$ is positive (resp. strictly positive).

A continuous function $f : J \to \mathbb{R}$ on an interval $J \subset \mathbb{R}$ is called *operator convex* if

$$f(\alpha A + (1-\alpha)B) \leqslant \alpha f(A) + (1-\alpha) f(B)$$

for all $\alpha \in [0,1]$ and every self-adjoint operators $A, B \in \mathbb{B}(H)$ with spectra $\sigma(A)$, $\sigma(B)$ contained in J.

A continuous function $f : J \to \mathbb{R}$ on an interval $J \subset \mathbb{R}$ is called *operator concave* if $-f$ is operator convex.

A linear map $\Phi : \mathscr{A} \to \mathscr{B}$ between C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ is said to be *positive*, if

$$0 \leqslant A \quad \text{implies} \quad 0 \leqslant \Phi(A) \quad \text{for } A \in \mathscr{A}. \tag{1}$$

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If (1) is replaced by the following condition

$$0 < A \quad \text{implies} \quad 0 < \Phi(A) \quad \text{for } A \in \mathscr{A},$$

then Φ is said to be *strictly positive*.

A linear map $\Phi : \mathscr{A} \to \mathscr{B}$ between unital C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ is called *unital* if $\Phi(I) = I$, where I denotes the unities of the algebras.

The Choi-Davis-Jensen (CDJ) inequality says that if f is an operator convex function on an interval J, and Φ is a unital positive linear map on $\mathbb{B}(H)$, then

$$f(\Phi(A)) \leqslant \Phi(f(A)) \tag{2}$$

for every self-adjoint operator $A \in \mathbb{B}(H)$ with spectrum $\sigma(A)$ contained in J [16, p. 128].

THEOREM A. [21] *If f is an operator convex function on an interval J, and $\Phi_1, \ldots, \Phi_n$ are positive linear maps on $\mathbb{B}(H)$ such that $\sum_{k=1}^{n} \Phi_k(I) = I$, then*

$$f\left(\sum_{k=1}^{n} \Phi_k(A_k)\right) \leqslant \sum_{k=1}^{n} \Phi_k(f(A_k)) \tag{3}$$

for every self-adjoint operators $A_k \in \mathbb{B}(H)$, $k = 1, 2, \ldots, n$, with spectra contained in J.

THEOREM B. [16] *Let unital C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ be closed *-subalgebras of $\mathbb{B}(H)$ and $\mathbb{B}(K)$, respectively, for some Hilbert spaces H and K.*

Let $\Phi_1, \ldots, \Phi_n$ be strictly positive linear maps from a unital C^*-algebra $\mathscr{A}$ into a unital C^*-algebra $\mathscr{B}$ and let $\Phi = \sum_{k=1}^{n} \Phi_k$ be unital.

If f is an operator convex function on an interval J, then

$$f(\Phi(A)) \leqslant \sum_{k=1}^{n} \Phi_k(I)^{1/2} f\left(\Phi_k(I)^{-1/2} \Phi_k(A) \Phi_k(I)^{-1/2}\right) \Phi_k(I)^{1/2} \leqslant \Phi(f(A)) \tag{4}$$

for every self-adjoint operator $A \in \mathscr{A}$ with spectrum contained in J.

In order to see recent theorems on Jensen type inequalities for functions without operator convexity, consult [13, 18, 19, 20].

The purpose of the present paper is to extend the above results to positive linear mappings Φ (not necessarily unital) equipped with auxilary positive linear mappings T. For an operator convex function f, we demonstrate Jensen type inequalities involving the transformation TfT^-, where T^- denotes a reflexive generalized inverse of T. In particular, we show that the transformation $TfT^-(B)$ is jointly subadditive in (T, B) and antimonotone in $T(I)$ with fixed B. Such an approach allows to obtain some complementary statements to (2), (3) and (4). We also provide corollaries describing some special cases. Finally, we prove corresponding results for indexed mappings T_p, when $f(0) \leqslant 0$ and the mapping $p \to T_p(I)$ is superadditive.

2. Results

A *generalized inverse* (in short, *g.i.*) of a linear map $T : V \to W$ between linear spaces V and W is a linear map $T^- : W \to V$ satisfying $TT^-T = T$. If in addition $T^-TT^- = T^-$ then T^- is said to be a *reflexive* generalized inverse of T [24] (cf. [2, pp. 819–820]).

EXAMPLE 2.1. A linear operator $T : V \to W$ between Hilbert spaces V and W is called *partial isometry* if $TT^*T = T$. Clearly, for a partial isometry T it holds that TT^* and T^*T are self-adjoint idempotents, i.e., projections.

It is also evident that $T^- = T^*$ is a generalized inverse of a partial isometry T. In particular, $T^- = T^{-1} = T^*$, whenever T is unitary.

On the other hand, $T^*TT^* = T^*$ holds for a partial isometry T. So, T^* is a reflexive generalized inverse of T.

EXAMPLE 2.2. Let $T \in \mathbb{M}_p$, where $\mathbb{M}_p$ is the space of $p \times p$ complex matrices. By Singular Value Decomposition, $T = U_2 \operatorname{diag} s(T) U_1{}^*$ with unitary U_1 and U_2 [12, p. 144]. Here $s(T) = (s_1(T), s_2(T), \ldots, s_p(T))$ is the vector of *singular values* of T. Define $T^- = U_1 \operatorname{diag} \sigma(T) U_2{}^*$, where $\sigma(T) = (\sigma_1(T), \sigma_2(T), \ldots, \sigma_p(T))$, with $\sigma_i(T) = \frac{1}{s_i(T)}$ if $s_i(T) \neq 0$, and $\sigma_i(T) = 0$ if $s_i(T) = 0$, $i = 1, 2, \ldots, p$. Then T^- is a reflexive generalized inverse of T.

It is known that if $T \in \mathbb{B}(H)$ with a Hilbert space H, then there exists a reflexive generalized inverse of T if and only if T has closed range (see [11], [15, p. 478]). See also [26, Theorem 2.2], [27, Theorems 2.1 and 2.3] for conditions for the existence of generalized inverses of so-called adjointable linear maps on Hilbert C^*-modules.

Remind that a linear map $T : \mathscr{A} \to \mathscr{B}$ between C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ is positive if T sends the set $P(\mathscr{A})$ of positive elements of $\mathscr{A}$ *into* the set $P(\mathscr{B})$ of positive elements of $\mathscr{B}$, i.e., $T(P(\mathscr{A})) \subset P(\mathscr{B})$ (see (1)).

It is easy to verify that a generalized inverse of a positive lineat map T need not be positive. For example, if $T = 0$ is the null map, then *each* linear map from $\mathscr{B}$ into $\mathscr{A}$ is a generalized inverse of T. However, among all linear maps from $\mathscr{B}$ into $\mathscr{A}$ there are non-positive linear maps.

We say that a linear map $T : \mathscr{A} \to \mathscr{B}$ is *strongly* positive if for $X \in \mathscr{A}$,

$$0 \leqslant X \quad \text{if and only if} \quad 0 \leqslant T(X). \tag{5}$$

In the rest of the paper, it is assumed that $\mathscr{A} = \mathbb{B}(H)$ with a Hilbert space H and $\mathscr{B}$ is a unital C^*-algebra. The symbol I stands for the identity operator in $\mathscr{A}$.

For a linear map $T : \mathscr{A} \to \mathscr{B}$, by $\operatorname{Ran}(T)$ we denote the *range* $\{TA \in \mathscr{B} : A \in \mathscr{A}\}$ of T.

LEMMA 2.3. *Let $T : \mathscr{A} \to \mathscr{B}$ be a positive linear map and $T^- : \mathscr{B} \to \mathscr{A}$ be a generalized inverse of T.*

If T is strongly positive and $T(P(\mathscr{A})) = P(\mathscr{B})$, then T^- is positive.

Proof. Take any $Y \in P(\mathscr{B})$. There exists $X \in P(\mathscr{A})$ such that $Y = T(X)$. It follows that $TT^-T(X) = T(X)$. Therefore we have $TT^-Y = Y$ with $Y \in P(\mathscr{B})$. Thus $TT^-Y \in P(\mathscr{B})$. So, in light of (5) we get $T^-Y \in P(\mathscr{A})$, as required.

In consequence, we conclude that T^- is positive. □

In what follows it is assumed that there exists a generalized inverse T^- of a linear map T, whenever the symbol T^- is used.

THEOREM 2.4. *Let $T : \mathscr{A} \to \mathscr{B}$ and $T_k : \mathscr{A} \to \mathscr{B}$, $k = 1,2,\dots,n$, be positive linear maps and $T^- : \mathscr{B} \to \mathscr{A}$ be a positive reflexive g.i. of T. Assume that*

$$T(I) = \sum_{k=1}^{n} T_k(I), \tag{6}$$

$$I \in \operatorname{Ran}(T^-), \tag{7}$$

$$\operatorname{Ran}(T_k) \subset \operatorname{Ran}(T), \quad k = 1,2,\dots,n. \tag{8}$$

If f is an operator convex function on an interval J then

$$TfT^-\left(\sum_{k=1}^{n} T_k A_k\right) \leqslant \sum_{k=1}^{n} T_k f(A_k) \tag{9}$$

for every self-adjoint operators $A_1,\dots,A_n \in \mathscr{A}$ with spectra contained in J.

Proof. Fix any self-adjoint operators $A_1,\dots,A_n \in \mathscr{A}$ with spectra contained in J. It follows that

$$TfT^-\left(\sum_{k=1}^{n} T_k A_k\right) = Tf\left(\sum_{k=1}^{n} T^- T_k A_k\right). \tag{10}$$

From (7) we have $T^-T(I) = I$ by the reflexivity of T^-. So, (6) implies

$$I = \sum_{k=1}^{n} T^- T_k(I).$$

That is, the map $\sum_{k=1}^{n} T^- T_k$ is unital. Moreover, $\Phi_k = T^- T_k : \mathscr{A} \to \mathscr{A}$, $k = 1,\dots,n$, are positive linear maps on $\mathscr{A} = \mathbb{B}(H)$.

In consequence, from Theorem A, eq. (3), we find that

$$f\left(\sum_{k=1}^{n} T^- T_k A_k\right) \leqslant \sum_{k=1}^{n} T^- T_k f(A_k).$$

Hence, by the positivity of T and (8), we conclude that

$$Tf\left(\sum_{k=1}^{n} T^- T_k A_k\right) \leqslant T\sum_{k=1}^{n} T^- T_k f(A_k) = \sum_{k=1}^{n} TT^- T_k f(A_k) = \sum_{k=1}^{n} T_k f(A_k). \tag{11}$$

Now, by combining (10) and (11) we obtain (9). This completes the proof. □

REMARK 2.5. The inequality (9) is motivated by the right-hand side inequality of (4) with $n=1$, and

$$T=\Phi(I)^{1/2}(\cdot)\Phi(I)^{1/2} \quad \text{and} \quad T^{-}=\Phi(I)^{-1/2}(\cdot)\Phi(I)^{-1/2}.$$

Observe that $T(I)=\Phi(I)$. However, in general $T\neq\Phi$.

An analog of the left-hand side inequality of (4) requires the additional assumption that Φ is unital.

EXAMPLE 2.6. Let $\mathscr{A}=\mathbb{M}_m$, $m=np$, and $\mathscr{B}=\mathbb{M}_p\oplus\ldots\oplus\mathbb{M}_p$. Assume $T:\mathscr{A}\to\mathscr{B}$ is a positive linear map given by $T(X)=X_{nn}\oplus X_{11}\oplus\ldots\oplus X_{n-1,n-1}$, where $X=(X_{ij})$ is the block form of $X\in\mathbb{M}_m$ with $p\times p$ blocks $X_{ij}\in\mathbb{M}_p$, $i,j=1,\ldots,n$.

Let $T_k(X)=0\oplus\ldots\oplus 0\oplus X_{kk}\oplus 0\oplus\ldots\oplus 0$, $k=1,\ldots,n$, where $0\in\mathbb{M}_p$ and $X=(X_{ij})$ is as above. Obviously, T_k is positive and $\operatorname{Ran}T_k\subset\operatorname{Ran}T$, $k=1,\ldots,n$. We also see that $\sum_{k=1}^{n}T_k(I)=T(I)$ for the identity matrix $I\in\mathbb{M}_m$. However, $\sum_{k=1}^{n}T_k\neq T$.

Finally, it follows from Theorem 2.4 that if f is an operator convex function on an interval J then inequality (9) is satisfied for any positive reflexive g.i. T^{-} of T satisfying (7). For instance, we can use $T^{-}=T^{n-1}$.

In next two corollaries we demonstrate a companion of CDJ inequality (2) for positive linear map Φ (not necessarily unital) endowed with auxilary positive map T (see Remark 2.8).

COROLLARY 2.7. *Let $\Phi:\mathscr{A}\to\mathscr{B}$ and $T:\mathscr{A}\to\mathscr{B}$ be positive linear maps and $T^{-}:\mathscr{B}\to\mathscr{A}$ be a positive reflexive g.i. of T such that*

$$\Phi(I)=T(I), \tag{12}$$
$$I\in\operatorname{Ran}(T^{-}), \tag{13}$$
$$\operatorname{Ran}(\Phi)\subset\operatorname{Ran}(T). \tag{14}$$

If f is an operator convex function on an interval J then

$$TfT^{-}\Phi(A)\leqslant\Phi(f(A)) \tag{15}$$

for every self-adjoint operator $A\in\mathscr{A}$ with spectrum contained in J.

Proof. It is sufficient to apply Theorem 2.4 for $n=1$ and $T_1=\Phi$, $A_1=A$. □

REMARK 2.8. In the case $\Phi:\mathscr{A}\to\mathscr{A}$ is a *unital* positive map, the CDJ inequality (2) can be recovered from (15), because conditions (12)–(14) hold trivially for T the identity on $\mathscr{A}$. In this sense, (15) can be thought of as an extension of (2).

COROLLARY 2.9. *Let $\Phi:\mathscr{A}\to\mathscr{B}$ be a positive linear map and $T:\mathscr{A}\to\mathscr{B}$ be an invertible positive linear map with positive inverse $T^{-1}:\mathscr{B}\to\mathscr{A}$ such that*

$$\Phi(I)=T(I). \tag{16}$$

If f is an operator convex function on an interval J then

$$T f T^{-1} \Phi(A) \leqslant \Phi(f(A)) \tag{17}$$

for every self-adjoint operator $A \in \mathscr{A}$ with spectrum contained in J.

Proof. Since T is invertible, we have $T^{-} = T^{-1}$, $\operatorname{Ran}(T) = \mathscr{B}$ and $\operatorname{Ran}(T^{-}) = \mathscr{A}$. Therefore conditions (13) and (14) are satisfied. Now, it is enough to apply Corollary 2.7. □

Remind that a map $T : \mathscr{A} \to \mathscr{B}$ with $\mathscr{B} \subset \mathscr{A}$ is called *tripotent* if $T^3 = T$. For intance, any *idempotent* ($T^2 = T$) is tripotent. Likewise, any *involution* ($T^2 = \mathrm{id}$) is tripotent.

COROLLARY 2.10. *Let $\Phi : \mathscr{A} \to \mathscr{B}$ be an unital positive linear map with $\mathscr{B} \subset \mathscr{A}$. Let $T : \mathscr{A} \to \mathscr{B}$ be a tripotent unital positive linear map such that*

$$\operatorname{Ran}(\Phi) \subset \operatorname{Ran}(T). \tag{18}$$

If f is an operator convex function on an interval J then

$$T f T \Phi(A) \leqslant \Phi(f(A)) \tag{19}$$

for every self-adjoint operator $A \in \mathscr{A}$ with spectrum contained in J.

Proof. Since $TTT = T$, we can put $T^{-} = T$. Hence T^{-} is a reflexive generalized inverse of T. Moreover, the map T^{-} is positive, because T is so.

Because $I = \Phi(I)$ and $\operatorname{Ran}(\Phi) \subset \operatorname{Ran}(T)$ by (18), we have $I \in \operatorname{Ran} T = \operatorname{Ran} T^{-}$. Thus condition (13) is fulfilled.

Furthermore, condition (12) is also met, since Φ and T are unital.

Now, by applying Corollary 2.7 we obtain (19). This completes the proof. □

EXAMPLE 2.11. Put $\mathscr{A} = \mathbb{M}_n$ with $n = 2k$ and $\mathscr{B} = \mathbb{M}_k \oplus \mathbb{M}_k$. Consider the unital positive linear maps $\Phi : \mathscr{A} \to \mathscr{B}$ and $T : \mathscr{A} \to \mathscr{B}$ given by

$$\Phi(X) = \begin{pmatrix} x_{11} & 0 & 0 & \dots & 0 \\ 0 & x_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x_{nn} \end{pmatrix} \quad \text{for} \quad X = (x_{ij}) \in \mathbb{M}_n,$$

$$T(X) = \begin{pmatrix} X_{22} & 0 \\ 0 & X_{11} \end{pmatrix} \quad \text{for} \quad X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathbb{M}_n$$

with $X_{11}, X_{12}, X_{21}, X_{22} \in \mathbb{M}_k$.

Then T is tripotent, and $\operatorname{Ran}(\Phi) \subset \operatorname{Ran}(T)$. In consequence, (19) holds.

In Corollary 2.12 we give an interpretation of Theorem 2.4, which extends some recent results by Effros [8] and Moslehian et al. [22].

COROLLARY 2.12. *Let $T : \mathscr{A} \to \mathscr{B}$ and $T_k : \mathscr{A} \to \mathscr{B}$, $k = 1,2,\ldots,n$, be positive linear maps and $T^- : \mathscr{B} \to \mathscr{A}$ be a positive reflexive g.i. of T. Assume that*

$$T(I) = \sum_{k=1}^{n} T_k(I), \tag{20}$$

$$I \in \operatorname{Ran}(T^-), \tag{21}$$

$$\operatorname{Ran}(T_k) \subset \operatorname{Ran}(T), \quad k = 1,2,\ldots,n. \tag{22}$$

If f is an operator convex function on an interval J then

$$TfT^-\left(\sum_{k=1}^{n} B_k\right) \leqslant \sum_{k=1}^{n} T_k f T_k^-(B_k) \tag{23}$$

for every self-adjoint operators $B_k \in \operatorname{Ran}(T_k)$ such that $T_k^-(B_k)$, $k = 1,2,\ldots,n$, have spectra contained in J.

Proof. By $B_k \in \operatorname{Ran}(T_k)$ we get $T_k T_k^-(B_k) = B_k$ for $k = 1,2,\ldots,n$. By putting $A_k = T_k^- B_k$ we have $T_k A_k = B_k$. Now, according to Theorem 2.4, eq. (9), we derive (23), as required. $\square$

Corollary 2.12 simplifies when the maps T and T_k, $k = 1,2,\ldots,n$, are invertible, as follows.

COROLLARY 2.13. *Let $T : \mathscr{A} \to \mathscr{B}$ and $T_k : \mathscr{A} \to \mathscr{B}$, $k = 1,2,\ldots,n$, be invertible positive linear maps and $T^{-1} : \mathscr{B} \to \mathscr{A}$ be a positive linear map. Assume that*

$$T(I) = \sum_{k=1}^{n} T_k(I). \tag{24}$$

If f is an operator convex function on an interval J then

$$TfT^{-1}\left(\sum_{k=1}^{n} B_k\right) \leqslant \sum_{k=1}^{n} T_k f T_k^{-1}(B_k) \tag{25}$$

for every self-adjoint operators $B_k \in \mathscr{B}$ such that $T_k^{-1}(B_k)$, $k = 1,2,\ldots,n$, have spectra contained in J.

Proof. Since T and T_k, $k = 1,\ldots,n$, are invertible, we have $T^- = T^{-1}$, $T_k^- = T_k^{-1}$ and $\operatorname{Ran}(T) = \mathscr{B}$, $\operatorname{Ran}(T_k) = \mathscr{B}$ and $\operatorname{Ran}(T^-) = \mathscr{A}$. Therefore conditions (21)–(22) are fulfilled. Now, it is enough to apply Corollary 2.12. $\square$

As can be seen in the previous results (cf. (9), (15), (17), (23), (25)), the key ingredient related to the Choi-Davis-Jensen's inequality (2) and to the operator Jensen's inequality (3) is the binary transformation

$$(B,T) \to TfT^-(B). \tag{26}$$

Such an operation plays an important role in many other problems. E.g., (26) can be viewed as an extension of the notion of *generalized perspective function* associated to (operator) convex function f (see [8, 11, 22]). In fact, for $T := C^{1/2}(\cdot)C^{1/2}$ with strictly positive operator C, from (26) we obtain

$$(B,C) \to C^{1/2} f(C^{-1/2}BC^{-1/2})C^{1/2}. \tag{27}$$

On the other hand, some sums (integrals) of mappings of type (26) and (27) lead to the definition f*-divergence* [7, 22] (cf. (31) below).

Furthermore, the version of (27) with operator concave function $f = (\cdot)^{1/2}$ provides the definition of the *geometric mean* of operators B and C [14, 17]. Similarly, for $f = \log(\cdot)$, the quantity in (27) becomes the *operator relative entropy* [1, 9].

To give another application of (26), we now recall the notion of sub- and superadditivity.

A nonempty subset $\mathscr{P}$ of a real linear space $\mathscr{V}$ is called *additive* if $p,q \in \mathscr{P}$ implies $p+q \in \mathscr{P}$.

Let $\mathscr{V}$ be a linear space and $\mathscr{B}$ be a C^*-algebra. A mapping $F : \mathscr{P} \to \mathscr{B}$ defined on an additive set $\mathscr{P} \subset \mathscr{V}$ is said to be *subadditive* (resp. *superadditive*) if

$$F(p+q) \leqslant (\geqslant)\, F(p) + F(q) \quad \text{for } p,q \in \mathscr{P}.$$

Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be linear spaces and $\mathscr{B}$ be a C^*-algebra. A mapping $F : \mathscr{P}_1 \times \mathscr{P}_2 \to \mathscr{B}$ with additive sets $\mathscr{P}_1 \subset \mathscr{V}_1$ and $\mathscr{P}_2 \subset \mathscr{V}_2$ is said to be *jointly subadditive* (resp. *jointly superadditive*) if

$$F(p_1+q_1, p_2+q_2) \leqslant (\geqslant)\, F(p_1,p_2) + F(q_1,q_2) \quad \text{for } p_1,q_1 \in \mathscr{P}_1,\ p_2,q_2 \in \mathscr{P}_2.$$

We are now in a position to present the joint subadditivity of the mapping $(T,B) \to TfT^-(B)$.

COROLLARY 2.14. *Let $T_k : \mathscr{A} \to \mathscr{B}$, $k = 1,2,\dots,n$, be positive linear maps and $\left(\sum_{k=1}^n T_k\right)^- : \mathscr{B} \to \mathscr{A}$ be a positive reflexive g.i. of $\sum_{k=1}^n T_k$. Assume that*

$$I \in \operatorname{Ran} \left(\sum_{k=1}^n T_k\right)^-, \tag{28}$$

$$\operatorname{Ran}(T_k) \subset \operatorname{Ran} \left(\sum_{k=1}^n T_k\right), \quad k = 1,2,\dots,n. \tag{29}$$

If f is an operator convex function on an interval J then

$$\left(\sum_{k=1}^n T_k\right) f \left(\sum_{k=1}^n T_k\right)^- \left(\sum_{k=1}^n B_k\right) \leqslant \sum_{k=1}^n T_k f T_k^-(B_k) \tag{30}$$

for every self-adjoint operators $B_k \in \operatorname{Ran}(T_k)$ such that $T_k^-(B_k)$, $k = 1,2,\dots,n$, have spectra contained in J.

Proof. Apply Corollary 2.12 with $T = \sum_{k=1}^{n} T_k$. □

REMARK 2.15. Condition (29) amounts to

$$\sum_{k=1}^{n} \operatorname{Ran}(T_k) = \operatorname{Ran}\left(\sum_{k=1}^{n} T_k\right).$$

To interpret inequality (30), we denote

$$\mathbf{T} = (T_1, \ldots, T_n) \quad \text{and} \quad \mathbf{B} = (B_1, \ldots, B_n),$$

and

$$I_f(\mathbf{T}, \mathbf{B}) = \sum_{k=1}^{n} T_k f T_k^{-}(B_k). \tag{31}$$

Following [22], the quantity (31) is called the *generalized f-divergence* of the pair $(\mathbf{T}, \mathbf{B})$.

Note that (30) can be restated as

$$I_f(\overline{\mathbf{T}}, \overline{\mathbf{B}}) \leqslant I_f(\mathbf{T}, \mathbf{B}), \tag{32}$$

where

$$\overline{\mathbf{T}} = (\overline{T}, \ldots, \overline{T}) \quad \text{and} \quad \overline{\mathbf{B}} = (\overline{B}, \ldots, \overline{B})$$

and

$$\overline{T} = \frac{1}{n} \sum_{k=1}^{n} T_k \quad \text{and} \quad \overline{B} = \frac{1}{n} \sum_{k=1}^{n} B_k.$$

In fact, (30) says that

$$\left(n\overline{T}\right) f \left(n\overline{T}\right)^{-} \left(n\overline{B}\right) \leqslant \sum_{k=1}^{n} T_k f T_k^{-}(B_k),$$

which gives

$$I_f(\overline{\mathbf{T}}, \overline{\mathbf{B}}) = n\overline{T} f \overline{T}^{-} \left(\overline{B}\right) = \left(n\overline{T}\right) f \left(n\overline{T}\right)^{-} \left(n\overline{B}\right) \leqslant \sum_{k=1}^{n} T_k f T_k^{-}(B_k) = I_f(\mathbf{T}, \mathbf{B}),$$

as claimed.

Statement (32) corresponds to a related result of Csiszár et al. [4] (see [7, pp. 159–160]).

EXAMPLE 2.16. (Csiszár divergence) Let $f : (0, \infty) \to (0, \infty)$ be a convex function, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be two n-tuples of positive numbers. Consider linear mappings defined by

$$\mathbb{R} \ni p \to T_k p = q_k p \in \mathbb{R}, \quad k = 1, 2, \ldots, n.$$

It is readily seen that

$$\mathbb{R} \ni p \to T_k^{-1} p = \frac{1}{q_k} p \in \mathbb{R}, \quad k = 1, 2, \ldots, n.$$

Therefore (31) with $\mathscr{A} = \mathscr{B} = \mathbb{R}$ reduces to the Csiszár f-divergence [3]:

$$I_f(\mathbf{p}, \mathbf{q}) = \sum_{k=1}^{n} q_k f\left(\frac{p_k}{q_k}\right). \tag{33}$$

In this context, inequality (32) takes the form

$$\sum_{k=1}^{n} q_k f\left(\frac{\sum_{j=1}^{n} p_j}{\sum_{j=1}^{n} q_j}\right) \leqslant I_f(\mathbf{p}, \mathbf{q}) \tag{34}$$

(see [7, p. 160]).

3. Results for operator convex functions f with $f(0) \leqslant 0$

In this section, we restrict ourselves to the class of operator convex functions f with the condition $f(0) \leqslant 0$. This allows to relax a restriction of type (12) or (20).

THEOREM 3.1. *Let $T : \mathscr{A} \to \mathscr{B}$ and $T_1 : \mathscr{A} \to \mathscr{B}$ be positive linear maps, and $T^- : \mathscr{B} \to \mathscr{A}$ be positive reflexive g.i. of T. Assume that*

$$I \in \operatorname{Ran}(T^-), \tag{35}$$

$$\operatorname{Ran} T_1 \subset \operatorname{Ran}(T). \tag{36}$$

If $f : J \to \mathbb{R}$ is an operator convex function such that $0 \in J$ and $f(0) \leqslant 0$, then

$$T_1(I) \leqslant T(I) \quad \textit{implies} \quad T f T^-(B) \leqslant T_1 f T_1^-(B) \tag{37}$$

for every self-adjoint operator $B \in \operatorname{Ran} T_1$ such that $T_1^- B$ has spectrum contained in J.

Proof. Assume that $T_1(I) \leqslant T(I)$. Then $T^- T_1(I) \leqslant T^- T(I) = I$ by the positivity of T^- and (35). Denote $\Phi_1 = T^- T_1$. Clearly, Φ_1 is a positive linear map on $\mathscr{A} = \mathbb{B}(H)$. We define $\Phi_2 = C^{1/2}(\cdot)C^{1/2}$, where $C = I - \Phi_1(I) \geqslant 0$. Hence $\Phi_2(I) = C$ and $\Phi_1(I) + \Phi_2(I) = I$.

In consequence, by operator Jensen's inequality (3) (see Theorem A), we get

$$f(\Phi_1(A) + \Phi_2(0)) \leqslant \Phi_1(f(A)) + \Phi_2(f(0))$$

for every self-adjoint operator $A \in \mathscr{A} = \mathbb{B}(H)$ with spectrum contained in J.

Therefore for each $A \in \mathscr{A}$ such that $\sigma(A) \subset J$, we obtain

$$f(\Phi_1(A)) \leqslant \Phi_1(f(A)) + \Phi_2(f(0)) \leqslant \Phi_1(f(A)) + \Phi_2(0) = \Phi_1(f(A)).$$

The second inequality follows from the assumptions that $0 \in J$ and $f(0) \leqslant 0$.

Thus we have

$$f(T^- T_1(A)) \leqslant T^- T_1(f(A)) \tag{38}$$

for each $A \in \mathscr{A}$ such that $\sigma(A) \subset J$.

By the fact that T preserves $\leqslant$ and from (36) and (38), we derive

$$T f T^- T_1(A) \leqslant T T^- T_1 f(A) = T_1 f(A) \quad \text{for each } A \in \mathscr{A} \text{ such that } \sigma(A) \subset J. \tag{39}$$

Consider any $B \in \operatorname{Ran} T_1 \subset \mathscr{B}$ such that $\sigma(T_1^-(B)) \subset J$. It is readily seen that $T_1 T_1^-(B) = B$. By putting $A = T_1^-(B)$ we find that $T_1(A) = B$ with $\sigma(A) \subset J$. Therefore (39) implies

$$T f T^-(B) \leqslant T_1 f T_1^-(B),$$

as claimed. □

Statement (37) has an interesting interpretation. Namely, let $\mathscr{P} \subset \mathscr{B}$ be an additive set. Consider indexed positive linear maps $T_p : \mathscr{A} \to \mathscr{B}$, $p \in \mathscr{P}$, satisfying

$$T_p(I) = p \quad \text{for each } p \in \mathscr{P}. \tag{40}$$

Then (37) means the *antimonotonicity* of the mapping $p \to T_p f T_p^-(B)$, as follows.

$$p_1 \leqslant p \quad \text{implies} \quad T_p f T_p^-(B) \leqslant T_{p_1} f T_{p_1}^-(B), \quad \text{for } p, p_1 \in \mathscr{P}. \tag{41}$$

REMARK 3.2. For some interesting relations between generalized inverses of a family $\{T_p : p \in \mathscr{P}\}$ and of their sum $\sum\limits_{p \in \mathscr{P}} T_p$, see [25, Theorems 3.4, 4.3, 4.9 and Corollary 3.5].

In the next theorem we prove joint subadditivity of the binary mapping $(p, B) \to T_p f T_p^-(B)$.

THEOREM 3.3. *Let $\mathscr{P}$ be an additive set in a linear space, and $T_p : \mathscr{A} \to \mathscr{B}$, $p \in \mathscr{P}$, be positive linear maps, and $(T_p)^- : \mathscr{B} \to \mathscr{A}$ be a positive reflexive g.i. of T_p. Assume that $\psi : p \to T_p(I)$, $p \in \mathscr{P}$, is a superadditive mapping. Let $p_1, \ldots, p_n \in \mathscr{P}$ and $\left(\sum\limits_{k=1}^n T_{p_k}\right)^- : \mathscr{B} \to \mathscr{A}$ be a positive reflexive g.i. of $\sum\limits_{k=1}^n T_{p_k}$ satisfying*

$$I \in \operatorname{Ran}\left(\sum_{k=1}^n T_{p_k}\right)^- \cap \operatorname{Ran}\left(T_{\sum\limits_{k=1}^n p_k}\right)^-, \tag{42}$$

$$\operatorname{Ran}(T_{p_k}) \subset \operatorname{Ran}\left(\sum_{k=1}^n T_{p_k}\right) \subset \operatorname{Ran}\left(T_{\sum\limits_{k=1}^n p_k}\right), \quad k = 1, 2, \ldots, n. \tag{43}$$

If $f: J \to \mathbb{R}$ is an operator convex function such that $0 \in J$ and $f(0) \leqslant 0$, then

$$\left(T_{\sum\limits_{k=1}^{n} p_k}\right) f \left(T_{\sum\limits_{k=1}^{n} p_k}\right)^{-} \left(\sum_{k=1}^{n} B_k\right) \leqslant \sum_{k=1}^{n} T_{p_k} f T_{p_k}^{-}(B_k) \tag{44}$$

for every self-adjoint operators $B_k \in \operatorname{Ran}(T_{p_k})$ such that $\sigma(T_{p_k}^{-}(B_k)) \subset J$, $k = 1, 2, \ldots, n$.

Proof. Since ψ is superadditive, we have

$$\psi\left(\sum_{k=1}^{n} p_k\right) \geqslant \sum_{k=1}^{n} \psi(p_k).$$

In other words,

$$T_{\sum\limits_{k=1}^{n} p_k}(I) \geqslant \sum_{k=1}^{n} T_{p_k}(I) = \left(\sum_{k=1}^{n} T_{p_k}\right)(I).$$

In light of Theorem 3.1 applied to $T := T_{\sum\limits_{k=1}^{n} p_k}$ and $T_1 := \sum\limits_{k=1}^{n} T_{p_k}$, we get

$$\left(T_{\sum\limits_{k=1}^{n} p_k}\right) f \left(T_{\sum\limits_{k=1}^{n} p_k}\right)^{-} (B) \leqslant \left(\sum_{k=1}^{n} T_{p_k}\right) f \left(\sum_{k=1}^{n} T_{p_k}\right)^{-} (B) \tag{45}$$

for every self-adjoint operators $B \in \operatorname{Ran} T_1$ such that $\sigma(T_1^{-}(B)) \subset J$.

Simultaneously, (43) gives

$$\operatorname{Ran}\left(\sum_{k=1}^{n} T_{p_k}\right) = \sum_{k=1}^{n} \operatorname{Ran}\left(T_{p_k}\right).$$

In consequence,

$$\sum_{k=1}^{n} B_k \in \operatorname{Ran}\left(\sum_{k=1}^{n} T_{p_k}\right) = \operatorname{Ran}(T_1),$$

because $B_k \in \operatorname{Ran}(T_{p_k})$ for $k = 1, 2, \ldots, n$.

Furthermore, we have $\sigma(T_1^{-}(B)) \subset J$ for $B = \sum\limits_{k=1}^{n} B_k$. In fact, by $B_k \in \operatorname{Ran} T_{p_k}$ it holds that $B_k = T_{p_k} T_{p_k}^{-} B_k$. Therefore one has

$$T_1^{-}(B) = \left(\sum_{k=1}^{n} T_{p_k}\right)^{-} \sum_{k=1}^{n} B_k = \left(\sum_{k=1}^{n} T_{p_k}\right)^{-} \sum_{k=1}^{n} T_{p_k} T_{p_k}^{-} B_k = \left(\sum_{k=1}^{n} T_{p_k}\right)^{-} \sum_{k=1}^{n} T_{p_k} A_k,$$

where $A_k := T_{p_k}^{-} B_k$. Notice that the map

$$(A_1, \ldots, A_n) \to \left(\sum_{k=1}^{n} T_{p_k}\right)^{-} \sum_{k=1}^{n} T_{p_k}(A_k) \tag{46}$$

is positive and unital on $\mathscr{A}^n$. Indeed, $\left(\sum_{k=1}^n T_{p_k}\right)^-$ and $\sum_{k=1}^n T_{p_k}(\cdot)$ are positive, and

$$\left(\sum_{k=1}^n T_{p_k}\right)^- \sum_{k=1}^n T_{p_k}(I) = \left(\sum_{k=1}^n T_{p_k}\right)^- \left(\sum_{k=1}^n T_{p_k}\right)(I) = I.$$

Here the last equality holds by the assumptions that $\left(\sum_{k=1}^n T_{p_k}\right)^-$ is a positive reflexive g.i. of $\sum_{k=1}^n T_{p_k}$ and that $I \in \operatorname{Ran}\left(\sum_{k=1}^n T_{p_k}\right)^-$ by (42). Consequently, the map (46) is unital and positive on $\mathscr{A}^n$.

So, the spectrum of

$$T_1^-(B) = \left(\sum_{k=1}^n T_{p_k}\right)^- \sum_{k=1}^n T_{p_k} A_k$$

lies in J, since the spectra of $A_k = T_{p_k}^- B_k$, $k = 1,2,\dots,n$, are contained in J.

Summarizing all of this, we deduce from (45) that

$$\left(T_{\sum_{k=1}^n p_k}\right) f\left(T_{\sum_{k=1}^n p_k}\right)^- \left(\sum_{k=1}^n B_k\right) \leqslant \left(\sum_{k=1}^n T_{p_k}\right) f\left(\sum_{k=1}^n T_{p_k}\right)^- \left(\sum_{k=1}^n B_k\right). \tag{47}$$

On the other hand, it follows from Corollary 2.14 that

$$\left(\sum_{k=1}^n T_{p_k}\right) f\left(\sum_{k=1}^n T_{p_k}\right)^- \left(\sum_{k=1}^n B_k\right) \leqslant \sum_{k=1}^n T_{p_k} f T_{p_k}^-(B_k). \tag{48}$$

Finally, by combining (47) and (48), we obtain (44), as required. □

COROLLARY 3.4. *Let $\mathscr{P}$ be an additive set in a linear space, and $T_p : \mathscr{A} \to \mathscr{B}$, $p \in \mathscr{P}$, be invertible positive linear maps, and $(T_p)^{-1} : \mathscr{B} \to \mathscr{A}$ be positive inverse of T_p. Assume that $\psi : p \to T_p(I)$, $p \in \mathscr{P}$, is a superadditive mapping. Let $p_1,\dots,p_n \in \mathscr{P}$ and $\left(\sum_{k=1}^n T_{p_k}\right)^{-1} : \mathscr{B} \to \mathscr{A}$ be positive inverse of $\sum_{k=1}^n T_{p_k}$.*

If $f : J \to \mathbb{R}$ is an operator convex function such that $0 \in J$ and $f(0) \leqslant 0$, then

$$\left(T_{\sum_{k=1}^n p_k}\right) f\left(T_{\sum_{k=1}^n p_k}\right)^{-1} \left(\sum_{k=1}^n B_k\right) \leqslant \sum_{k=1}^n T_{p_k} f T_{p_k}^{-1}(B_k) \tag{49}$$

for every self-adjoint operators $B_k \in \mathscr{B}$ such that $\sigma(T_{p_k}^{-1}(B_k)) \subset J$, $k = 1,2,\dots,n$.

Proof. Use Theorem 3.3. □

In order to interpret the last result, we now employ positive linear maps induced by Schur product of matrices.

EXAMPLE 3.5. Remind that the *Schur product* of $k\times k$ matrices $A=(a_{ij})$ and $B=(b_{ij})$ is the $k\times k$ matrix $A\circ B=(a_{ij}b_{ij})$. The Schur product theorem says that

$$A,B\geqslant 0 \ \text{ implies } \ A\circ B\geqslant 0 \tag{50}$$

(see [12, Theorem 5.2.1]).

The m-th Schur-power of A is defined by $A^{[m]}=\underbrace{A\circ\ldots\circ A}_{m\,\text{times}}$, $m=1,2,\ldots$. In addition, $A^{[0]}=E$, where $E=ee^*\geqslant 0$ is the $k\times k$ matrix of ones and e is the $k\times 1$ vector of ones.

Consider the following situation. Let $\mathscr{A}=\mathscr{B}=\mathbb{M}_k(\mathbb{C})$, the linear space of $k\times k$ complex matrices. Denote by $\mathscr{P}$ the set of all $k\times k$ positive semidefinite matrices $P=(p_{ij})$ with $0<p_{ij}$, $i,j=1,2,\ldots,k$.

Take

$$T_P(X)=P\circ X=(p_{ij}x_{ij}) \ \text{ for } \ X=(x_{ij})\in\mathbb{M}_k(\mathbb{C}) \ \text{ and } \ P=(p_{ij})\in\mathscr{P}.$$

By (50), T_P is a positive linear map, i.e.,

$$X\geqslant 0 \ \text{ implies } \ T_P(X)\geqslant 0,$$

provided $P\in\mathscr{P}$. Moreover, the mapping $\psi: P\to T_P(I)$, $P\in\mathscr{P}$, is additive.

It is not hard to verify that T_P is invertible with

$$T_P^{-1}(Y)=P^{[-1]}\circ Y \ \text{ for } \ Y=(y_{ij})\in\mathbb{M}_k(\mathbb{C}), \tag{51}$$

where $P^{[-1]}=(1/p_{ij})$.

Under the hypothesis that $P=(p_{ij})$ with $0<p_{ij}<2$, $i,j=1,2,\ldots,k$, we have

$$P^{[-1]}=E+(E-P)+(E-P)^{[2]}+(E-P)^{[3]}+\ldots, \tag{52}$$

the convergent (geometric) Schur-power series (see [12, pp. 449-450]).

So, it is a simple consequence of (50) that $(E-P)^{[m]}\geqslant 0$, $m=0,1,2,3,\ldots$, provided that $E-P\geqslant 0$. For this reason we deduce that

$$P\in\mathscr{P}_0 \ \text{ imply } \ P^{[-1]}\geqslant 0, \tag{53}$$

where

$$\mathscr{P}_0:=\{P\in\mathscr{P}: P=(p_{ij}),\ 0<p_{ij}<2,\ 0\leqslant P\leqslant E\}$$

(see [12, Theorem 6.3.5]). In other words, T_P^{-1} is a positive linear map from $\mathbb{M}_k(\mathbb{C})$ to $\mathbb{M}_k(\mathbb{C})$, whenever $P\in\mathscr{P}_0$ (see (51) and (53)).

Take any $P_1,\ldots,P_n\in\mathscr{P}_0$ so that $\sum\limits_{k=1}^n P_k\in\mathscr{P}_0$. Then $\left(\sum\limits_{k=1}^n T_{P_k}\right)^{-1}:\mathbb{M}_k(\mathbb{C})\to\mathbb{M}_k(\mathbb{C})$ is positive inverse of $\sum\limits_{k=1}^n T_{P_k}=T_{\sum\limits_{k=1}^n P_k}$. Therefore by Corollary 3.4, we conclude that if $f:J\to\mathbb{R}$ is an operator convex function such that $0\in J$ and $f(0)\leqslant 0$, then

$$\left(\sum_{k=1}^n P_k\right)\circ f\left(\left(\sum_{k=1}^n P_k\right)^{[-1]}\circ\left(\sum_{k=1}^n B_k\right)\right)\leqslant\sum_{k=1}^n P_k\circ f(P_k^{[-1]}\circ B_k) \tag{54}$$

for every self-adjoint operators $B_k \in \mathbb{M}_k(\mathbb{C})$ such that $\sigma(T_{P_k}^{-1}(B_k)) \subset J$, $k = 1,2,\dots,n$.

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Marek Niezgoda
Department of Applied Mathematics and Computer Science
University of Life Sciences in Lublin
Akademicka 13, 20-950 Lublin, Poland
e-mail: marek.niezgoda@up.lublin.pl