Mathematical
Inequalities
& Applications
Volume 18, Number 1 (2015), 207–215 doi:10.7153/mia-18-15

ON SOLUTIONS OF A COMPOSITE TYPE FUNCTIONAL INEQUALITY

ELIZA JABŁOŃSKA

(*Communicated by Z. Páles*)

Abstract. In the paper we consider a composite type inequality $f(x+f(x)y) \leqslant f(x)f(y)$ in the class of real continuous functions. Our paper refers to the paper [12].

In 1985 W. Sander [16] considered the following composite type functional equation:

$$f(F(x,y)) = H(g(x),h(y)), \tag{1}$$

which "connected" the well-known Pexider equations:

$$f(x+y) = g(x)+h(y),$$

$$f(x+y) = g(x)h(y)$$

(for information on it we refer the reader to [10, Chapter XIII §3] or [2, pp. 42–46]), with the pexiderized Gołąb-Schinzel equation

$$f(x+g(x)y) = H(h(x),k(y)) \tag{2}$$

introduced by E. Vincze [17] in 1966.

The Gołąb-Schinzel equation

$$f(x+f(x)y) = f(x)f(y), \tag{3}$$

has been introduced in [6] in connection with looking for subgroups of the centroaffine group of a field. This equation and its generalizations have also others applications, especially in algebra ([3], [4], [11]), as well as in differential equations in meteorology and fluid mechanics [9] and in the theory of geometric objects [1] (an extensive bibliography concerning the Gołąb-Schinzel equation, its generalizations and applications can be found in [5] and [8]).

The inequality

$$f(F(x,y)) \leqslant H(g(x),h(y)) \tag{4}$$

Mathematics subject classification (2010): 39B62, 26A06.
Keywords and phrases: Gołąb–Schinzel equation, Gołąb–Schinzel inequality, functional inequality.
Some parts of proofs of these results can be derive thanks to kind help of Prof. L. Olszowy. I would like to thank him for it.

is a generalization of functional inequalities fulfilled by convex or subadditive functions. This inequality has applications in the theory of generalized convex or subadditive functions, as well as in the theory of semi-groups of linear operators in the strong operator topology (see [7], [12], [13]).

For the first time such composite type functional inequality has been introduced by C. Ionescu Tulcea [7] in 1960. In fact he studied a special case of the inequality (4), i.e.

$$f(F(x,y)) \leqslant g(x) + h(y).$$

W. Sander [12] was first who studied more general inequality (4).

In the paper we consider the Gołąb-Schinzel inequality

$$f(x+f(x)y) \leqslant f(x)f(y), \tag{5}$$

which is also a special case of the inequality (4).

S. Gołąb and A. Schinzel [6] proved that real continuous not constant solutions of (3) have one of the following forms:

$$f(x) = cx+1 \text{ for } x \in \mathbb{R}, \quad \text{or} \quad f(x) = \max\{0, cx+1\} \text{ for } x \in \mathbb{R}$$

with a $c \in \mathbb{R} \setminus \{0\}$. It means that $f^{-1}(\{0\})$ is equal to one of the following sets:

- $\{-\frac{1}{c}\}$ with a $c \in \mathbb{R} \setminus \{0\}$;
- $[-\frac{1}{c}, \infty)$ with a $c < 0$;
- $(-\infty, -\frac{1}{c}]$ with a $c > 0$.

Here, we prove that the set $f^{-1}(\{0\})$ has to be one of these three forms even if we assume that $f : \mathbb{R} \to \mathbb{R}$ is a nonconstant solution of the Gołąb-Schinzel inequality such that $0 \in f(\mathbb{R})$. Moreover, we show that then f satisfies one of the following four conditions:

(i) $f^{-1}(\{0\}) = [-\frac{1}{c}, +\infty)$ and $f(x) \geqslant cx+1$ for $x < -\frac{1}{c}$ with a $c < 0$;

(ii) $f^{-1}(\{0\}) = (-\infty, -\frac{1}{c}]$ and $f(x) \geqslant cx+1$ for $x > -\frac{1}{c}$ with a $c > 0$;

(iii) $f^{-1}(\{0\}) = \{-\frac{1}{c}\}$, $f(x) = cx+1$ for $x \geqslant -\frac{1}{c}$ and $f(x) \geqslant cx+1$ for $x < -\frac{1}{c}$ with some $c < 0$;

(iv) $f^{-1}(\{0\}) = \{-\frac{1}{c}\}$, $f(x) = cx+1$ for $x \leqslant -\frac{1}{c}$ and $f(x) \geqslant cx+1$ for $x > -\frac{1}{c}$ with some $c > 0$.

In the whole paper we use the following notation:

$$G = f^{-1}(\{0\}), \; F = \mathbb{R} \setminus G.$$

1. Preliminaries

REMARK 1. Clearly, if $f:\mathbb{R}\to\mathbb{R}$ is a constant function satisfying inequality (5), then $f=c$ with any $c\in(-\infty,0]\cup[1,+\infty)$. That is why we study nonconstant solutions of (5).

Let us start with some basic properties of functions satisfying (5).

LEMMA 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a nonconstant function satisfying inequality* (5). *Then one of the following conditions holds:*

(i) $f\leqslant 0$*;*

(ii) $f(0)\geqslant 1$ *and* $f\geqslant 0$*;*

(iii) $f(0)=1$ *and there is an* $x_0\in\mathbb{R}$ *with* $f(x_0)<0$.

Proof. Setting $x=y=0$ in (5), we obtain that either $f(0)\leqslant 0$, or $f(0)\geqslant 1$. Moreover, setting $y=0$ in (5), we have

$$f(x)\leqslant f(x)f(0) \text{ for each } x\in\mathbb{R}. \tag{6}$$

If $f(0)\leqslant 0$, then condition (i) holds. To see it, contrary suppose that there is an $x_0\in\mathbb{R}$ with $f(x_0)>0$. Then, by (6), $f(x_0)\leqslant f(0)f(x_0)\leqslant 0$, what is a contradiction.

Now, let $f(0)\geqslant 1$. Then either $f\geqslant 0$ and condition (ii) holds, or there is an $x_0\in\mathbb{R}$ such that $f(x_0)<0$. But in this case, using (6) for $x=x_0$ we obtain $f(0)\leqslant 1$. Consequently, $f(0)=1$ and condition (iii) holds. □

REMARK 2. Everyone can easily see that an arbitrary real function, which does not take any positive values, satisfies (5). That is why the case (i) of Lemma 1 is not interesting for us.

LEMMA 2. *Let $f:\mathbb{R}\to\mathbb{R}$ be a nonconstant function satisfying inequality* (5) *such that $f(\bar{x})>0$ for some $\bar{x}\in\mathbb{R}$. Then*

$$f\left(\frac{y-x}{f(x)}\right)\geqslant f(y)f(x)^{-1}$$

for every $x,y\in\mathbb{R}$ with $f(x)>0$.

Proof. Take any $x,y\in\mathbb{R}$ with $f(x)>0$. Setting $z=\frac{y-x}{f(x)}$ and using (5) we have

$$f(y)=f(x+f(x)z)\leqslant f(x)f(z)=f(x)f\left(\frac{y-x}{f(x)}\right).$$

Since $f(x)>0$, we obtain the thesis. □

2. Continuous solutions

In further considerations we study continuous solutions of (5).

PROPOSITION 1. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a nonconstant continuous function satisfying inequality* (5)*. If* $0\notin f(\mathbb{R})$*, then either* $f<0$*, or* $f\geqslant 1$*.*

Proof. By Lemma 1 and continuity of f, we obtain that either $f<0$, or $f>0$. But if $f>0$, then:

$$f\left(\frac{x}{1-f(x)}\right)=f\left(x+f(x)\frac{x}{1-f(x)}\right)\leqslant f(x)f\left(\frac{x}{1-f(x)}\right)$$

for each $x\in\mathbb{R}$ with $f(x)\neq 1$. Consequently, since $f\left(\frac{x}{1-f(x)}\right)>0$, $f(x)\geqslant 1$ for $x\in\mathbb{R}$, what ends the proof. □

Now, we give some farther information on solutions of (5). By Lemma 1, in view of Remark 2, we consider two ”types” of functions satisfying (5) (in two theorems).

THEOREM 1. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a continuous function satisfying inequality* (5) *such that* $f(0)\geqslant 1$*,* $f\geqslant 0$ *and* $0\in f(\mathbb{R})$*. Then one of the following conditions holds:*

(i) $G=[x_0,+\infty)$ *with an* $x_0>0$*;*

(ii) $G=(-\infty,x_0]$ *with an* $x_0<0$

and $f(x)\geqslant -\frac{x}{x_0}+1$ *for* $x\in F$*.*

Proof. Since f is continuous, the set G is closed. First we prove that either there exists $\max G\cap(-\infty,0)=y<0$, or there is $\min G\cap(0,\infty)=z>0$. Contrary suppose that both of them exist. Since $f(0)\geqslant 1$ and $f(u)\neq 0$ for each $u\in(y,z)$,

$$f(u)>0 \text{ for } u\in(y,z). \tag{7}$$

In view of (5),

$$0\leqslant f(x+f(x)t)\leqslant 0 \text{ for every } x\in\mathbb{R},\ t\in G. \tag{8}$$

Hence $x+f(x)y\in G$ for $x\in\mathbb{R}$; i.e.

$$\text{either } x+f(x)y\geqslant z, \text{ or } x+f(x)y\leqslant y \tag{9}$$

for each $x\in\mathbb{R}$. Thus, for $x\in\mathbb{R}$, the following condition holds:

$$\text{either } f(x)\geqslant -\frac{x}{y}+1, \text{ or } f(x)\leqslant -\frac{x}{y}+\frac{z}{y}. \tag{10}$$

Now, we prove that

$$f(x)\geqslant -\frac{x}{y}+1 \text{ for each } x\in(y,z). \tag{11}$$

Contrary suppose that

$$f(y_0) < -\frac{y_0}{y} + 1 \text{ for some } y_0 \in (y,z).$$

Then, since $y < 0$ and $y_0 < z$, by (10) we obtain

$$f(y_0) \leqslant -\frac{y_0}{y} + \frac{z}{y} < 0,$$

what contradicts (7). Consequently (11) holds.

Now, take a sequence $(x_n)_{n\in\mathbb{N}} \subset (y,z)$ such that $\lim_{n\to\infty} x_n = z$. Then, by (11),

$$f(x_n) \geqslant -\frac{x_n}{y} + 1 \text{ for } n \in \mathbb{N}$$

and thus, by the continuity of f,

$$0 = f(z) \geqslant -\frac{z}{y} + 1,$$

what contradicts $y < 0 < z$.

In this way we proved that either there exists $\max G \cap (-\infty, 0) = y < 0$, or there is $\min G \cap (0,\infty) = z > 0$. Now, assume that there exists $\min G \cap (0,\infty) = \min G = x_0 > 0$ (the case, when $x_0 = \max G \cap (-\infty,0) = \max G < 0$ is analogous). Define a function $g : [x_0,\infty) \to [x_0,\infty)$ as follows:

$$g(x) = x + f(x)x_0 \text{ for } x \in [x_0,\infty).$$

Since $f \geqslant 0$, $g(x) \geqslant x$ for each $x \geqslant x_0$. Hence g is a continuous surjection. It means that for each $y \geqslant x_0$ there is an $x_1 \geqslant x_0$ such that $y = g(x_1) = x_1 + f(x_1)x_0$. Consequently, by (8), $f(y) = 0$ for each $y \geqslant x_0$, i.e. $G = [x_0,\infty)$ (in the case, when $x_0 = \max G \cap (-\infty,0) = \max G < 0$, we obtain that $G = (-\infty, x_0]$).

Then, by (8), $x + f(x)x_0 \geqslant x_0$ for each $x \in F$, what ends the proof. □

THEOREM 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying inequality* (5) *such that $f(0) = 1$ and $f(\bar{x}) < 0$ for some $\bar{x} \in \mathbb{R}$. Then one of the following conditions holds:*

(i) $G = \{x_0\}$, $f(x) = -\frac{x}{x_0} + 1$ *for each* $x \geqslant x_0$ *and* $f(x) \geqslant -\frac{x}{x_0} + 1$ *for each* $x < x_0$ *with some* $x_0 > 0$,

(ii) $G = \{x_0\}$, $f(x) = -\frac{x}{x_0} + 1$ *for each* $x \leqslant x_0$ *and* $f(x) \geqslant -\frac{x}{x_0} + 1$ *for each* $x > x_0$ *with some* $x_0 < 0$.

Proof. Since f is continuous, the set G is closed. First we prove that either there exists $\max G \cap (-\infty,0) = y < 0$, or there is $\min G \cap (0,\infty) = z > 0$.

Contrary suppose that both of them exist. Since $f(0) = 1$ and $f(u) \neq 0$ for each $u \in (y,z)$, condition (7) holds. In view of (5)

$$f(x + f(x)t) \leqslant 0 \text{ for every } x \in \mathbb{R},\ t \in G. \tag{12}$$

Hence either $x+f(x)t \leqslant y$, or $x+f(x)t \geqslant z$ for every $x \in \mathbb{R}$ and $t \in G$. Thus (9) holds for $x \in \mathbb{R}$ and, in the same way as in the proof of Theorem 1 (using (7)), we obtain a contradiction.

In this way we proved that either there exists $\max G \cap (-\infty, 0) = y < 0$, or there is $\min G \cap (0, \infty) = z > 0$. Now assume that there exists $\min G \cap (0, \infty) = \min G = x_0 > 0$ (the case, when $x_0 = \max G \cap (-\infty, 0) = \max G < 0$ is analogous). Contrary suppose that $G \neq \{x_0\}$. By assumption $f(\bar{x}) < 0$ for some $\bar{x} \in \mathbb{R}$, so, using (12), $f(\bar{x}+f(\bar{x})y) \leqslant 0$ for each $y \in G$. Since $x_0 = \min G > 0$, f is continuous and $f(0) = 1$, we obtain that $f(x) > 0$ for each $x < x_0$ and $\bar{x}+f(\bar{x})y \geqslant x_0$. Consequently

$$y \leqslant \frac{x_0 - \bar{x}}{f(\bar{x})} \text{ for } y \in G.$$

It means that G is upper bounded, i.e. there is a $\max G = y_0 > x_0$. Moreover, one of the following two conditions holds:

$$f(t) < 0 \text{ for each } t > y_0, \text{ or } f(t) > 0 \text{ for each } t > y_0.$$

First consider the case, when $f(t) > 0$ for each $t > y_0$. Since $f(t) > 0$ for each $t < x_0$, $f^{-1}((-\infty, 0]) \subset [x_0, y_0]$. Thus, by (12),

$$x_0 \leqslant x + f(x)x_0 \leqslant y_0 \text{ and } x_0 \leqslant x + f(x)y_0 \leqslant y_0$$

for each $x \in \mathbb{R}$. Hence

$$\begin{cases} -\frac{x}{x_0} + 1 \leqslant f(x) \leqslant -\frac{x}{x_0} + \frac{y_0}{x_0}, \\ -\frac{x}{y_0} + \frac{x_0}{y_0} \leqslant f(x) \leqslant -\frac{x}{y_0} + 1 \end{cases}$$

for $x \in \mathbb{R}$. Consequently, for each $x < 0$ we have

$$-\frac{x}{x_0} + 1 \leqslant f(x) \leqslant -\frac{x}{y_0} + 1 < -\frac{x}{x_0} + 1,$$

a contradiction.

Next consider the case, when $f(t) < 0$ for each $t > y_0$. Since $f(t) > 0$ for $t < x_0$, $f^{-1}([0, \infty)) \subset (-\infty, y_0]$. Then, by Lemma 2,

$$f\left(\frac{y_0 - x}{f(x)}\right) \geqslant f(y_0)f(x)^{-1} = 0 \text{ for } x < x_0.$$

Hence

$$\frac{y_0 - x}{f(x)} \leqslant y_0 \text{ for } x < x_0$$

and, consequently,

$$f(x) \geqslant -\frac{x}{y_0} + 1 \text{ for } x < x_0.$$

Now, take a sequence $(x_n)_{n \in \mathbb{N}} \subset (0, x_0)$ such that $\lim_{n \to \infty} x_n = x_0$. Then

$$f(x_n) \geqslant -\frac{x_n}{y_0} + 1 \text{ for } n \in \mathbb{N}$$

and thus, by the continuity of f,

$$0 = f(x_0) \geqslant -\frac{x_0}{y_0} + 1,$$

what contradicts $x_0 < y_0$.

Hence $G = \{x_0\}$. Since $x_0 > 0$, f is continuous and $f(0) = 1$, we obtain that $f(x) > 0$ for each $x < x_0$. Thus, by (12), $x + f(x)x_0 \geqslant x_0$ for $x \in \mathbb{R}$ and hence

$$f(x) \geqslant -\frac{x}{x_0} + 1 \text{ for } x \in \mathbb{R}. \tag{13}$$

Now we prove that $f(x) = -\frac{x}{x_0} + 1$ for $x \geqslant x_0$. Contrary suppose that there is a $z_0 > x_0$ such that

$$0 > f(z_0) > -\frac{z_0}{x_0} + 1. \tag{14}$$

First we show that there is a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{n\to\infty} y_n = \infty$, and

$$\lim_{n\to\infty} \frac{f(y_n)}{y_n} = -\frac{1}{x_0}. \tag{15}$$

Take an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = \infty$. According to (5) and (13),

$$-\frac{z_0}{x_0 x_n} - \frac{f(z_0)}{x_0} + \frac{1}{x_n} \leqslant \frac{f(z_0 + f(z_0)x_n)}{x_n} \leqslant f(z_0)\frac{f(x_n)}{x_n}.$$

Hence

$$-\frac{f(z_0)}{x_0} \leqslant \limsup_{n\to\infty} f(z_0)\frac{f(x_n)}{x_n}. \tag{16}$$

On the other hand, by (13), we have

$$\liminf_{n\to\infty} \frac{f(x_n)}{x_n} \geqslant -\frac{1}{x_0}.$$

Consequently, since $f(z_0) < 0$,

$$\limsup_{n\to\infty} f(z_0)\frac{f(x_n)}{x_n} = f(z_0)\liminf_{n\to\infty} \frac{f(x_n)}{x_n} \leqslant -\frac{f(z_0)}{x_0}$$

and, in view of (16),

$$\limsup_{n\to\infty} f(z_0)\tfrac{f(x_n)}{x_n} = f(z_0)\liminf_{n\to\infty} \tfrac{f(x_n)}{x_n} = -\tfrac{f(z_0)}{x_0}.$$

It means that

$$\liminf_{n\to\infty} \frac{f(x_n)}{x_n} = -\frac{1}{x_0}.$$

Hence there is a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} y_n = \infty$ and (15) holds.

Now, by (13) and (5), we obtain

$$\frac{-y_n - f(y_n)z_0}{x_0} + 1 \leqslant f(y_n + f(y_n)z_0) \leqslant f(y_n)f(z_0).$$

Thus

$$-\frac{y_n}{x_0} + 1 \leqslant f(y_n)\left(\frac{z_0}{x_0} + f(z_0)\right)$$

and, consequently,

$$-\frac{1}{x_0} + \frac{1}{y_n} \leqslant \frac{f(y_n)}{y_n}\left(\frac{z_0}{x_0} + f(z_0)\right).$$

Whence, in view of (15),

$$-\frac{1}{x_0} \leqslant -\frac{1}{x_0}\left(\frac{z_0}{x_0} + f(z_0)\right).$$

In this way we obtain that $f(z_0) \leqslant -\frac{z_0}{x_0} + 1$, what contradicts (14) and ends the proof of theorem. □

REMARK 3. Clearly, there exist continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfying one of the following conditions:

(i) $G = [x_0, +\infty)$ with an $x_0 > 0$ and $f(x) \geqslant -\frac{x}{x_0} + 1$ for $x \in F$;

(ii) $G = (-\infty, x_0]$ with an $x_0 < 0$ and $f(x) \geqslant -\frac{x}{x_0} + 1$ for $x \in F$;

(iii) $G = \{x_0\}$, $f(x) = -\frac{x}{x_0} + 1$ for $x \geqslant x_0$ and $f(x) \geqslant -\frac{x}{x_0} + 1$ for $x < x_0$ with some $x_0 > 0$;

(iv) $G = \{x_0\}$, $f(x) = -\frac{x}{x_0} + 1$ for $x \leqslant x_0$ and $f(x) \geqslant -\frac{x}{x_0} + 1$ for $x > x_0$ with some $x_0 < 0$

and such that the inequality (5) does not hold.

For example, fix $x_0 = 1$. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function, such that:

- $f(0) = 1$, $f\left(\frac{1}{2}\right) = \frac{3}{5}$, $f\left(\frac{2}{3}\right) = \frac{2}{5}$, $f\left(\frac{9}{10}\right) = \frac{7}{25}$,
- $f(x) \geqslant -x + 1$ for each $x \leqslant 1$,
- either $f(x) = -x + 1$ for $x \geqslant 1$, or $f|_{[1,\infty)} = 0$.

Then, for $x = \frac{1}{2}$ and $y = \frac{2}{3}$ we have:

$$f(x + f(x)y) = f\left(\frac{9}{10}\right) = \frac{7}{25} > \frac{6}{25} = f(x)f(y).$$

EXAMPLE 1. We can check that function $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(x) = \begin{cases} -2x + 1 & \text{for } x < 0; \\ -x^2 + 1 & \text{for } x \in [0,1); \\ g(x) & \text{for } x \geqslant 1, \end{cases}$$

where either $g(x) = 0$ for $x \geqslant 1$, or $g(x) = -x + 1$ for $x \geqslant 1$, satisfy inequality (5).

REFERENCES

[1] J. ACZÉL, *Beiträge zur Theorie der geometrischen Objekte*, Acta Math. Acad. Sci. Hungar., **8**, (1957), 19–52.

[2] J. ACZÉL AND J. DHOMBRES (Eds), *Functional equations in several variables*, Encyclopedia of Mathematics and its Applications **31**, Cambridge University Press, New York-New Rochelle-Melbourne-Sydney, 1989.

[3] J. ACZÉL AND S. GOŁĄB, *Remarks on one-parameter subsemigroups of the affine group and their homo- and isomorphism*, Aequationes Math., **4**, (1970), 1–10.

[4] E. AICHINGER AND M. FARAG, *On when the multiplicative center of a near-ring is a subnear-ring*, Aequationes Math., **68**, (2004), 46–59.

[5] J. BRZDĘK, *The Gołąb–Schinzel equation and its generalizations*, Aequationes Math., **70**, (2005), 14–24.

[6] S. GOŁĄB AND A. SCHINZEL, *Sur l'équation fonctionnelle* $f(x+f(x)y)=f(x)f(y)$, Publ. Math. Debrecen, **6**, (1959), 113–125.

[7] C. IONESCU TULCEA, *Suboperative functions and semigroup of operators*, Ark. Mat., **4**, (1960), 55–61.

[8] E. JABŁOŃSKA, *On solutions of some generalizations of the Gołąb-Schinzel equation*, In: J. Brzdęk and Th. M. Rassias (Eds), *Functional Equations in Mathematical Analysis*, pp. 509–521. Springer Optimization and its Applications **52**, Springer, New York-Dordrecht-Heidelberg-London, 2012.

[9] P. KAHLIG AND J. MATKOWSKI, *A modified Gołąb-Schinzel equation on a restricted domain (with applications to meteorology and fluid mechanics)*, Österreich. Akad. Wiss. Math.-Natur. Kl Sitzungsber. II, **211**, (2002), 117–136.

[10] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, In: A. Gilányi (ed.), 2nd edition, Birkhäuser Verlag, Basel, 2009.

[11] P. PLAUMANN AND S. STRAMBACH, *Zweidimensionale Quasialgebren mit Nullteilern*, Aequationes Math., **15**, (1977), 249–264.

[12] W. SANDER, *Regularitätseigenschaften von Funktionalungleichungen*, Glasnik Mat., **13**, 33 (1978), 237–247.

[13] W. SANDER, *Eine Funktionalgleichung für operatorwertige Funktionen*, Monatsh. Math., **92**, (1981), 61–73.

[14] W. SANDER, *Boundedness properties for functional inequalities*, Manuscripta Math., **39**, (1982), 271–276.

[15] W. SANDER, *Some Functional Inequalities*, Monatsh. Math., **95**, (1983), 149–157.

[16] W. SANDER, *About solutions of a functional equation*, Aequationes Math., **29**, (1985), 135–144.

[17] E. VINCZE, *Über die Lösung der Funktionalgleichung* $f(y+xg(y))=L(h(x),k(y))$, Ann. Polon. Math., **18**, (1966), 115–119.

(Received June 30, 2013)

Eliza Jabłońska
Department of Mathematics
Rzeszów University of Technology
Powstańców Warszawy 12
35-959 Rzeszów, Poland
e-mail: elizapie@prz.edu.pl