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WEAK MONOTONICITY AND CHEBYSHEV TYPE INEQUALITY

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Abstract. The weak monotonic function is defined in this paper. We will study the relationship between the weak monotonic function and the Schur-function. We show that a Schur-convex function is a weak increasing function under the proper hypotheses. By means of the theory of weak monotonic function with appropriate assumptions, we have established a Chebyshev type inequality as follows:

$$\frac{\langle \mathbf{a},\mathbf{b}\rangle}{\langle \mathbf{a}^*,\mathbf{b}^*\rangle} \geqslant \frac{\|\mathbf{a}\|_p}{\|\mathbf{a}^*\|_p}\cdot\frac{\|\mathbf{b}\|_q}{\|\mathbf{b}^*\|_q}.$$

As the application of the inequality, a new proof of Marshall's inequality is obtained.

1. Introduction

There are many approaches to prove inequalities (see [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13]). The use of monotonicity to prove inequalities is one of the basic approaches.

There are many different types of monotonicity. In the references [1] and [2], the authors studied the problems of monotonicity as follows: Let $\mathbf{x} = (x_1,\cdots,x_n) \in I^n$, where I is an interval and $n \geqslant 2$, be a monotonic n-tuple and $\mathbf{p} = (p_1,\cdots,p_n)$ a real n-tuple such that

$$0 \leqslant P_k \leqslant P_n,\ \ k = 1,\cdots,n-1,\ \ P_n > 0 \tag{1}$$

hold. Then for a convex function $f : I \to \mathbb{R}$ we have

$$F\left([\mathbf{x}]^1\right) \geqslant \cdots \geqslant F\left([\mathbf{x}]^k\right) \geqslant F\left([\mathbf{x}]^{k+1}\right) \geqslant \cdots \geqslant F\left([\mathbf{x}]^n\right) = 0, \tag{2}$$

and

$$F\left([\mathbf{x}]_1\right) \geqslant \cdots \geqslant F\left([\mathbf{x}]_k\right) \geqslant F\left([\mathbf{x}]_{k+1}\right) \geqslant \cdots \geqslant F\left([\mathbf{x}]_n\right) = 0, \tag{3}$$

where (2) is given in [1] and (3) is given in [2] and

$$P_k = \sum_{i=1}^{k} p_i,\ \ k = 1,2,\cdots,n,$$

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$$F : I^n \to \mathbb{R},\ \ F(\mathbf{x}) \triangleq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right), \tag{4}$$

$$[\mathbf{x}]^k = (x_1, x_2, \cdots, x_{n-k}, \underbrace{x_{n-k+1}, x_{n-k+1}, \cdots, x_{n-k+1}}_{k}),$$

$$[\mathbf{x}]_k = (\underbrace{x_k, x_k, \cdots, x_k}_{k}, x_{k+1}, x_{k+2}, \cdots, x_n). \tag{5}$$

In other words, the sequences

$$\left\{F\left([\mathbf{x}]^k\right)\right\}_{k=1}^{n} \text{ and } \left\{F\left([\mathbf{x}]_k\right)\right\}_{k=1}^{n}$$

are monotonically decreasing.

In this paper, we will introduce a new approach to prove inequalities by means of the theory of weak monotonic function. This approach was first used by Guang Xing Li and Ji Cheng (see [3]). Unfortunately, Li and Cheng did not generalize the approach.

In Section 2, we will define the weak monotonic function, as well as study some basic properties of the weak monotonic function. In Section 3, we will study the relationship between the weak monotonic function and the Schur-function. We show that a Schur-convex function is a weak increasing function under the proper hypotheses. As the application of the theory of weak monotonic function, in Section 4, we will establish a Chebyshev type inequality. As an application of the Chebyshev type inequality, in Section 5, we will give a new proof of Marshall's inequality.

2. Theory of weak monotonic function

The following notations will be used throughout the paper:

$$\mathbb{R} = (-\infty, \infty),\ \mathbb{R}_+ = [0, \infty),\ \mathbb{R}_{++} = (0, \infty),$$

$$I^n = \underbrace{I \times \cdots \times I}_{n},\ \mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n,\ \mathbf{e} = (1, \cdots, 1) \in \mathbb{R}^n,$$

$$[\mathbf{x}]_k^{'} = (\underbrace{x_k^{'}, x_k^{'}, \cdots, x_k^{'}}_{k}, x_{k+1}, x_{k+2}, \cdots, x_n) \in \mathbb{R}^n,$$

$$\Omega\downarrow = \left\{\mathbf{x} \in \Omega \subseteq \mathbb{R}^n | x_1 \geqslant \cdots \geqslant x_n\right\},\ \Omega\uparrow = \left\{\mathbf{x} \in \Omega \subseteq \mathbb{R}^n | x_1 \leqslant \cdots \leqslant x_n\right\},$$

and we assume that $n \geqslant 2$ throughout the paper.

We first define the weak monotonic function as follows.

DEFINITION 2.1. Let $\Omega \subset \mathbb{R}^n$ be a symmetrical and convex set. If the function $f : \Omega \to \mathbb{R}$ satisfies the conditions:

$$[\mathbf{x}]_k \in \Omega\downarrow,\ [\mathbf{x}]_k^{'} \in \Omega\downarrow,\ x_k \geqslant x_k^{'} \Rightarrow f([\mathbf{x}]_k) \geqslant f\left([\mathbf{x}]_k^{'}\right),\ \ k = 1, 2, \cdots, n-1, \tag{6}$$

where $[\mathbf{x}]_k$ is defined in (5), then we call $f:\Omega\to\mathbb{R}$ a *weak increasing function*. If $-f$ is a weak increasing function, then we call $f:\Omega\to\mathbb{R}$ a *weak decreasing function*. A weak increasing function or a weak decreasing function is called a *weak monotonic function*.

By Definition 2.1 it is easy to get

PROPOSITION 2.1. *If $f:\Omega\to\mathbb{R}$ is a weak increasing function, then for any $\mathbf{x}\in\Omega\downarrow$ we have*

$$f(\mathbf{x})=f([\mathbf{x}]_1)\geqslant f([\mathbf{x}]_2)\geqslant\cdots\geqslant f([\mathbf{x}]_n)=f(x_n\mathbf{e}). \tag{7}$$

By Definition 2.1 and some facts of mathematical analysis we easily get

PROPOSITION 2.2. *Let $\Omega\subset\mathbb{R}^n$ be a symmetrical and convex domain, and let $f:\Omega\to\mathbb{R}$ be a differentiable function. If*

$$\frac{\partial f([\mathbf{x}]_k)}{\partial x_k}\geqslant 0,\ \ k=1,2,\cdots,n-1 \tag{8}$$

for any $\mathbf{x}\in\Omega\downarrow$, then $f:\Omega\to\mathbb{R}$ is a weak increasing function.

PROPOSITION 2.3. *Let $\Omega\subset\mathbb{R}^n_{++}$ be a symmetrical and convex domain, and let $f:\Omega\to\mathbb{R}$ be a homogeneous of degree γ ($\gamma\in\mathbb{R}$) and symmetrical function , as well as let $f(\mathbf{e})\geqslant 0$. If there is a symmetric function $f_*:\Omega\to\mathbb{R}_{++}$ such that the function f_*f is a weak increasing function, then*

$$f(\mathbf{x})\geqslant 0,\ \ \forall\mathbf{x}\in\Omega. \tag{9}$$

Proof. Since $f:\Omega\to\mathbb{R}$ is a symmetric function, we can assume that $\mathbf{x}\in\Omega\downarrow$ in (9). Since $f:\Omega\to\mathbb{R}$ is a homogeneous function of degree γ, we have

$$f(t\mathbf{x})\equiv t^{\gamma}f(\mathbf{x}),\ \ \text{where}\ \ t\in\mathbb{R}_{++}\ \ \text{and}\ \ \mathbf{x},\ t\mathbf{x}\in\Omega. \tag{10}$$

By Definition 2.1, (10) and (7) in Proposition 2.1 we have

$$f_*(\mathbf{x})f(\mathbf{x})\geqslant f_*(x_n\mathbf{e})f(x_n\mathbf{e})=(x_n)^{\gamma}f_*(x_n\mathbf{e})f(\mathbf{e})\geqslant 0. \tag{11}$$

From (11) and $f_*(\mathbf{x})>0,\forall\mathbf{x}\in\Omega$ we get (9). This completes the proof of Proposition 2.3. □

THEOREM 2.1. *Let the function $f:I\to\mathbb{R}$ be a differentiable convex function and let (1) hold. Then the function $F:I^n\to\mathbb{R}$, defined in (4), is a weak increasing function. That is to say,*

$$[\mathbf{x}]_k\in I^n\downarrow,\ [\mathbf{x}]_k'\in I^n\downarrow,\ x_k\geqslant x_k'\Rightarrow F([\mathbf{x}]_k)\geqslant F\left([\mathbf{x}]_k'\right),\ \ k=1,2,\cdots,n-1. \tag{12}$$

Proof. Let $\mathbf{x} \in I^n \downarrow$. Then

$$x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n. \tag{13}$$

According to Proposition 2.2, we just need to prove that

$$\frac{\partial F([\mathbf{x}]_k)}{\partial x_k} \geqslant 0, \quad k = 1, 2, \cdots, n-1. \tag{14}$$

Note that

$$F([\mathbf{x}]_k) = \frac{1}{P_n}\left[P_k f(x_k) + \sum_{i=k+1}^{n} p_i f(x_i)\right] - f\left[\frac{1}{P_n}\left(P_k x_k + \sum_{i=k+1}^{n} p_i x_i\right)\right], \tag{15}$$

and

$$\frac{\partial F([\mathbf{x}]_k)}{\partial x_k} = \frac{P_k}{P_n}\left\{f'(x_k) - f'\left[\frac{1}{P_n}\left(P_k x_k + \sum_{i=k+1}^{n} p_i x_i\right)\right]\right\}. \tag{16}$$

From (13) and $f : I \to \mathbb{R}$ being a differentiable convex function we get

$$x_k \geqslant \frac{1}{P_n}\left(P_k x_k + \sum_{i=k+1}^{n} p_i x_i\right) \in I, \tag{17}$$

and

$$f'(x_k) \geqslant f'\left[\frac{1}{P_n}\left(P_k x_k + \sum_{i=k+1}^{n} p_i x_i\right)\right]. \tag{18}$$

Combining with (1), (16) and (18) we get the inequalities (14).

According to Proposition 2.2 and (14) we know that the function F is a weak increasing function.

This completes the proof of Theorem 2.1. □

REMARK 2.1. Proposition 2.3 gives a new approach to prove inequalities. We call the function $f_* : \Omega \to \in \mathbb{R}_{++}$ in Proposition 2.3 a *weak increasing function factor* of the function $f : \Omega \to \mathbb{R}$.

REMARK 2.2. Theorem 2.1 gives a new proof of the inequalities (3). Indeed, according to Theorem 2.1 and Proposition 2.1 we know that the inequalities (3) hold for $\mathbf{x} \in I^n \downarrow$. If $\mathbf{x} \in I^n \uparrow$, then the inequalities (14) are reversed by the above proof. from the inequalities $x_k \leqslant x_{k+1}, \ k = 1, 2, \cdots, n-1$ we get $F([\mathbf{x}]_k) \geqslant F([\mathbf{x}]_{k+1}), \ k = 1, 2, \cdots, n-1$, hence the inequalities (3) still hold. So we have the following Jensen inequality

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \geqslant f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right), \quad \forall \mathbf{x} \in I^n \downarrow \ \text{ or } \ I^n \uparrow \tag{19}$$

under the assumption (1), where the function $f : I \to \mathbb{R}$ is a differentiable convex function.

REMARK 2.3. ZhongLie Wang in [4] has proved the sequence $\{F_k(\mathbf{x},p)\}_{k=1}^n$ is also monotonically decreasing, i.e.

$$F_1(\mathbf{x},p) \geqslant \cdots \geqslant F_k(\mathbf{x},p) \geqslant F_{k+1}(\mathbf{x},p) \geqslant \cdots \geqslant F_n(\mathbf{x},p) = 0, \tag{20}$$

where

$$F_k(\mathbf{x},p) = \sum_{i=k}^n p_i f(x_i) - \overline{P}_k f\left(\frac{1}{\overline{P}_k}\sum_{i=k}^n p_i x_i\right), \ \ \overline{P}_k = \sum_{i=k}^n p_i,$$

$\mathbf{x} \in I^n$, $p_k > 0$, $k = 1,2,\cdots,n$ and $f : I \to \mathbb{R}$ is a convex function. If $f : I \to \mathbb{R}$ is a differentiable convex function, then we can give a new proof of the inequalities (20) as follows: Set

$$\overline{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i \in I, \ x_1' = \frac{1}{\overline{P}_2}\sum_{i=2}^n p_i x_i \in I.$$

If we set $k = 1$ in (16), then we get

$$\frac{\partial F([\mathbf{x}]_1)}{\partial x_1} = \frac{p_1}{P_n}\left[f'(x_1) - f'(\overline{x})\right]. \tag{21}$$

Since f' is increasing, $p_k > 0$, $k = 1,2,\cdots,n$ and

$$x_1 > x_1' \Rightarrow x_1 > \overline{x} \Rightarrow \frac{\partial F([\mathbf{x}]_1)}{\partial x_1} \geqslant 0,$$
$$x_1 < x_1' \Rightarrow x_1 < \overline{x} \Rightarrow \frac{\partial F([\mathbf{x}]_1)}{\partial x_1} \leqslant 0,$$

we have

$$F_1(\mathbf{x},p) = F(x_1,x_2,\cdots,x_n) \geqslant F\left(x_1',x_2,\cdots,x_n\right) = F_2(\mathbf{x},p). \tag{22}$$

That is to say, the inequalities (20) hold for $k = 1$. Similarly, we can prove that the inequalities (20) hold for $k = 1,2,\cdots,n-1$. This ends the proof.

REMARK 2.4. The reference [5] gives the applications of the function $F : I^n \to \mathbb{R}$, defined in (4) and where $P_n = 1$, in statistics and space science.

3. The relationship between the weak monotonic function and Schur-function

Let $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ be such that $x_{[1]} \geqslant \cdots \geqslant x_{[n]}$ and $y_{[1]} \geqslant \cdots \geqslant y_{[n]}$. If

$$\sum_{i=1}^k x_{[i]} \leqslant \sum_{i=1}^k y_{[i]}, \ \ k = 1,\cdots,n-1 \ \text{ and } \ \sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]},$$

then we say $\mathbf{x}$ is majorized by $\mathbf{y}$ and denote this by $\mathbf{x} \prec \mathbf{y}$, where $[1],\cdots,[n]$ is a permutation of $1,\cdots,n$.

DEFINITION 3.1. (see [1], [6], [10] and [12]) Let $\Omega \subset \mathbb{R}^n$ be a symmetrical and convex set, and let $f : \Omega \to \mathbb{R}$ be a symmetric function. If

$$\mathbf{x},\ \mathbf{y} \in \Omega \ \text{ and } \ \mathbf{x} \prec \mathbf{y} \Rightarrow f(\mathbf{x}) \leqslant f(\mathbf{y}),$$

then we call $f : \Omega \to \mathbb{R}$ a *Schur-convex function*. If $-f$ is a Schur-convex function, then we call $f : \Omega \to \mathbb{R}$ a *Schur-concave function*. A Schur-convex function or a Schur-concave function is called a *Schur-function*.

LEMMA 3.1. (see [6, p. 57]) *Let $\Omega \subset \mathbb{R}^n$ be a symmetrical and convex domain, and let $f : \Omega \to \mathbb{R}$ be a symmetrical and differentiable function. Then $f : \Omega \to \mathbb{R}$ is a Schur-convex function if and only if*

$$(x_i - x_j)\left[\frac{\partial f(\mathbf{x})}{\partial x_i} - \frac{\partial f(\mathbf{x})}{\partial x_j}\right] \geqslant 0, \quad i, j = 1, 2, \cdots, n \tag{23}$$

for all $\mathbf{x} \in \Omega$.

In this section, our main result is as follows.

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^n_{++}$ be a symmetrical and convex domain, and let $f : \Omega \to \mathbb{R}$ be a homogeneous of degree $\gamma \geqslant 0$, symmetrical and differentiable function. If $f : \Omega \to \mathbb{R}$ is a Schur-convex function and $f(\mathbf{e}) \geqslant 0$, then $f : \Omega \to \mathbb{R}$ is a weak increasing function.*

Proof. Assume that $f : \Omega \to \mathbb{R}$ is a Schur-convex function and $f(\mathbf{e}) \geqslant 0$. Since $f : \Omega \to \mathbb{R}$ is a homogeneous function, (10) holds. From (10) we see that

$$\gamma t^{\gamma-1} f(\mathbf{x}) = \frac{\partial f(t\mathbf{x})}{\partial t} = \sum_{i=1}^{n} \frac{\partial f(t\mathbf{x})}{\partial (tx_i)} \frac{\partial (tx_i)}{\partial t} = \sum_{i=1}^{n} x_i \frac{\partial f(t\mathbf{x})}{\partial (tx_i)} \tag{24}$$

holds for any $t \in \mathbb{R}_{++}$, $\mathbf{x} \in \Omega$ and $t\mathbf{x} \in \Omega$. Set $t = 1$ in (24), we get

$$\gamma f(\mathbf{x}) = \sum_{i=1}^{n} x_i f_i'(\mathbf{x}), \tag{25}$$

where

$$f_i'(\mathbf{x}) \triangleq \frac{\partial f(\mathbf{x})}{\partial x_i}, \quad i = 1, 2, \cdots, n.$$

Since $f : \Omega \to \mathbb{R}$ is a Schur-convex function and from Lemma 3.1 we know that the inequalities (23) hold for any $\mathbf{x} \in \Omega\downarrow$. That is

$$f_1'(\mathbf{x}) \geqslant f_2'(\mathbf{x}) \geqslant \cdots \geqslant f_n'(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega\downarrow. \tag{26}$$

By (25) and (26) we know that for any $\mathbf{x} \in \Omega\downarrow$ and any $k : 1 \leqslant k \leqslant n-1$ we have

$$\gamma f(\mathbf{x}) = \sum_{i=1}^{n} x_i f_i'(\mathbf{x}) \leqslant \sum_{i=1}^{k} x_i f_i'(\mathbf{x}) + \sum_{i=k+1}^{n} x_i f_k'(\mathbf{x}). \tag{27}$$

Set $\mathbf{x} \mapsto [\mathbf{x}]_k$ in (27). Since $f : \Omega \to \mathbb{R}$ is a symmetric function and $x_1 = x_2 = \cdots = x_k$ we get

$$\frac{\partial f([\mathbf{x}]_k)}{\partial x_k} = \sum_{i=1}^{k} \frac{\partial f([\mathbf{x}]_k)}{\partial x_i} \frac{\partial x_i}{\partial x_k} = \sum_{i=1}^{k} f_i'([\mathbf{x}]_k), \ \ k = 1, 2, \cdots, n-1, \tag{28}$$

$$f_1'([\mathbf{x}]_k) = \cdots = f_k'([\mathbf{x}]_k) = \frac{1}{k} \frac{\partial f([\mathbf{x}]_k)}{\partial x_k}, \ \ k = 1, 2, \cdots, n-1, \tag{29}$$

$$\begin{aligned} \gamma f([\mathbf{x}]_k) &\leqslant \sum_{i=1}^{k} x_i f_i'([\mathbf{x}]_k) + \sum_{i=k+1}^{n} x_i f_k'([\mathbf{x}]_k) \\ &= x_k \frac{\partial f([\mathbf{x}]_k)}{\partial x_k} + \sum_{i=k+1}^{n} x_i \frac{1}{k} \frac{\partial f([\mathbf{x}]_k)}{\partial x_k} \\ &= \frac{1}{k} \left(k x_k + \sum_{i=k+1}^{n} x_i \right) \frac{\partial f([\mathbf{x}]_k)}{\partial x_k}, \ \ k = 1, 2, \cdots, n-1. \end{aligned} \tag{30}$$

Note that (see [6])

$$\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \mathbf{e} \prec \mathbf{x}, \ \ \forall \mathbf{x} \in \Omega \ \text{ and } \ f(\mathbf{e}) \geqslant 0. \tag{31}$$

Since $f : \Omega \to \mathbb{R}$ is a Schur-convex function and from (31) we get

$$f(\mathbf{x}) \geqslant f\left[\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \mathbf{e} \right] = \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^{\gamma} f(\mathbf{e}) \geqslant 0, \quad \forall \mathbf{x} \in \Omega. \tag{32}$$

From (32) we get

$$f([\mathbf{x}]_k) \geqslant 0, \ \ \forall \mathbf{x} \in \Omega\downarrow, \ \ k = 1, 2, \cdots, n-1. \tag{33}$$

By (30), (33) and $\Omega\downarrow \subset \Omega \subset \mathbb{R}^n_{++}$ we get

$$\frac{\partial f([\mathbf{x}]_k)}{\partial x_k} \geqslant \gamma f([\mathbf{x}]_k) \left[\frac{1}{k} \left(k x_k + \sum_{i=k+1}^{n} x_i \right) \right]^{-1} \geqslant 0$$

for any $\mathbf{x} \in \Omega\downarrow$ and $k \in \{1, 2, \cdots, n-1\}$. By Proposition 2.2, the function $f : \Omega \to \mathbb{R}$ is a weak increasing function. The proof of Theorem 3.1 is completed. □

REMARK 3.1. Theorem 3.1 shows that the class of weak monotonic functions is more extensive than the class of Schur-functions.

4. A Chebyshev type inequality

The following notations (see [7, 8]) will be used in this section:

$$\mathbf{a}=\mathbf{a}^1,\ \mathbf{a}^p=\left(a_1^p,\cdots,a_n^p\right),\ C^n=\{t\mathbf{e}|t\in\mathbb{R}\},$$

$$\|\mathbf{a}\|_p=\begin{cases}\left(\sum_{i=1}^n|a_i|^p\right)^{1/p}, & \text{if } 0<p<\infty\\ \max\limits_{1\leqslant i\leqslant n}\{|a_i|\}, & \text{if } p=\infty\end{cases},$$

$$\mathbf{ab}=(a_1b_1,\cdots,a_nb_n),\ \frac{\mathbf{a}}{\mathbf{b}}=\left(\frac{a_1}{b_1},\cdots,\frac{a_n}{b_n}\right),\ \langle\mathbf{a},\ \mathbf{b}\rangle=\sum_{i=1}^n a_ib_i,$$

where $\langle\mathbf{a},\mathbf{b}\rangle$ is the inner product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

DEFINITION 4.1. (See [7] and [13]) Two row vectors $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{y}\in\mathbb{R}^n$ are said to be *similarly ordered*, denoted by $\mathbf{x}\uparrow\mathbf{y}$, if and only if for any $i,j:1\leqslant i,j\leqslant n$ we have

$$(x_i-x_j)(y_i-y_j)\geqslant 0.$$

If the inequality is reversed, then $\mathbf{x}$ and $\mathbf{y}$ are said to *oppositely ordered*, denoted by $\mathbf{x}\downarrow\mathbf{y}$.

A well-known Chebyshev's inequality in $\mathbb{R}^n$ states (see [7, 8]):

Let $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$. If $\mathbf{a}\uparrow\mathbf{b}$, then

$$\frac{\langle\mathbf{a},\mathbf{b}\rangle}{\langle\mathbf{e},\mathbf{e}\rangle}\geqslant\frac{\langle\mathbf{e},\mathbf{a}\rangle}{\langle\mathbf{e},\mathbf{e}\rangle}\cdot\frac{\langle\mathbf{e},\mathbf{b}\rangle}{\langle\mathbf{e},\mathbf{e}\rangle}. \tag{34}$$

The inequality is reversed if $\mathbf{a}\downarrow\mathbf{b}$. The equality in (34) holds if and only if $\mathbf{a}\in C^n$ or $\mathbf{b}\in C^n$.

A large number of generalizations and applications of the inequality (34) had been obtained in [7, 8, 9, 10, 11, 12]. An interesting generalization of (34) was given by Wen and Wang under the proper hypotheses in [11]:

$$\frac{\operatorname{per}(A\odot B)}{n!}\geqslant\frac{\operatorname{per}A}{n!}\cdot\frac{\operatorname{per}B}{n!}, \tag{35}$$

and

$$\frac{\operatorname{per}A}{\prod_{i=1}^n\sum_{j=1}^n a_{i,j}}\leqslant\frac{\operatorname{per}B}{\prod_{i=1}^n\sum_{j=1}^n b_{i,j}}, \tag{36}$$

where $A,B\in\mathbb{R}_{++}^{n\times n}$ are two matrices.

In this section, we will generalize the inequality (34). Our main result is as follows.

THEOREM 4.1. *Let* $\mathbf{a},\ \mathbf{b},\ \mathbf{a}^*,\ \mathbf{b}^* \in \mathbb{R}^n_{++}$. *If*

$$\mathbf{a}^* \uparrow \frac{\mathbf{a}}{\mathbf{a}^*},\ \mathbf{b}^* \uparrow \frac{\mathbf{b}}{\mathbf{b}^*},\ \mathbf{a}^* \uparrow \mathbf{b}^*,\ (p,q) \in (0,1]^2,$$

then

$$\frac{\langle \mathbf{a},\ \mathbf{b}\rangle}{\langle \mathbf{a}^*,\ \mathbf{b}^*\rangle} \geqslant \frac{\|\mathbf{a}\|_p}{\|\mathbf{a}^*\|_p} \cdot \frac{\|\mathbf{b}\|_q}{\|\mathbf{b}^*\|_q}. \tag{37}$$

The equality in (37) *holds if and only if*

$$\frac{\mathbf{a}}{\mathbf{a}^*},\ \frac{\mathbf{b}}{\mathbf{b}^*} \in C^n \ \ or\ \ p=1,\ \mathbf{b},\ \mathbf{b}^* \in C^n \ \ or\ \ q=1,\ \mathbf{a},\ \mathbf{a}^* \in C^n. \tag{38}$$

Proof. Set

$$\mathbf{x} = \frac{\mathbf{a}}{\mathbf{a}^*},\ \mathbf{y} = \frac{\mathbf{b}}{\mathbf{b}^*} \Leftrightarrow \mathbf{a} = \mathbf{a}^*\mathbf{x},\ \mathbf{b} = \mathbf{b}^*\mathbf{y}.$$

Without loss of generality, we can assume

$$\mathbf{a}^*,\ \mathbf{x},\ \mathbf{b}^*,\ \mathbf{y} \in \mathbb{R}^n_{++}\downarrow. \tag{39}$$

Then the inequality (37) is equivalent to

$$\frac{\langle \mathbf{a}^*\mathbf{x}, \mathbf{b}^*\mathbf{y}\rangle}{\langle \mathbf{a}^*, \mathbf{b}^*\rangle} \geqslant \frac{\|\mathbf{a}^*\mathbf{x}\|_p}{\|\mathbf{a}^*\|_p} \cdot \frac{\|\mathbf{b}^*\mathbf{y}\|_q}{\|\mathbf{b}^*\|_q}. \tag{40}$$

We define an auxiliary function $F : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}$ as follows:

$$\begin{aligned} F(\mathbf{x},'\mathbf{y}) &\triangleq \ln \frac{\langle \mathbf{a}^*\mathbf{x}, \mathbf{b}^*\mathbf{y}\rangle}{\|\mathbf{a}^*\mathbf{x}\|_p \cdot \|\mathbf{b}^*\mathbf{y}\|_q} \\ &= \ln\langle \mathbf{a}^*\mathbf{x}, \mathbf{b}^*\mathbf{y}\rangle - \ln\left(\|\mathbf{a}^*\mathbf{x}\|_p \cdot \|\mathbf{b}^*\mathbf{y}\|_q\right) \\ &= \ln\left(\sum_{i=1}^n a_i^* b_i^* x_i y_i\right) - \frac{1}{p}\ln\left(\sum_{i=1}^n a_i^{*p} x_i^p\right) - \frac{1}{q}\ln\left(\sum_{i=1}^n b_i^{*q} y_i^q\right). \end{aligned}$$

It follows from the above that the inequality (40) is equivalent to

$$F(\mathbf{x},\mathbf{y}) \geqslant F(\mathbf{e},\mathbf{e}). \tag{41}$$

For any fixed $\mathbf{y} \in \mathbb{R}^n_{++}\downarrow$, we prove that the auxiliary function

$$f : \mathbb{R}^n_{++}\downarrow \to \mathbb{R},\ f(\mathbf{x}) = F(\mathbf{x},\ \mathbf{y})$$

is a weak increasing function.

Set

$$A \triangleq \left(\sum_{i=1}^{k} a_i^* b_i^* y_i\right)\left(x_k^p \sum_{i=1}^{k} a_i^{*p} + \sum_{i=k+1}^{n} a_i^{*p} x_i^p\right)$$
$$- \left(x_k^{p-1} \sum_{i=1}^{k} a_i^{*p}\right)\left(x_k \sum_{i=1}^{k} a_i^* b_i^* y_i + \sum_{i=k+1}^{n} a_i^* b_i^* x_i y_i\right)$$
$$= \left(\sum_{i=1}^{k} a_i^* b_i^* y_i\right)\left(\sum_{i=k+1}^{n} a_i^{*p} x_i^p\right) - \left(x_k^{p-1} \sum_{i=1}^{k} a_i^{*p}\right)\left(\sum_{i=k+1}^{n} a_i^* b_i^* x_i y_i\right)$$
$$= \sum_{j=k+1}^{n} \sum_{i=1}^{k} a_j^{*p} x_j^p a_i^* b_i^* y_i - \sum_{j=k+1}^{n} \sum_{i=1}^{k} a_j^* b_j^* x_j y_j x_k^{p-1} a_i^{*p}$$
$$= \sum_{j=k+1}^{n} \sum_{i=1}^{k} \left(a_j^{*p} x_j^p a_i^* b_i^* y_i - a_j^* b_j^* x_j y_j x_k^{p-1} a_i^{*p}\right)$$
$$= \sum_{j=k+1}^{n} \sum_{i=1}^{k} a_j^* b_j^* x_j y_j x_k^{p-1} a_i^{*p} \left(\frac{a_j^{*p} x_j^p a_i^* b_i^* y_i}{a_j^* b_j^* x_j y_j x_k^{p-1} a_i^{*p}} - 1\right)$$
$$= \sum_{j=k+1}^{n} \sum_{i=1}^{k} a_j^* b_j^* x_j y_j x_k^{p-1} a_i^{*p} \left[\left(\frac{a_j^*}{a_i^*}\right)^{p-1} \cdot \left(\frac{x_j}{x_k}\right)^{p-1} \cdot \frac{b_i^*}{b_j^*} \cdot \frac{y_i}{y_j} - 1\right],$$

and

$$B \triangleq \left(x_k \sum_{i=1}^{k} a_i^* b_i^* y_i + \sum_{i=k+1}^{n} a_i^* b_i^* x_i y_i\right)\left(x_k^p \sum_{i=1}^{k} a_i^{*p} + \sum_{i=k+1}^{n} a_i^{*p} x_i^p\right).$$

From

$$f([\mathbf{x}]_k) = F(\underbrace{x_k, x_k, \cdots, x_k}_{k}, x_{k+1}, x_{k+2}, \cdots, x_n, y)$$
$$= \ln\left(x_k \sum_{i=1}^{k} a_i^* b_i^* y_i + \sum_{i=k+1}^{n} a_i^* b_i^* x_i y_i\right) \tag{42}$$
$$- \frac{1}{p} \ln\left(x_k^p \sum_{i=1}^{k} a_i^{*p} + \sum_{i=k+1}^{n} a_i^{*p} x_i^p\right) - \frac{1}{q} \ln\left(\sum_{i=1}^{n} b_i^{*q} y_i^q\right)$$

we get

$$\frac{\partial f([\mathbf{x}]_k)}{\partial x_k} = \frac{\sum_{i=1}^{k} a_i^* b_i^* y_i}{x_k \sum_{i=1}^{k} a_i^* b_i^* y_i + \sum_{i=k+1}^{n} a_i^* b_i^* x_i y_i} - \frac{x_k^{p-1} \sum_{i=1}^{k} a_i^{*p}}{x_k^p \sum_{i=1}^{k} a_i^{*p} + \sum_{i=k+1}^{n} a_i^{*p} x_i^p} = \frac{A}{B}. \tag{43}$$

Since $0 < p \leqslant 1$, $1 \leqslant i \leqslant k < j \leqslant n$ with (39), we have that

$$\left(\frac{a_j^*}{a_i^*}\right)^{p-1} \geqslant 1,\ \left(\frac{x_j}{x_k}\right)^{p-1} \geqslant 1,\ \frac{b_i^*}{b_j^*} \geqslant 1,\ \frac{y_i}{y_j} \geqslant 1, \tag{44}$$

$$\left(\frac{a_j^*}{a_i^*}\right)^{p-1}\cdot\left(\frac{x_j}{x_k}\right)^{p-1}\cdot\frac{b_i^*}{b_j^*}\cdot\frac{y_i}{y_j}-1\geqslant 0. \tag{45}$$

Hence

$$A\geqslant 0,\quad B>0. \tag{46}$$

From (43) and (46) we get

$$\frac{\partial f([\mathbf{x}]_k)}{\partial x_k}\geqslant 0,\ \ \forall \mathbf{x}\in\mathbb{R}_{++}^n\downarrow,\ \ k=1,2,\cdots,n-1. \tag{47}$$

From (47) and Proposition 2.2 we know that the function

$$f:\mathbb{R}_{++}^n\downarrow\to\mathbb{R},\quad f(\mathbf{x})=F(\mathbf{x},\mathbf{y})$$

is a weak increasing function.

By (7) in Proposition 2.1 we get

$$F(\mathbf{x},\mathbf{y})=f(\mathbf{x})\geqslant f(x_n\mathbf{e})=F(\mathbf{e},\mathbf{y}). \tag{48}$$

Similarly, we can prove that the auxiliary function

$$g:\mathbb{R}_{++}^n\downarrow\to\mathbb{R},\quad g(\mathbf{y})=F(\mathbf{e},\mathbf{y})$$

is also a weak increasing function.

Indeed, if we use the following exchange:

$$\mathbf{a}^*\leftrightarrow\mathbf{b}^*,\ \mathbf{x}\leftrightarrow\mathbf{y},\ p\leftrightarrow q$$

in the proof of the above, and set $\mathbf{x}=C\mathbf{e},C\in\mathbb{R}_{++}$, we get

$$\frac{\partial g([\mathbf{y}]_k)}{\partial y_k}=\frac{A^*}{B^*},$$

where

$$A^*=\sum_{j=k+1}^{n}\sum_{i=1}^{k}b_j^*a_j^*y_jx_jy_k^{q-1}b_i^{*q}\left[\left(\frac{b_j^*}{b_i^*}\right)^{q-1}\cdot\left(\frac{y_j}{y_k}\right)^{q-1}\cdot\frac{a_i^*}{a_j^*}\cdot\frac{x_i}{x_j}-1\right]\geqslant 0,$$

and

$$B^*=\left(y_k\sum_{i=1}^{k}b_i^*a_i^*x_i+\sum_{i=k+1}^{n}b_i^*a_i^*y_ix_i\right)\left(y_k^q\sum_{i=1}^{k}b_i^{*q}+\sum_{i=k+1}^{n}b_i^{*q}y_i^q\right)>0.$$

Hence the auxiliary function

$$g:\mathbb{R}_{++}^n\downarrow\to\mathbb{R},\quad g(\mathbf{y})=F(\mathbf{e},\mathbf{y})$$

is also a weak increasing function by the above proof.

By (7) in Proposition 2.1 and (48) we get

$$F(\mathbf{x},\mathbf{y}) \geqslant F(\mathbf{e},\mathbf{y}) = g(\mathbf{y}) \geqslant g(y_n\mathbf{e}) = F(\mathbf{e},\mathbf{e}). \tag{49}$$

that is, (41) holds.

We will discuss the equality condition of the inequality (37) as follows.

Based on the above analysis, these conditions should be

$$\mathbf{x} \in C^n \text{ and } \mathbf{y} \in C^n, \text{ or } \frac{\partial f([\mathbf{x}]_k)}{\partial x_k} \equiv 0, \text{ or } \frac{\partial g([\mathbf{y}]_k)}{\partial y_k} \equiv 0, \ k = 1,2,\cdots,n-1, \tag{50}$$

that is (38) hold.

Indeed, assume that the equality in (37) holds. If $\mathbf{x} \in C^n$ does not hold, then

$$\frac{\partial f([\mathbf{x}]_k)}{\partial x_k} \equiv 0, \ k = 1,2,\cdots,n-1 \Leftrightarrow p = 1, \ \mathbf{b}, \ \mathbf{b}^* \in C^n \tag{51}$$

by the above prove. If $\mathbf{y} \in C^n$ does not hold, then

$$\frac{\partial g([\mathbf{y}]_k)}{\partial y_k} \equiv 0, \ k = 1,2,\cdots,n-1 \Leftrightarrow q = 1, \ \mathbf{a}, \ \mathbf{a}^* \in C^n \tag{52}$$

by the above prove. hence (50) hold, i.e. (38) hold.

Assume that the (50) hold, i.e. (38) hold, we can easily prove that the equality in (37) holds.

This completes the proof of Theorem 4.1. □

REMARK 4.1. We remark here if $p = q = 1, \mathbf{a}^* = \mathbf{b}^* = \mathbf{e}$, then the inequality (37) can be rewritten as inequality (34). Therefore, the inequality (37) is a generalization of the inequality (34).

REMARK 4.2. Theorem 4.1 implies the following interesting result: If the four continuous functions $f^*, \ f/f^*, \ g^*, \ g/g^* : I \to \mathbb{R}_{++}$, where I is an interval, are increasing (or decreasing) , and $(p,q) \in (0,1]^2$, then we have the following inequality:

$$\frac{\int_I fg}{\int_I f^* g^*} \leqslant \left(\frac{\int_I f^p}{\int_I f^{*p}}\right)^{1/p} \times \left(\frac{\int_I g^q}{\int_I g^{*q}}\right)^{1/q}. \tag{53}$$

5. A new proof of Marshall's inequality

Now we give an application of Theorem 4.1 as follows.

THEOREM 5.1. (Marshall's inequality, see [8, 13]) *Let*

$$\mathbf{a}, \ \mathbf{a}^* \in \mathbb{R}^n_{++} \ \text{and} \ \mathbf{a}^* \uparrow \frac{\mathbf{a}}{\mathbf{a}^*}.$$

If the real numbers α, β *are such that* $\alpha < \beta$*, then*

$$\frac{M_n^{[\alpha]}(\mathbf{a})}{M_n^{[\alpha]}(\mathbf{a}^*)} \leqslant \frac{M_n^{[\beta]}(\mathbf{a})}{M_n^{[\beta]}(\mathbf{a}^*)}, \tag{54}$$

the equality in (54) holds if and only if

$$\frac{\mathbf{a}}{\mathbf{a}^*} \in C^n, \tag{55}$$

where

$$M_n^{[t]}(\mathbf{a}) = \begin{cases} \left(n^{-1}\sum_{i=1}^n a_i^t\right)^{1/t}, & \text{if } 0 < |t| < \infty \\ \left(\prod_{i=1}^n a_i\right)^{1/n}, & \text{if } t = 0 \end{cases}.$$

Proof. Since

$$\mathbf{x} \uparrow \mathbf{e}, \ \forall \mathbf{x} \in \mathbb{R}^n, \ \mathbf{a}, \ \mathbf{a}^* \in \mathbb{R}^n_{++} \text{ and } \mathbf{a}^* \uparrow \frac{\mathbf{a}}{\mathbf{a}^*},$$

we see that

$$\mathbf{a}^{*\gamma} \uparrow \frac{\mathbf{a}^\gamma}{\mathbf{a}^{*\gamma}}, \ \mathbf{e} \uparrow \frac{\mathbf{e}}{\mathbf{e}} \text{ and } \mathbf{a}^\gamma \uparrow \mathbf{e}, \ \forall \gamma \in \mathbb{R}.$$

(i) if $0 < \alpha < \beta$, then $0 < \frac{\alpha}{\beta} < 1$. Since the inequality (54) can be written as

$$\frac{\langle \mathbf{a}^\beta, \mathbf{e}\rangle}{\langle \mathbf{a}^{*\beta}, \mathbf{e}\rangle} \geqslant \frac{\|\mathbf{a}^\beta\|_{\alpha/\beta}}{\|\mathbf{a}^{*\beta}\|_{\alpha/\beta}} \cdot \frac{\|\mathbf{e}\|_1}{\|\mathbf{e}\|_1}, \tag{56}$$

according to Theorem 4.1, the inequality (56) holds, hence the inequality (54) holds.

(ii) if $\alpha < \beta < 0$, then $0 < -\beta < -\alpha$. According to the proof of the assertion (i) we have

$$\frac{M_n^{[-\beta]}(\mathbf{a}^{-1})}{M_n^{[-\beta]}((\mathbf{a}^*)^{-1})} \leqslant \frac{M_n^{[-\alpha]}(\mathbf{a}^{-1})}{M_n^{[-\alpha]}((\mathbf{a}^*)^{-1})}. \tag{57}$$

That is to say, the inequality (54) holds.

(iii) if $\alpha \leqslant 0 \leqslant \beta$ and $\alpha \neq \beta$, then

$$\frac{M_n^{[\alpha]}(\mathbf{a})}{M_n^{[\alpha]}(\mathbf{a}^*)} \leqslant \frac{M_n^{[0]}(\mathbf{a})}{M_n^{[0]}(\mathbf{a}^*)} \leqslant \frac{M_n^{[\beta]}(\mathbf{a})}{M_n^{[\beta]}(\mathbf{a}^*)} \tag{58}$$

by the proof of the assertion (i) and the proof of the assertion (ii), hence the inequality (54) still holds.

From Theorem 4.1 it follows that equality in (54) holds if and only if (55) holds.

The proof of Theorem 5.1 is completed. □

REMARK 5.1. Theorem 5.1 implies the following interesting result: Let

$$\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n_{++},\ \mathbf{y} \uparrow \frac{\mathbf{x}}{\mathbf{y}}.$$

If $\sum_{i=1}^n x_i \geqslant \sum_{i=1}^n y_i$, then we have

$$\sum_{i=1}^n x_i^p \geqslant \sum_{i=1}^n y_i^p,\ \ \forall p \in (1,\infty). \tag{59}$$

If $\sum_{i=1}^n x_i \leqslant \sum_{i=1}^n y_i$, then we have

$$\sum_{i=1}^n x_i^p \leqslant \sum_{i=1}^n y_i^p,\ \ \forall p \in (0,1). \tag{60}$$

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