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THE BEST BOUND FOR n–DIMENSIONAL FRACTIONAL HARDY OPERATORS

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Abstract. In this note, we precisely evaluate the operator norm of the fractional Hardy operator $\mathbb{H}_\beta$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $0<\beta<n$, $1<p<q<\infty$ and $1/p-1/q=\beta/n$. By this we extend the result of Bliss [1] to the case of high dimension and improve our result in [7].

1. Introduction

Recall that, for a nonnegative locally integrable function f on $\mathbb{R}^n$, the n-dimensional fractional Hardy operator $\mathbb{H}_\beta$ is defined by

$$\mathbb{H}_\beta f(x)=\frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}}\int_{|y|<|x|}f(y)dy,\ x\in\mathbb{R}^n\setminus\{0\}, \tag{1}$$

where $0<\beta<n$ (cf. [4]). Given $0<\beta<n$ and a locally integrable function f on $\mathbb{R}^n$, the fractional Hardy operator is closely related to the fractional Hardy-Littlewood maximal operator M_β defined by

$$M_\beta f(x)=\sup_{r>0}\frac{1}{|B(x,r)|^{1-\frac{\beta}{n}}}\int_{|y-x|<r}|f(y)|dy,\quad x\in\mathbb{R}^n,$$

and to the Riesz potential I_β defined by

$$I_\beta f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\beta}},\quad x\in\mathbb{R}^n.$$

It is then easy to see that

$$M_\beta f(x)=\sup_{y\in\mathbb{R}^n}(\mathbb{H}_\beta(|f(\cdot+x)|))(y),\quad x\in\mathbb{R}^n,$$

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and

$$\mathbb{H}_\beta(|f|)(x) \leqslant 2^{n-\beta} M_\beta(f)(x) \leqslant 2^{n-\beta}|B(0,1)|^{\frac{\beta}{n}-1} I_\beta(|f|)(x),\ x \in \mathbb{R}^n \setminus \{0\}.$$

Let us note some results of the fractional Hardy operator. For the one-dimensional case, Bliss in [1] worked out the best possible constant C_0 in the inequality

$$\|\mathbb{H}_\beta f\|_{L^q(\mathbb{R}_+)} \leqslant C_0 \|f\|_{L^p(\mathbb{R}_+)},$$

where $0<\beta<1,\ 1<p<\infty,\ \frac{1}{p}-\frac{1}{q}=\beta,\ C_0=\left(\frac{p'}{q}\right)^{1/q}\left(\frac{1}{q\beta}\cdot B\left(\frac{1}{q\beta},\frac{1}{q'\beta}\right)\right)^{-\beta}$, and $\mathbb{R}_+=(0,\infty)$. A natural question is to consider the case of high dimension. As we know, the theory in higher dimensions is perceived to be much more difficult and indeed there are significant problems in higher dimensions for which the one-dimensional techniques are not adequate. For the n-dimensional case, we in [7] showed that C_1, the bound of operator $\mathbb{H}_\beta$ from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\beta},\infty}(\mathbb{R}^n)$, is 1, and the constant C_1 is best possible. We also worked out that the constant C_2, the bound of the operator $\mathbb{H}_\beta$ from from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, satisfies

$$\left(\frac{p}{q}\right)^{1/q}\left(\frac{p}{p-1}\right)^{1/q}\left(\frac{q}{q-1}\right)^{1-1/q}\left(1-\frac{p}{q}\right)^{1/p-1/q} \leqslant C_2 \leqslant \left(\frac{p}{p-1}\right)^{\frac{p}{q}},$$

with $0<\beta<n,\ 1<p<q<\infty$ and $1/p-1/q=\beta/n$. For more information about the fractional Hardy operator, we refer to ([2], [3], [6], [8]) and references therein.

The purpose of this note is to completely fix the gap in [7], and give the norm of $\|\mathbb{H}_\beta\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)}$. Our result is:

THEOREM 1. *Suppose that* $0<\beta<n,\ 1<p<q<\infty$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\beta}{n}$. *If* $f\in L^p(\mathbb{R}^n)$, *then we have*

$$\|\mathbb{H}_\beta f\|_{L^q(\mathbb{R}^n)} \leqslant A\|f\|_{L^p(\mathbb{R}^n)}. \tag{2}$$

Moreover,

$$\|\mathbb{H}_\beta\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} = A,$$

where

$$A=\left(\frac{p'}{q}\right)^{1/q}\left(\frac{n}{q\beta}\cdot B\left(\frac{n}{q\beta},\frac{n}{q'\beta}\right)\right)^{-\beta/n}.$$

It is worth mentioning that the proof from [5] is not applicable to the fractional Hardy operator. Although our idea partly come from [1], there are some essential difficulties. The first difficulty is how to deal with the high-dimensional case. In this paper, we shall use the rotation method as in [5] to reduce the n-dimensional case to the one-dimensional case. The second difficulty here is how to reconstruct some auxiliary functions to achieve the sharp bound, which is quite different from [1].

Throughout the note, we use the following notation. The definition of the usual beta function is defined by $B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt$, where z and w are complex numbers with positive real parts. The set $B(0,|x|)$ denotes a ball with center at the original point and radius $|x|$, and $|B(0,|x|)|$ denotes the volume of the ball $B(0,|x|)$. For a real number $p,\ 1<p<\infty,\ p'$ is the conjugate number of p, that is, $1/p+1/p'=1$.

2. Preliminaries

To reduce the dimension of function space, we need the following lemma which was obtained in [7, p. 7].

LEMMA 1. *For a function* $f \in L^p(\mathbb{R}^n)$, *let*

$$g_f(y) = \frac{1}{\omega_n} \int_{|\xi|=1} |f(|y|\xi)|\, d\xi,\ y \in \mathbb{R}^n,$$

where $\omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$. *Then*

$$\mathbb{H}_\beta(|f|)(x) = \mathbb{H}_\beta(g_f)(x)$$

and

$$\|g_f\|_{L^p(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)}.$$

REMARK 1. It follows from the above lemma that

$$\frac{\|\mathbb{H}_\beta(f)\|_{L^q(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} \leqslant \frac{\|\mathbb{H}_\beta(g_f)\|_{L^q(\mathbb{R}^n)}}{\|g_f\|_{L^p(\mathbb{R}^n)}}.$$

Therefore, the norm of the operator $\mathbb{H}_\beta$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ is equal to the norm that $\mathbb{H}_\beta$ restricts to radial functions.

In order to prove Theorem 1, we need to construct an auxiliary function $W(x,y,z)$ and study the continuity and differentiability of the function $W(x,y,z)$. These properties will be applied in Section 3. In order to show the properties of $W(x,y,z)$, we first introduce another auxiliary function $\phi(u)$, which is closely related with the function $W(x,y,z)$.

Assume that f is a nonnegative continuous function on $\mathbb{R}$. By Hölder's inequality,

$$\int_0^s f(r) r^{n-1} dr \leqslant \left(\int_0^s f^p(r) r^{n-1} dr\right)^{1/p} s^{\frac{n(p-1)}{p}} n^{(1-p)/p}.$$

If we denote $y = n\int_0^s f(r) r^{n-1} dr$ and $z = n\int_s^\infty f^p(r) r^{n-1} dr$, then

$$\lim_{s\to 0} s^{-\frac{n(p-1)}{p}} y = 0. \tag{3}$$

Let $\phi(u)$ be the function defined by the equation

$$\phi(u) = u^p \int_0^1 \frac{\eta^{q-2}}{\left(1-u+u\eta^{q\beta/n}\right)^{n/\beta}} d\eta = \frac{1}{[(1-u)u^{p-1}]^{\frac{n}{q\beta}}} \int_0^U \frac{\zeta^{q-2}}{\left(1+\zeta^{q\beta/n}\right)^{n/\beta}} d\zeta, \tag{4}$$

where $U = [u/(1-u)]^{n/q\beta}$.

In order to easily describe the point in the three-dimensional rectangular coordinate system, we shall use x instead of s. For each point (x,y,z) in the octant R of xyz-space where $x>0$, $y>0$, $z>0$, the equation

$$\phi(u) = x^{n(p-1)}y^{-p}z \tag{5}$$

has a unique solution $u(x,y,z)$, since

$$\phi(0)=0,\ \phi(1)=\infty,\ \phi'(u)>0 \text{ for } 0<u<1. \tag{6}$$

If (x,y,z) approaches a point $(x_1,0,1)$, where $x_1>0$, along a continuous curve in R, then by (6), we know that $u(x,y,z)$ approaches 1. It follows from the second form of $\phi(u)$, (6), (5) and (3) that

$$\lim_{x\to x_1} \frac{x^{(p-1)n}}{y^p}(1-u)^{\frac{n}{q\beta}} = \int_0^\infty \frac{\zeta^{q-2}}{(1+\zeta^{\frac{q\beta}{n}})^{n/\beta}}d\zeta = \frac{n}{q\beta}\cdot\frac{\Gamma(\frac{n}{q\beta})\Gamma((1-\frac{1}{q})\frac{n}{\beta})}{\Gamma(\frac{n}{\beta})}. \tag{7}$$

If (x,y,z) approaches a point $(x_2,y_2,0)$, where $x_2>0$, $y_2>0$, along a continuous curve in R. It follows from the first form of $\phi(u)$, (5) and (6) that $u(x,y,z)\to 0$ and

$$\lim_{x\to x_2}\frac{u^p}{z} = (q-1)\frac{x_2^{(p-1)n}}{y_2^p}. \tag{8}$$

Let $W(x,y,z)$ be the function defined by the equation

$$W(x,y,z) = \frac{n}{q(\beta-n)+n}\left(\frac{1}{1-u}\cdot\frac{y^q}{x^{q(n-\beta)-n}} + \frac{z}{(1-u)u^{p-1}}\cdot\frac{y^{q-p}}{x^{n(q-p)-q\beta}}\right),$$

where the function $W(x,y,z)$ is determined when u is replaced by the function $u(x,y,z)$ in (4).

In order to calculate the derivatives of W with respect to x, y and z, we need to give some necessary estimates. It follows from the second form for $\phi(u)$ that

$$\phi'(u) = \frac{n}{q\beta}\left(\frac{1+pu-p}{u(1-u)}\cdot\frac{x^{n(p-1)}z}{y^p} + \frac{u^{p-1}}{1-u}\right).$$

When W is treated as a function of x, y, z, u, the partial derivative of W with respect to u is

$$\begin{aligned}\frac{\partial W}{\partial u} &= \frac{n}{q(\beta-n)+n}\left\{\frac{1}{(1-u)^2}\cdot\frac{y^q}{x^{q(n-\beta)-n}}\right.\\ &\qquad\left.+\frac{y^{q-p}z}{x^{n(q-p)-q\beta}}\left(\frac{1}{(1-u)^2u^{p-1}} - \frac{(p-1)u^{p-2}}{(1-u)u^{2(p-1)}}\right)\right\}\\ &= \frac{n}{q(\beta-n)+n}\left\{\frac{1}{(1-u)^2}\cdot\frac{y^q}{x^{q(n-\beta)-n}} + \frac{y^{q-p}z}{x^{n(q-p)-q\beta}}\left(\frac{1}{1-u}-\frac{p-1}{u}\right)\frac{1}{(1-u)u^{p-1}}\right\}\end{aligned}$$

$$= \frac{q\beta}{q(\beta-n)+n}\cdot\frac{1}{(1-u)u^{p-1}}\cdot\frac{y^q}{x^{q(n-\beta)-n}}\left\{\frac{u^{p-1}}{1-u}+\frac{x^{n(p-1)}}{y^p}\cdot z\cdot\frac{1+pu-p}{u(1-u)}\right\}\frac{n}{q\beta}$$

$$= \frac{q\beta}{q(\beta-n)+n}\cdot\frac{1}{(1-u)u^{p-1}}\cdot\frac{y^q}{x^{q(n-\beta)-n}}\phi'(u).$$

In view of (5), we have

$$x=\left(y^p z^{-1}\phi(u)\right)^{\frac{1}{n(p-1)}},\ y=x^{\frac{n(p-1)}{p}}z^{\frac{1}{p}}\phi(u)^{-\frac{1}{p}},\ z=x^{-n(p-1)}y^p\phi(u).$$

Then

$$\frac{\partial x}{\partial u}=\frac{1}{n(p-1)}y^{\frac{p}{n(p-1)}}z^{-\frac{1}{n(p-1)}}(\phi(u))^{\frac{1}{n(p-1)}-1}\phi'(u),$$

$$\frac{\partial y}{\partial u}=-\frac{1}{p}x^{\frac{n(p-1)}{p}}z^{\frac{1}{p}}(\phi(u))^{-\frac{1}{p}-1}\phi'(u),\ \frac{\partial z}{\partial u}=x^{-n(p-1)}y^p\phi'(u).$$

Using the above calculations and (5), we can get the derivatives of W with respect to x, y and z as follows:

$$\begin{aligned} W_x &= \frac{\partial W}{\partial x}+\frac{\partial W}{\partial u}\frac{\partial u}{\partial x}\\ &= \frac{n}{q(\beta-n)+n}\left((n-q(n-\beta))\frac{y^q}{1-u}x^{n-q(n-\beta)-1}\right.\\ &\quad\left.+\frac{q\beta-n(q-p)}{(1-u)u^{p-1}}\cdot\frac{y^{q-p}z}{x^{n(q-p)-q\beta+1}}\right)\\ &\quad-\frac{p\beta}{(1-u)u^{p-1}}\cdot\frac{y^{q-p}z}{x^{n(q-p)-q\beta+1}}\\ &= \frac{ny^q}{1-u}\cdot\frac{1}{x^{q(n-\beta)-n+1}}, \end{aligned}$$

and analogously,

$$W_y=\frac{p}{1-p}\cdot\frac{y^{q-1}}{(1-u)x^{q(n-\beta)-n}},\ W_z=\frac{1}{1-p}\cdot\frac{1}{(1-u)u^{p-1}}\cdot\frac{y^{q-p}}{x^{q(n-\beta)-np}}.$$

Let $\lambda=\frac{1}{1-p}\cdot\frac{1}{(1-u)u^{p-1}}\cdot\frac{y^{q-p}}{x^{q(n-\beta)-np}}$ and $g=\frac{yu}{x^n}$. Then W_x, W_y, W_z can be rewritten as

$$W_x=nx^{n-q(n-\beta)-1}y^q-n(p-1)\lambda g^p x^{n-1},\ W_y=p\lambda g^{p-1},\ W_z=\lambda.$$

When (x,y,z) approaches $(x_1,0,1)$ or $(x_2,y_2,0)$, the expressions (7) and (8) show that

$$\lim_{x\to x_1}W=-A,\ \lim_{x\to x_2}W=\frac{n}{q(\beta-n)+n}\cdot\frac{y_2^q}{x_2^{q(n-\beta)-n}}. \tag{9}$$

3. Proof of Theorem 1

It follows from Lemma 1 that the norm of the operator $\mathbb{H}_\beta$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ is equal to the norm that $\mathbb{H}_\beta$ restricts to radial functions. Consequently, without loss of generality, it suffices to carry out the proof of the theorem by assuming that f is a nonnegative, radial, smooth function with compact support on $\mathbb{R}^n$.

Using the polar coordinate transformation, we can rewrite (2) as

$$n\int_0^\infty \left(n\int_0^s f(r)r^{n-1}dr\right)^q s^{q(\beta-n)}s^{n-1}ds \leqslant A^q\left(n\int_0^\infty f^p(r)r^{n-1}dr\right)^{q/p}. \qquad (10)$$

It reduces to prove that the (10) holds.

Without loss of generality, we assume that $n\int_0^\infty f^p(r)r^{n-1}dr = 1$.

If $n\int_0^\infty f^p(r)r^{n-1}dr \neq 1$, we can replace f by $(n\int_0^\infty f^p(r)r^{n-1}dr)^{-1}f$.

Consider the curve C in xyz-space defined by the equations $y = n\int_0^x f(r)r^{n-1}dr$, $z = n\int_x^\infty f^p(r)r^{n-1}dr$ and $z(0) = 1$. For such a curve there exists a smallest interval (x_1,x_2), such that

$$0 \leqslant x_1 < x_2 \leqslant \infty,$$

$$y(x) \equiv 0,\ z(x) \equiv 1 \ \text{ on } 0 \leqslant x \leqslant x_1,$$

$$y(x) \equiv \text{const.},\ z(x) \equiv 0 \text{ on } x_2 \leqslant x < \infty.$$

Considering the function $W(x,y(x),z(x))$, we get the derivative of W with respect to x, $W'(x)$,

$$W'(x) = nx^{n-1}\left\{x^{-q(n-\beta)}y^q - \lambda\left((p-1)h^p - ph^{p-1}f + f^p\right)\right\}.$$

Let $\psi(h,f) = (p-1)h^p - ph^{p-1}f + f^p$. Since $h \geqslant 0$ and $f \geqslant 0$, then $\psi(h,f)$ is always positive except at its root $h = f$. And $W'(x)$ can be rewritten as

$$W'(x) = nx^{n-1}x^{-q(n-\beta)}y^q - nx^{n-1}\lambda\psi(h,f).$$

Let $I = n\int_0^\infty \left(\int_0^x f(r)r^{n-1}dr\right)^q x^{q(\beta-n)+n-1}dx$ and $x_2 < \infty$, then on the sub-curves C_{01}, C_{12}, $C_{2\infty}$ of C corresponding to the intervals $(0,x_1)$, (x_1,x_2), (x_2,∞),

$$I(C_{01}) = 0, I(C_{2\infty}) = -\frac{n}{q(\beta-n)+n}\cdot\frac{y_2^q}{x_2^{q(n-\beta)-n}},$$

In view of (9), we get

$$\begin{aligned} I(C_{12}) &= \int_{x_1}^{x_2} dW + \lambda\int_{x_1}^{x_2} nx^{n-1}\psi(h,f)dx \\ &= W(x_2) - W(x_1) + \lambda\int_{x_1}^{x_2} nx^{n-1}\psi(h,f)dx \\ &= A + \frac{n}{q(\beta-n)+n}\cdot\frac{y_2^q}{x_2^{q(n-\beta)-n}} + \lambda\int_{x_1}^{x_2} nx^{n-1}\psi(h,f)dx, \end{aligned}$$

where $y_2 = y(x_2)$. It follows from $\lambda < 0$ and $\psi(h,f) \geqslant 0$ that

$$I(C) = I(C_{12}) + I(C_{2\infty}) \leqslant A.$$

When $x_2 = \infty$, the value of $I(C)$ can be calculated by taking the limit. We omit the details.

Therefore, by the the equivalence of (2) and (10) we obtain

$$\|\mathbb{H}_\beta\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} = \sup_{\|f\|_{L^p(\mathbb{R}^n)}\neq 0} \frac{\|\mathbb{H}_\beta f\|_{L^q(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} \leqslant A.$$

On the other hand, take $\tilde{f}(x) = \frac{1}{\left(1+|x|^{q\beta}\right)^{1+\frac{n}{q\beta}}}$. It follows from

$$n\int_0^s \tilde{f}(r)r^{n-1}dr = \frac{s^n}{\left(1+s^{q\beta}\right)^{\frac{n}{q\beta}}}$$

that the left side of (10) is $\frac{n}{q\beta} \cdot B\left(\frac{n}{q\beta}+1, \frac{n}{q'\beta}-1\right)$. It is easy to verify that

$$n\int_0^\infty \tilde{f}^p(r)r^{n-1}dr = \frac{n}{q\beta} \cdot B\left(\frac{n}{q\beta}, \frac{n}{q'\beta}\right).$$

Therefore,

$$\|\mathbb{H}_\beta\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} = \sup_{\|f\|_{L^p(\mathbb{R}^n)}\neq 0} \frac{\|\mathbb{H}_\beta f\|_{L^q(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} \geqslant \frac{\|\mathbb{H}_\beta \tilde{f}\|_{L^q(\mathbb{R}^n)}}{\|\tilde{f}\|_{L^p(\mathbb{R}^n)}} = A,$$

which completes the proof. □

REMARK 2. If a modified form was as follows:

$$\tilde{\mathbb{H}}_\beta f(x) = \frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}} \int_{|y|<|x|} f(y)dy,\ x \in \mathbb{R}^n \setminus \{0\},$$

where $0 < \beta < n$ and f is a locally integrable function on $\mathbb{R}^n$. In view of $|\tilde{\mathbb{H}}_\beta f| \leqslant \mathbb{H}_\beta(|f|)$ and Lemma 1, Theorem 1 still holds for the operator $\tilde{\mathbb{H}}_\beta$.

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