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ASYMPTOTIC EXPANSIONS OF INTEGRAL MEAN OF POLYGAMMA FUNCTIONS

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Abstract. Let s,t be two given real numbers, $s \neq t$ and $m \in \mathbb{N}$. We determine the coefficients $a_j(s,t)$ in the asymptotic expansion of integral (or differential) mean of polygamma functions $\psi^{(m)}(x)$:

$$\frac{1}{t-s}\int_s^t \psi^{(m)}(x+u)\,\mathrm{d}u \sim \psi^{(m)}\left(x\sum_{j=0}^{\infty}\frac{a_j(s,t)}{x^j}\right), \qquad x \to \infty.$$

We derive the recursive relations for polynomials $a_j(t,s)$, and also as polynomials in intrinsic variables $\alpha = \frac{1}{2}(s+t-1)$, $\beta = \frac{1}{4}[1-(t-s)^2]$. We derive also the main properties of these polynomials and as a consequence the asymptotic formula for shifted variables.

1. Introduction

Euler's gamma function:

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,\mathrm{d}t, \qquad x > 0$$

is one of the most important functions in mathematical analysis and its applications in various diverse areas. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{or} \quad \log\Gamma(x) = \int_1^x \psi(t)\,\mathrm{d}t$$

is known as the psi (or digamma) function. The successive derivatives of the psi function $\psi(x)$:

$$\psi^{(n)}(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}\{\psi(x)\}, \qquad n \in \mathbb{N} := \{1,2,3,\ldots\}$$

are called the polygamma functions.

In 1959, Gautschi [8] presented the remarkable inequality:

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp\big((1-s)\psi(n+1)\big) \tag{1.1}$$

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for $0 < s < 1$ and $n \in \mathbb{N}$. In 1983, Kershaw [10] gave the following closer bounds:

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s}, \tag{1.2}$$

$$\exp[(1-s)\psi(x+\sqrt{s})] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right] \tag{1.3}$$

for real $x > 0$ and $0 < s < 1$.

In 2005, Kershaw [11] proved a more general result:

$$\psi(x+\sqrt{st}) < \frac{1}{t-s}\ln\frac{\Gamma(x+t)}{\Gamma(x+s)} < \psi\left(x+\frac{s+t}{2}\right) \tag{1.4}$$

for $x \geqslant 0$ and $0 < s \leqslant t$, which includes the inequality (1.3) as its special cases, but the better bound was already proved in [6]:

$$\psi(x+I_\psi(s,t)) < \frac{1}{t-s}\ln\frac{\Gamma(x+t)}{\Gamma(x+s)} < \psi\left(x+\frac{s+t}{2}\right), \tag{1.5}$$

where

$$I_\psi(s,t) = \psi^{-1}\left(\frac{1}{t-s}\int_s^t \psi(u)du\right)$$

is integral ψ-mean of s and t, ψ^{-1} denotes the inverse function of ψ.

The reader can find further information on these inequalities in [12, 13, 15]. In [13] it is pointed out that the Gautschi's result and its generalization to real positive x and any positive parameter s is a very simple consequence of the mean value theorem. It is also observed that the lower bound of Gautschi is an old inequality due to Wendel [17].

In [7] it is proved that

$$I_\psi(x+s,x+t) - x \to \frac{s+t}{2} \quad \text{as} \quad x\to\infty,$$

which implies that

$$\frac{1}{t-s}\ln\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim \psi\left(x+\frac{s+t}{2}\right) \quad \text{as} \quad x\to\infty. \tag{1.6}$$

Stolarsky's mean $S_p(a,b)$ of two positive numbers a,b is defined in [16] for $a=b$ by $S_p(a,b) = a$ and for $a \neq b$ by

$$S_p(a,b) = \left(\frac{b^p-a^p}{p(b-a)}\right)^{1/(p-1)}, \quad p \neq 0,\ 1;$$

$$S_0(a,b) = \frac{b-a}{\ln b - \ln a} = L(a,b);$$

$$S_1(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} = I(a,b).$$

Clearly,

$$S_2(a,b) = A(a,b), \quad S_{-1}(a,b) = G(a,b).$$

Where A, G, L, I are arithmetic, geometric, logarithmic and identric means, respectively. It is known that $S_p(a,b)$ for $a \neq b$ is a strictly increasing function of p. Clearly, for $a \neq b$,

$$G(a,b) < L(a,b) < I(a,b) < A(a,b).$$

It was shown [7, Lemma 1] that for $s,t > 0$,

$$\psi(L(s,t)) < \frac{\log\Gamma(t) - \log\Gamma(s)}{t-s} < \psi(A(s,t)). \tag{1.7}$$

N. Batir [2, Theorem 2.7] established an extended form of (1.7) as follows:

$$\begin{aligned}(-1)^n\psi^{(n+1)}\left(\frac{x+y}{2}\right) &< \frac{(-1)^n[\psi^{(n)}(x) - \psi^{(n)}(y)]}{x-y} \\ &< (-1)^n\psi^{(n+1)}\left(S_{-(n+1)}(x,y)\right).\end{aligned} \tag{1.8}$$

where x and y are positive real numbers and n is a positive integer.

Recently, Burić and Elezović [3, Theorem 2.1] gave the following complete asymptotic expansion for the Wallis power function

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim \sum_{n=0}^{\infty} P_n(t,s)x^{-n+1}, \tag{1.9}$$

where $P_n(t,s)$ are polynomials of order n defined by

$$\begin{aligned}&P_0(t,s) = 1,\\ &P_n(t,s) = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\frac{B_{k+1}(t) - B_{k+1}(s)}{(k+1)(t-s)}P_{n-k}(t,s), \qquad n \in \mathbb{N}.\end{aligned} \tag{1.10}$$

Here $B_k(t)$ stands for the Bernoulli polynomials.

Polynomials $P_n(t,s)$ have complicated form so by the change of variables

$$\alpha = \frac{s+t-1}{2}, \quad \beta = \frac{1-(t-s)^2}{4},$$

authors in [3, Theorem 5.1] presented the following expansion for (1.9):

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim x + \sum_{n=0}^{\infty} Q_{n+1}(\alpha,\beta)\frac{1}{x^n}, \tag{1.11}$$

where $Q_n(\alpha,\beta)$ is a polynomial obtained from $P_n(t,s)$ and has much more natural form than $P_n(t,s)$. Moreover, the authors gave an efficient recursive formula for determining the coefficients $Q_n(\alpha,\beta)$ and finally derived closed form for polynomials $Q_n(\alpha,\beta)$.

Very recently, Chen *et al.* [5] extended the formula (1.6) and obtained full asymptotic expansion of the form

$$\frac{1}{t-s}\ln\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim \psi\left(x+\sum_{j=0}^{\infty}a_j(s,t)x^{-j}\right), \qquad x\to\infty. \tag{1.12}$$

Moreover, the authors gave a relation for successively determining the coefficients $a_j(s,t)$. This paper is a continuation of our earlier work [5], we extend the formula (1.12) and obtain full asymptotic expansion of the form

$$\frac{\psi^{(m-1)}(x+t)-\psi^{(m-1)}(x+s)}{t-s} \sim \psi^{(m)}\left(x+\sum_{j=0}^{\infty}\beta_j(s,t)x^{-j}\right) \tag{1.13}$$

for $x\to\infty$ and $m\in\mathbb{N}$ (see Theorem 2.1).

The following lemma is required in the sequel.

LEMMA 1.1. (see [9, 5]) *Let $a_0\neq 0$ and let $g(x)$ be a function with asymptotic expansion*

$$g(x)\sim\sum_{n=0}^{\infty}a_nx^{-n}, \qquad x\to\infty.$$

Then for all real r it holds

$$[g(x)]^r\sim\sum_{n=0}^{\infty}P_n(r)x^{-n}, \qquad x\to\infty,$$

where

$$\begin{aligned} P_0(r)&=a_0^r,\\ P_n(r)&=\frac{1}{na_0}\sum_{k=1}^{n}[k(1+r)-n]a_kP_{n-k}(r), \qquad n\in\mathbb{N}. \end{aligned} \tag{1.14}$$

2. Main Result

Let us prove the main theorem of this paper.

THEOREM 2.1. *Let $m\in\mathbb{N}$ and*

$$\frac{1}{t-s}\int_s^t\psi^{(m)}(x+u)\,\mathrm{d}u=\psi^{(m)}\left(x\sum_{j=0}^{\infty}\frac{a_j(s,t)}{x^j}\right), \quad x\to\infty. \tag{2.1}$$

Then the coefficients a_n can be calculated by following recursive formula:

$$a_0=1,$$

$$a_n = (-1)^{n+m}\frac{(n+m-1)!}{b_0 mn!}\Delta_n + \frac{1}{b_0 m}\sum_{j=0}^{n-1} b_{n-j}P_j(-n+j-m)$$
$$+\frac{1}{mn}\sum_{k=1}^{n-1}[k(1-m)-n]\,a_k P_{n-k}(-m), \qquad n \in \mathbb{N},$$

where

$$\Delta_n(t,s) = \frac{B_{n+1}(t)-B_{n+1}(s)}{(n+1)(t-s)},$$
$$b_k = (-1)^{k+m-1}\frac{(k+m-1)!}{k!}B_k,$$

and P_n are connected with (a_n) by (1.14).

Proof. The proof of this theorem is based on calculating asymptotic expansion of the left and right sides and then equating coefficients of equal powers of x. Asymptotic expansion of polygamma function is known:

$$\psi^{(m)}(x+t) \sim \sum_{n=0}^{\infty}(-1)^{m+n-1}\frac{(m+n-1)!}{n!}B_n(t)x^{-(n+m)}$$

and

$$\psi^{(m)}(x) \sim x^{-m}\sum_{k=0}^{\infty} b_k x^{-k}.$$

We can write (2.1) in the form

$$\frac{\psi^{(m-1)}(x+t)-\psi^{(m-1)}(x+s)}{t-s} = \psi^{(m)}\left(x\sum_{j=0}^{\infty}\frac{a_j}{x^j}\right),$$

wherefrom it follows:

$$\sum_{k=0}^{\infty} b_k\left(x\sum_{j=0}^{\infty}\frac{a_j}{x^j}\right)^{-k-m} = \sum_{j=1}^{\infty}(-1)^{j+m}\frac{(j+m-2)!}{j!}\frac{B_j(t)-B_j(s)}{t-s}x^{-j-m+1},$$

$$\sum_{k=0}^{\infty} b_k x^{-k-m}\sum_{j=0}^{\infty}P_j(-k-m)x^{-j}$$
$$= \sum_{j=0}^{\infty}(-1)^{j+m-1}\frac{(j+m-1)!}{j!}\frac{B_{j+1}(t)-B_{j+1}(s)}{(j+1)(t-s)}x^{-j-m},$$

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} b_{n-j}P_j(-n+j-m)x^{-n} = \sum_{n=0}^{\infty}(-1)^{n+m-1}\frac{(n+m-1)!}{n!}\Delta_n x^{-n},$$

$$\sum_{j=0}^{n} b_{n-j}P_j(-n+j-m) = (-1)^{n+m-1}\frac{(n+m-1)!}{n!}\Delta_n.$$

Coefficient a_n is contained in P_n. For $n=0$ we obtain $a_0=1$, and for $n\geqslant 1$ we have

$$\sum_{j=0}^{n-1} b_{n-j}P_j(-n+j-m) + b_0P_n(-m) = (-1)^{n+m-1}\frac{(n+m-1)!}{n!}\Delta_n$$

and

$$\begin{aligned}P_n(-m) &= \frac{1}{n}\sum_{k=1}^{n}[k(1-m)-n]a_kP_{n-k}(-m)\\ &= \frac{1}{n}\sum_{k=1}^{n-1}[k(1-m)-n]a_kP_{n-k}(-m) - ma_nP_0(-m).\end{aligned}$$

Combining last two equalities yields

$$\begin{aligned}\sum_{j=0}^{n-1} b_{n-j}P_j(-n+j-m) + \frac{b_0}{n}\sum_{k=1}^{n-1}[k(1-m)-n]a_kP_{n-k}(-m) - b_0ma_n\\ = (-1)^{n+m-1}\frac{(n+m-1)!}{n!}\Delta_n,\end{aligned}$$

whence

$$\begin{aligned}a_n = (-1)^{n+m}\frac{(n+m-1)!}{b_0mn!}\Delta_n + \frac{1}{b_0m}\sum_{j=0}^{n-1} b_{n-j}P_j(-n+j-m)\\ + \frac{1}{mn}\sum_{k=1}^{n-1}[k(1-m)-n]a_kP_{n-k}(-m),\end{aligned}$$

which had to be proved. □

The first few coefficients in this expansions are

$$\begin{aligned}a_0 &= 1,\\ a_1 &= \tfrac{1}{2}(s+t),\\ a_2 &= -\tfrac{1}{24}(m+1)(s-t)^2,\\ a_3 &= \tfrac{1}{48}(1+m)(s-t)^2(s+t-1),\\ a_4 &= \tfrac{1}{5760}(1+m)(s-t)^2(20+20m+120s-73s^2-5ms^2+2m^2s^2\\ &\quad +120t-94st+10mst-4m^2st-73t^2-5mt^2+2m^2t^2),\\ a_5 &= -\tfrac{1}{3840}(1+m)(s-t)^2(-1+s+t)(20+20m+40s-33s^2-5ms^2+2m^2s^2\\ &\quad +40t-14st+10mst-4m^2st-33t^2-5mt^2+2m^2t^2),\\ a_6 &= -\tfrac{1}{2903040}(1+m)(s-t)^2(-10248-3192m+3192m^2+168m^3+30240s\\ &\quad +30240ms+19404s^2-16128ms^2-2016m^2s^2+252m^3s^2\end{aligned}$$

$$
\begin{aligned}
&-49896s^3-7560ms^3+3024m^2s^3+18125s^4+4270ms^4-1629m^2s^4\\
&-70m^3s^4+16m^4s^4+30240t+30240mt+21672st-28224mst\\
&+4032m^2st-504m^3st-71064s^2t+7560ms^2t-3024m^2s^2t\\
&+27292s^3t-1960ms^3t+468m^2s^3t+280m^3s^3t-64m^4s^3t\\
&+19404t^2-16128mt^2-2016m^2t^2+252m^3t^2-71064st^2+7560mst^2\\
&-3024m^2st^2+30126s^2t^2-4620ms^2t^2+2322m^2s^2t^2-420m^3s^2t^2\\
&+96m^4s^2t^2-49896t^3-7560mt^3+3024m^2t^3+27292st^3-1960mst^3\\
&+468m^2st^3+280m^3st^3-64m^4st^3\\
&+18125t^4+4270mt^4-1629m^2t^4-70m^3t^4+16m^4t^4).
\end{aligned}
$$

The coefficients a_n will have much simpler form if we introduce variables $a=\frac{s+t}{2}$ and $b=\frac{t-s}{2}$. Let $b_k(a,b)=a_k(s,t)$. Then

$$
\begin{aligned}
b_0&=1,\\
b_1&=\alpha,\\
b_2&=-\tfrac{1}{6}(m+1)\beta^2,\\
b_3&=\tfrac{1}{12}(m+1)(2\alpha-1)\beta^2,\\
b_4&=\tfrac{1}{360}(m+1)(\beta^2(2m^2-5m-13)-60\alpha(\alpha-1)+5m-5)\beta^2,\\
b_5&=-\tfrac{1}{240}(m+1)(2\alpha-1)(\beta^2(2m^2-5m-13)-20\alpha(\alpha-1)+5m+5)\beta^2,\\
b_6&=-\tfrac{1}{90720}(m+1)\beta^2\Big(21(m^3+19m^2-19m-61)+7560\alpha(\alpha-1)(2\alpha^2-2\alpha-m-1)\\
&\quad+126\beta^2(m^3-8m^2-4m+17)-1512\alpha\beta^2(\alpha-1)(2m^2-5m-13)\\
&\qquad+2\beta^4(16m^4-70m^3-117m^2+490m+737)\Big).
\end{aligned}
$$

3. Asymptotic expansions using intrinsic variables

Coefficients of asymptotic expansion of polygamma functions are expressed in terms of Bernoulli polynomials. Integral mean of polygamma functions are in the same time differential mean:

$$
\frac{\psi^{(m-1)}(t)-\psi^{(m-1)}(s)}{t-s}\sim\psi^{(m)}\left(x\sum_{j=0}^{\infty}\frac{a_j(s,t)}{x^j}\right),\qquad x\to\infty. \tag{3.1}
$$

Therefore coefficients of these expansions will be connected with the difference

$$
\Delta_n(t,s)=\frac{B_{n+1}(t)-B_{n+1}(s)}{(n+1)(t-s)} \tag{3.2}
$$

as explained in [3]. This function has natural expression through inner variables obtained through substitution:

$$\alpha=\frac{1}{2}(s+t-1), \qquad \beta=\frac{1}{4}[1-(s-t)^2]. \tag{3.3}$$

In this section we will analyse coefficients in the expansion (3.1) through variables α and β For the convenience of the reader we give here the first few coefficients from the table above. Denoting $c_k(\alpha,\beta)=a_k(s,t)$ we have

$$\begin{aligned}
c_0 &= 1,\\
c_1 &= \tfrac{1}{2}+\alpha,\\
c_2 &= \tfrac{1}{24}(m+1)(4\beta-1),\\
c_3 &= -\tfrac{1}{24}(m+1)\alpha(4\beta-1),\\
c_4 &= \tfrac{1}{5760}(m+1)(4\beta-1)(-27-15m-2m^2+240\alpha^2+4\beta(2m^2-5m-13)),\\
c_5 &= -\tfrac{1}{1920}(1+m)\alpha(4\beta-1)(-27-15m-2m^2+80\alpha^2+4\beta(2m^2-5m-13)),\\
c_6 &= \tfrac{1}{2903040}(1+m)(4\beta-1)(7625+7630m+2571m^2+350m^3+16m^4\\
&\quad -81648\alpha^2-45360m\alpha^2-6048m^2\alpha^2+120960\alpha^4+16280\beta+15232m\beta\\
&\quad +2952m^2\beta-448m^3\beta-128m^4\beta-157248\alpha^2\beta-60480m\alpha^2\beta\\
&\quad +24192m^2\alpha^2\beta+11792\beta^2+7840m\beta^2-1872m^2\beta^2-1120m^3\beta^2+256m^4\beta^2).
\end{aligned}$$

Denote

$$G(x)\sim\sum_{n=0}^{\infty}c_n(\alpha,\beta)x^{-n+1}. \tag{3.4}$$

Then (2.1) reads as

$$H(x,\alpha,\beta):=\frac{1}{t-s}\int_s^t \psi^{(m)}(x+u)\,\mathrm{d}u=\psi^{(m)}(G(x)), \quad x\to\infty. \tag{3.5}$$

THEOREM 3.1. *It holds*

1.

$$\frac{\partial c_n(\alpha,\beta)}{\partial\alpha}=-(n-2)c_{n-1}(\alpha,\beta),\ \textit{for}\ n\geqslant 2. \tag{3.6}$$

2.

$$c_n(\alpha,\beta)=\sum_{k=2}^{n}(-1)^{n-k}\binom{n-2}{n-k}c_k(0,\beta)\alpha^{n-k},\ \textit{for}\ n\geqslant 2. \tag{3.7}$$

Proof. For proving (1) it suffices to see that the function H is of the form

$$\tilde{H}(x+\alpha,\beta)=\frac{1}{2\beta}\int_{x+s}^{x+t}\psi^{(m)}(u)\,\mathrm{d}u$$

since we have

$$x+s=x+\alpha+\frac{1}{2}-\sqrt{1-4\beta}, \qquad x+t=x+\alpha+\frac{1}{2}+\sqrt{1-4\beta}.$$

Hence it holds

$$\frac{\partial H}{\partial x}=\frac{\partial H}{\partial \alpha},$$

which implies

$$\frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty} c_k(\alpha,\beta)x^{-k+1}\right)=\frac{\partial}{\partial \alpha}\left(\sum_{k=0}^{\infty} c_k(\alpha,\beta)x^{-k+1}\right),$$

that is

$$\sum_{k=0}^{\infty} c_k(\alpha,\beta)(-k+1)x^{-k}=\sum_{k=0}^{\infty}\frac{\partial c_{k+1}(\alpha,\beta)}{\partial \alpha}x^{-k},$$

wherefrom follows (1).

Explicit formula for coefficient c_n easily follows from (1) and Taylor expansion of $c_n(\alpha,\beta)$:

$$\begin{aligned} c_n(\alpha,\beta)&=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial^k c_n(\alpha,\beta)}{\partial\alpha^k}\bigg|_{\alpha=0}\alpha^k\\ &=\sum_{k=0}^{n-2}\frac{(n-2)!}{j!(n-2-k)!}c_{n-k}(0,\beta)\alpha^k\\ &=\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}c_{n-k}(0,\beta)\alpha^k. \quad \square\end{aligned}$$

THEOREM 3.2. *Let $d_n(\beta)=c_n(0,\beta)$. Then function G has the following asymptotic expansion*

$$G(x)\sim(x+\alpha)+\tfrac{1}{2}+\sum_{k=1}^{\infty} d_{2k}(\beta)(x+\alpha)^{-2k+1}. \tag{3.8}$$

Coefficients d_n satsfy following recursive relation

$$\begin{aligned} &d_0(\beta)=1,\\ &d_n(\beta)=(-1)^{n+m}\frac{(n+m-1)!}{b_0mn!}\nabla_n(\beta)+\frac{1}{b_0m}\sum_{j=0}^{n-1} b_{n-j}P_j(-n+j-m)\\ &\qquad+\frac{1}{mn}\sum_{k=1}^{n-1}[k(1-m)-n]\,d_kP_{n-k}(-m), \qquad n\in\mathbb{N},\end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \nabla_{2n}(\beta)&=0,\\ \nabla_{2n+1}(\beta)&=\sum_{k=0}^{n+1}\binom{2n+2}{2k}\frac{B_{2k}(\frac{1}{2})}{2n+2}\beta^{n-2k}.\end{aligned}$$

Proof. Asymptotic expansion

$$G(x)\sim\sum_{n=0}^{\infty}d_n(\beta)(x+\alpha)^{-j+1}$$

of function G and recursive relations for coefficients are direct consequences of Theorem 2.1. It remains to show that d_{2k+1} for $k\geqslant 1$. Asymptotic expansion of polygamma function with shifted variable reads

$$\psi^{(m)}(x+r)\sim\sum_{n=0}^{\infty}(-1)^{m+n-1}\frac{(m+n-1)!}{n!}B_n(r)x^{-(n+m)}. \tag{3.10}$$

On the left side of (2.1) we have

$$\begin{aligned}\frac{\psi^{(m-1)}(x+t)-\psi^{(m-1)}(x+s)}{t-s}\\ &\sim\sum_{n=0}^{\infty}(-1)^{m+n}\frac{(m+n-2)!}{n!}\frac{B_n(t-\alpha)-B_n(s-\alpha)}{t-s}(x+\alpha)^{-(n+m-1)}\\ &\sim\sum_{n=0}^{\infty}k_{2n+1}(x+\alpha)^{-m-2n}. \end{aligned}\tag{3.11}$$

On the other side we use (3.10) with $r=\frac{1}{2}$

$$\begin{aligned}\psi^{(m)}(x+\tfrac{1}{2})&\sim\sum_{n=0}^{\infty}(-1)^{m+n-1}\frac{(m+n-1)!}{n!}B_n(\tfrac{1}{2})x^{-(n+m)}\\ &\sim\sum_{n=0}^{\infty}(-1)^{m-1}\frac{(m+2n-1)!}{(2n)!}B_{2n}(\tfrac{1}{2})x^{-(2n+m)}\\ &\sim\sum_{n=0}^{\infty}l_{2n}x^{-m-2n},\end{aligned}$$

which gives

$$\begin{aligned}\psi^{(m)}(G(x))&=\psi^{(m)}((G(x)-\tfrac{1}{2})+\tfrac{1}{2})\\ &\sim\sum_{n=0}^{\infty}l_{2n}\Big(d_0(\beta)(x+\alpha)+d_2(\beta)(x+\alpha)^{-1}\\ &\qquad+d_3(\beta)(x+\alpha)^{-2}+d_4(\beta)(x+\alpha)^{-3}+\cdots\Big)^{-m-2n}.\end{aligned}\tag{3.12}$$

Finally, equating (3.11) and (3.12) and including $\alpha=0$ yields

$$\sum_{n=0}^{\infty}k_{2n+1}x^{-2n}=\sum_{n=0}^{\infty}l_{2n}x^{-2n}\Big(d_0(\beta)+d_2(\beta)x^{-2}+d_3(\beta)x^{-3}+d_4(\beta)x^{-4}+\cdots\Big)^{-m-2n}.$$

Since there exist only even powers of x on the left side, Lemma 1.1 implies that d_3 must be equal to 0. The same conclusion follows by induction for any other d_{2k+1}, $k\geqslant 1$. $\square$

The first few coefficients d_n are:

$$\begin{aligned}
d_0 &= 1,\\
d_1 &= \frac{1}{2},\\
d_2 &= \frac{1}{24}(m+1)(4\beta-1),\\
d_4 &= -\frac{1}{5760}(m+1)(4\beta-1)(27+15m+2m^2+52\beta+20m\beta-8m^2\beta),\\
d_6 &= \frac{1}{2903040}(m+1)(4\beta-1)\big(16\beta^2(16m^4-70m^3-117m^2+490m+737)\\
&\quad -8\beta(16m^4+56m^3-369m^2-1094m-2035)\\
&\quad +(m+5)(16m^3+270m^2+1221m+1525)\big).
\end{aligned}$$

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