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ON A MORE ACCURATE MULTIDIMENSIONAL HILBERT–TYPE INEQUALITY WITH PARAMETERS

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Abstract. In this paper, by using the way of weight coefficients and technique of real analysis and complex analysis, a more accurate multidimensional discrete Hilbert-type inequality with a best possible constant factor and some parameters is given. The equivalent form, the operator expression with the norm are also considered.

1. Introduction

Assuming that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x),\ g(y)\geqslant 0$, $f\in L^p(\mathbf{R}_+)$, $g\in L^q(\mathbf{R}_+)$, $||f||_p=\{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}}>0$, $||g||_q>0$, we have the following Hardy-Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}||f||_p||g||_q, \tag{1}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$. If $a_m,b_n\geqslant 0$, $a=\{a_m\}_{m=1}^\infty\in l^p$, $b=\{b_n\}_{n=1}^\infty\in l^q$, $||a||_p=\{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}}>0$, $||b||_q>0$, then we have the following Hardy-Hilbert's inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$ (cf.[1]):

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (cf. [1], [2], [3], [4], [6], [7]).

In 1998, by introducing an independent parameter $\lambda\in(0,1]$, Yang [5] gave an extension of (1) for $p=q=2$. Following the results of [5], Yang [6] gave some best extensions of (1) and (2) as follows:

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If $\lambda_1,\lambda_2,\lambda \in \mathbf{R}$, $\lambda_1+\lambda_2=\lambda, k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1)=\int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}dt \in \mathbf{R}_+,$$

$\phi(x)=x^{p(1-\lambda_1)-1}$, $\psi(x)=x^{q(1-\lambda_2)-1}$, $f(x)$, $g(y)\geqslant 0$,

$$f\in L_{p,\phi}(\mathbf{R}_+)=\left\{f;||f||_{p,\phi}:=\{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}}<\infty\right\},$$

$g\in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}$, $||g||_{q,\psi}>0$, then

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy<k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x>0(y>0)$, then for $a_m,b_n\geqslant 0$,

$$a\in l_{p,\phi}=\left\{a;||a||_{p,\phi}:=\{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}}<\infty\right\},$$

$b=\{b_n\}_{n=1}^\infty\in l_{q,\psi}$, $||a||_{p,\phi}$, $||b||_{q,\psi}>0$, it follows

$$\sum_{m=1}^\infty\sum_{n=1}^\infty k_\lambda(m,n)a_m b_n<k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{4}$$

where, the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda=1$, $k_1(x,y)=\frac{1}{x+y}$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$, inequality (3) reduces to (1), while (4) reduces to (2). Some other results including the multidimensional Hilbert-type integral inequalities are provided by [8]–[22].

About half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the the constant factors are the best possible. However, Yang [23] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ $(0<\lambda\leqslant 2)$ by introducing a variable and proved that the constant factor is the best possible. In 2011 Yang [24] gave the following half-discrete Hardy-Hilbert's inequality with the best possible constant factor $B(\lambda_1,\lambda_2)$:

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx<B(\lambda_1,\lambda_2)||f||_{p,\phi}||a||_{q,\psi}, \tag{5}$$

where, $\lambda_1>0$, $0<\lambda_2\leqslant 1$, $\lambda_1+\lambda_2=\lambda$,

$$B(u,v)=\int_0^\infty \frac{1}{(1+t)^{u+v}}t^{u-1}dt\,(u,v>0)$$

is the beta function. Zhong et *al* ([25]–[17]) investigated several half-discrete Hilbert-type inequalities with particular kernels.

Using the way of weight functions and the techniques of discrete and integral Hilbert-type inequalities with some additional conditions on the kernel, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ is obtained as follows:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} k_{\lambda}(x,n) a_n dx < k(\lambda_1) ||f||_{p,\phi} ||a||_{q,\psi}, \tag{6}$$

which is an extension of (5) (see Yang and Chen [33]). At the same time, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [34].

In this paper, by using the way of weight coefficients and technique of real analysis, a multidimensional discrete Hilbert's inequality with parameters and a best possible constant factor is given, which is an extension of (4) for $k_{\lambda}(m,n) = \frac{1}{\prod_{k=1}^{s}(m^{\lambda/s} + c_k n^{\lambda/s})}$. The equivalent form, the operator expression with the norm are also considered.

2. Some lemmas

If $i_0, j_0 \in \mathbf{N}$($\mathbf{N}$ is the set of positive integers), $\alpha,\ \beta > 0,$ we put

$$||x||_{\alpha} := \left(\sum_{k=1}^{i_0} |x_k|^{\alpha} \right)^{\frac{1}{\alpha}} \ (x = (x_1, \cdots, x_{i_0}) \in \mathbf{R}^{i_0}), \tag{7}$$

$$||y||_{\beta} := \left(\sum_{k=1}^{j_0} |y_k|^{\beta} \right)^{\frac{1}{\beta}} \ (y = (y_1, \cdots, y_{j_0}) \in \mathbf{R}^{j_0}). \tag{8}$$

LEMMA 1. *If* $s \in \mathbf{N},\ \gamma,\ M > 0,\ \Psi(u)$ *is a non-negative measurable function in* $(0,1]$, *and*

$$D_M := \left\{ x \in \mathbf{R}_+^s; \sum_{i=1}^{s} x_i^{\gamma} \leqslant M^{\gamma} \right\},$$

then we have (cf. [35]*)*

$$\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{s} \left(\frac{x_i}{M} \right)^{\gamma} \right) dx_1 \cdots dx_s$$
$$= \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{9}$$

LEMMA 2. *If* $s \in \mathbf{N}, \gamma > 0,\ \varepsilon > 0,\ c = (c_1, \cdots, c_s) \in [0,1)^s$, *then we have*

$$\sum_{m} ||m-c||_{\gamma}^{-s-\varepsilon} = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} + O(1)(\varepsilon \to 0^+). \tag{10}$$

Proof. For $M > s^{1/\gamma}$, we set

$$\Psi(u) = \begin{cases} 0, 0 < u < \frac{s}{M^\gamma}, \\ (Mu^{1/\gamma})^{-s-\varepsilon}, \frac{s}{M^\gamma} \leqslant u \leqslant 1. \end{cases}$$

Then by (9), it follows

$$\begin{aligned} \sum_m ||m-c||_\gamma^{-s-\varepsilon} &\geqslant \int_{\{x\in \mathbf{R}_+^s; x_i \geqslant 1+c_i\}} ||x-c||_\gamma^{-s-\varepsilon} dx \\ &= \int_{\{u\in \mathbf{R}_+^s; u_i \geqslant 1\}} ||u||_\gamma^{-s-\varepsilon} du \\ &= \lim_{M\to\infty} \int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s \\ &= \lim_{M\to\infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{s/M^\gamma}^1 (Mu^{1/\gamma})^{-s-\varepsilon} u^{\frac{s}{\gamma}-1} du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})}. \end{aligned}$$

For $s = 1$, it follows $0 < \sum_{m=1}^2 ||m-c||_\gamma^{-1-\varepsilon} < \infty$; for $s \geqslant 2$,

$$\begin{aligned} 0 &< \sum_{\{m\in \mathbf{N}^s; \exists i_0, m_{i_0} = 1,2\}} ||m-c||_\gamma^{-s-\varepsilon} \leqslant a + \sum_{\{m\in \mathbf{N}^{s-1}; m_i \geqslant 3\}} ||m-c||_\gamma^{-(s-1)-(1+\varepsilon)} \\ &\leqslant a + \frac{\Gamma^{s-1}(\frac{1}{\gamma})}{(1+\varepsilon)(s-1)^{(1+\varepsilon)/\gamma} \gamma^{s-2} \Gamma(\frac{s-1}{\gamma})} < \infty (a \in \mathbf{R}_+). \end{aligned}$$

Then we have

$$\begin{aligned} 0 &< \sum_{\{m\in \mathbf{N}^s; m_i \geqslant 1\}} ||m-c||_\gamma^{-s-\varepsilon} \\ &= \sum_{\{m\in \mathbf{N}^s; \exists i_0, m_{i_0} = 1,2\}} ||m-c||_\gamma^{-s-\varepsilon} + \sum_{\{m\in \mathbf{N}^s; m_i \geqslant 3\}} ||m-c||_\gamma^{-s-\varepsilon} \\ &\leqslant \widetilde{O}(1) + \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} (\varepsilon \to 0^+). \end{aligned}$$

Hence we have (10). □

LEMMA 3. *If* $\mathbf{C}$ *is the set of complex numbers and* $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, $z_k \in \mathbf{C} \backslash \{z | \mathrm{Re} z \geqslant 0,\ \mathrm{Im} z = 0\}$ $(k = 1, 2, \cdots, n)$ *are different points, the function* $f(z)$ *is analytic in* $\mathbf{C}_\infty$ *except for* $z_i (i = 1, 2, \cdots, n)$, *and* $z = \infty$ *is a zero point of* $f(z)$ *whose order is not less than 1, then for* $\alpha \in \mathbf{R}$, *we have*

$$\int_0^\infty f(x) x^{\alpha-1} dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}} \sum_{k=1}^n \mathrm{Res}[f(z) z^{\alpha-1}, z_k], \tag{11}$$

where, $0 < \operatorname{Im}\ln z = \arg z < 2\pi$. *In particular, if* z_k $(k = 1, \cdots, n)$ *are all poles of order 1, setting* $\varphi_k(z) = (z - z_k)f(z)(\varphi_k(z_k) \neq 0)$, *then*

$$\int_0^\infty f(x)x^{\alpha-1}dx = \frac{\pi}{\sin\pi\alpha}\sum_{k=1}^{n}(-z_k)^{\alpha-1}\varphi_k(z_k). \tag{12}$$

Proof. By [36] (P. 118), we have (11). We find

$$\begin{aligned} 1 - e^{2\pi\alpha i} &= 1 - \cos 2\pi\alpha - i\sin 2\pi\alpha \\ &= -2i\sin\pi\alpha(\cos\pi\alpha + i\sin\pi\alpha) = -2ie^{i\pi\alpha}\sin\pi\alpha. \end{aligned}$$

In particular, since $f(z)z^{\alpha-1} = \frac{1}{z-z_k}(\varphi_k(z)z^{\alpha-1})$, it is obvious that

$$\operatorname{Res}[f(z)z^{\alpha-1}, -a_k] = z_k{}^{\alpha-1}\varphi_k(z_k) = -e^{i\pi\alpha}(-z_k)^{\alpha-1}\varphi_k(z_k).$$

Then by (11), we obtain (12). □

EXAMPLE 1. For $s \in \mathbf{N}$, we set

$$k_\lambda(x,y) = \prod_{k=1}^{s}\frac{1}{(x^{\lambda/s} + c_k y^{\lambda/s})} \quad (0 < c_1 < \cdots < c_s, 0 < \lambda \leqslant s).$$

For $0 < \lambda_1 \leqslant i_0$, $0 < \lambda_2 \leqslant j_0$, $\lambda_1 + \lambda_2 = \lambda$, by (12), we find

$$\begin{aligned} k_s(\lambda_1) &:= \int_0^\infty \prod_{k=1}^{s}\frac{1}{t^{\lambda/s} + c_k}t^{\lambda_1 - 1}dt \\ &\overset{u=t^{\lambda/s}}{=} \frac{s}{\lambda}\int_0^\infty \prod_{k=1}^{s}\frac{1}{u + c_k}u^{\frac{s\lambda_1}{\lambda}-1}du \\ &= \frac{\pi s}{\lambda\sin(\frac{\pi s\lambda_1}{\lambda})}\sum_{k=1}^{s}c_k^{\frac{s\lambda_1}{\lambda}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{c_j - c_k} \in \mathbf{R}_+. \end{aligned} \tag{13}$$

In particular, for $s = 1$, we obtain

$$k_1(\lambda_1) = \frac{1}{\lambda}\int_0^\infty \frac{u^{(\lambda_1/\lambda)-1}}{u + c_1}du = \frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}c_1^{\frac{\lambda_1}{\lambda}-1}.$$

LEMMA 4. *If* $(-1)^i h^{(i)}(t) > 0$ $(t > 0;\ i = 1,2)$, *then for* $b > 0$, $0 < \alpha \leqslant 1$, *we have*

$$(-1)^i\frac{d^i}{dx^i}h((b + x^\alpha)^{\frac{1}{\alpha}}) > 0(x > 0; i = 1,2). \tag{14}$$

Proof. We find

$$\begin{aligned}
\frac{d}{dx}h((b+x^{\alpha})^{\frac{1}{\alpha}}) &= h'((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{1}{\alpha}-1}x^{\alpha-1} < 0,\\
\frac{d^2}{dx^2}h((b+x^{\alpha})^{\frac{1}{\alpha}}) &= \frac{d}{dx}[h'((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{1}{\alpha}-1}x^{\alpha-1}]\\
&= h''((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{2}{\alpha}-2}x^{2\alpha-2}\\
&\quad +\alpha\left(\frac{1}{\alpha}-1\right)h'((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{1}{\alpha}-2}x^{2\alpha-2}\\
&\quad +(\alpha-1)h'((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{1}{\alpha}-1}x^{\alpha-2}\\
&= h''((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{2}{\alpha}-2}x^{2\alpha-2}\\
&\quad +b(\alpha-1)h'((b+x^{\alpha})^{\frac{1}{\alpha}})(b+x^{\alpha})^{\frac{1}{\alpha}-2}x^{\alpha-2} > 0.
\end{aligned}$$

The lemma is proved. □

DEFINITION 1. For $s\in\mathbf{N}$, $0<\alpha,\beta\leqslant 1$, $0<c_1<\cdots<c_s$, $0<\lambda\leqslant s$, $0<\lambda_1\leqslant i_0$, $0<\lambda_2\leqslant j_0$, $\lambda_1+\lambda_2=\lambda$, $\tau=(\tau_1,\cdots,\tau_{i_0})\in(0,\frac{1}{2}]^{i_0}$, $\sigma=(\sigma_1,\cdots,\sigma_{j_0})\in(0,\frac{1}{2}]^{j_0}$, $m-\tau=(m_1-\tau_1,\cdots,m_{i_0}-\tau_{i_0})\in\mathbf{R}_+^{i_0}$, $n-\sigma=(n_1-\sigma_1,\cdots,n_{j_0}-\sigma_{j_0})\in\mathbf{R}_+^{j_0}$, define two weight coefficients $w_\lambda(\lambda_2,n)$ and $W_\lambda(\lambda_1,m)$ as follows:

$$w_\lambda(\lambda_2,n) := \sum_m \frac{||n-\sigma||_\beta^{\lambda_2}||m-\tau||_\alpha^{\lambda_1-i_0}}{\prod_{k=1}^s(||m-\tau||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})}, \tag{15}$$

$$W_\lambda(\lambda_1,m) := \sum_n \frac{||m-\tau||_\alpha^{\lambda_1}||n-\sigma||_\beta^{\lambda_2-j_0}}{\prod_{k=1}^s(||m-\tau||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})}, \tag{16}$$

where, $\sum_m=\sum_{m_{i_0}=1}^\infty\cdots\sum_{m_1=1}^\infty$ and $\sum_n=\sum_{n_{j_0}=1}^\infty\cdots\sum_{n_1=1}^\infty$.

LEMMA 5. *Let the assumptions as in Definition 1 are fulfilled. Then, we have (i)*

$$w_\lambda(\lambda_2,n) < K_2(n\in\mathbf{N}^{j_0}), \tag{17}$$
$$W_\lambda(\lambda_1,m) < K_1(m\in\mathbf{N}^{i_0}), \tag{18}$$

where,

$$K_1 := \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}k_s(\lambda_1), \qquad K_2 := \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k_s(\lambda_1), \tag{19}$$

where, $k_s(\lambda_1)$ is indicated by (13);

(ii) for $p>1$, $0<\varepsilon<p\min\{\lambda_1,1-\lambda_2\}$, setting $\widetilde{\lambda}_1=\lambda_1-\frac{\varepsilon}{p},\widetilde{\lambda}_2=\lambda_2+\frac{\varepsilon}{p}$, we have

$$0<\widetilde{K}_2(1-\widetilde{\theta}_\lambda(n))<w_\lambda(\widetilde{\lambda}_2,n), \tag{20}$$

where,

$$\widetilde{\theta}_\lambda(n) = \frac{1}{k_s(\widetilde{\lambda}_1)} \int_0^{i_0^{\lambda/(\alpha s)}/||n-\sigma||_\beta^{\lambda/s}} \frac{v^{\frac{s\lambda_1}{\lambda}-1}dv}{\prod_{k=1}^s (v+c_k)} = O\left(\frac{1}{||n-\sigma||_\beta^{\widetilde{\lambda}_1}}\right), \tag{21}$$

$$\widetilde{K}_2 = \frac{\Gamma^{i_0}(\frac{1}{\alpha})k_s(\widetilde{\lambda}_1)}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \in \mathbf{R}_+. \tag{22}$$

Proof. By Lemma 4, Hermite-Hadamard's inequality (cf. [37]), (9) and (13), it follows

$$\begin{aligned}
w_\lambda(\lambda_2,n) &< \int_{(\frac{1}{2},\infty)^{i_0}} \frac{||n-\sigma||_\beta^{\lambda_2}||x-\tau||_\alpha^{\lambda_1-i_0}}{\prod_{k=1}^s (||x-\tau||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})}dx \\
&= \int_{\{u\in\mathbf{R}_+^{i_0};u_i>\frac{1}{2}-\tau_i\}} \frac{||n-\sigma||_\beta^{\lambda_2}||u||_\alpha^{\lambda_1-i_0}}{\prod_{k=1}^s (||u||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})}du \\
&\leqslant \int_{\mathbf{R}_+^{i_0}} \frac{||n-\sigma||_\beta^{\lambda_2}||u||_\alpha^{\lambda_1-i_0}}{\prod_{k=1}^s (||u||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})}du \\
&= \lim_{M\to\infty} \int_{\mathbf{D}_M} \frac{||n-\sigma||_\beta^{\lambda_2}M^{\lambda_1-i_0}[\sum_{i=1}^{j_0}(\frac{u_i}{M})^\alpha]^{(\lambda_1-i_0)/\alpha}}{\prod_{k=1}^s \{M^{\frac{\lambda}{s}}[\sum_{i=1}^{i_0}(\frac{u_i}{M})^\alpha]^{\frac{\lambda}{\alpha s}}+c_k||n-\sigma||_\beta^{\frac{\lambda}{s}}\}}du \\
&= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{||n-\sigma||_\beta^{\lambda_2}M^{\lambda_1-i_0}t^{(\lambda_1-i_0)/\alpha}dt}{\prod_{k=1}^s (M^{\frac{\lambda}{s}}t^{\frac{\lambda}{\alpha s}}+c_k||n-\sigma||_\beta^{\frac{\lambda}{s}})} \\
&= \lim_{M\to\infty} \frac{M^{\lambda_1}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{||n-\sigma||_\beta^{\lambda_2}t^{\frac{\lambda_1}{\alpha}-1}dt}{\prod_{k=1}^s (M^{\frac{\lambda}{s}}t^{\frac{\lambda}{\alpha s}}+c_k||n-\sigma||_\beta^{\frac{\lambda}{s}})} \\
&\overset{t=||n-\sigma||_\beta^\alpha M^{-\alpha}v^{\alpha s/\lambda}}{=} \frac{s\Gamma^{i_0}(\frac{1}{\alpha})}{\lambda\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_0^\infty \frac{v^{\frac{s\lambda_1}{\lambda}-1}}{\prod_{k=1}^s (v+c_k)}dv \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k_s(\lambda_1) = K_2.
\end{aligned}$$

Hence, we have (17). By the same way, we have (18).

By the decreasing property and the same way of obtaining (10), we have

$$\begin{aligned}
w_\lambda(\widetilde{\lambda}_2,n) &> \int_{\{x\in\mathbf{R}_+^{i_0};x_i\geqslant 1+\tau_i\}} \frac{||n-\sigma||_\beta^{\widetilde{\lambda}_2}||x-\tau||_\alpha^{\widetilde{\lambda}_1-i_0}dx}{\prod_{k=1}^s (||x-\tau||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})} \\
&= ||n-\sigma||_\beta^{\widetilde{\lambda}_2} \int_{\{u\in\mathbf{R}_+^{i_0};u_i\geqslant 1\}} \frac{||u||_\alpha^{\widetilde{\lambda}_1-i_0}du}{\prod_{k=1}^s (||u||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{s\Gamma^{i_0}(\frac{1}{\alpha})}{\lambda\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\int_{i_0^{\lambda/(\alpha s)}/||n-\sigma||_\beta^{\lambda/s}}^{\infty}\frac{v^{\frac{s\lambda_1}{\lambda}-1}}{\prod_{k=1}^{s}(v+c_k)}dv \\
&= \widetilde{K}_2(1-\widetilde{\theta}_\lambda(n))>0, \\
0<\widetilde{\theta}_\lambda(n) &= \frac{s}{\lambda k_s(\widetilde{\lambda}_1)}\int_0^{i_0^{\lambda/(\alpha s)}/||n-\sigma||_\beta^{\lambda/s}}\frac{v^{\frac{s\lambda_1}{\lambda}-1}}{\prod_{k=1}^{s}(v+c_k)}dv \\
&\leqslant \frac{s}{\lambda k_s(\widetilde{\lambda}_1)\prod_{k=1}^{s}c_k}\int_0^{i_0^{\lambda/(\alpha s)}/||n-\sigma||_\beta^{\lambda/s}}v^{\frac{s\widetilde{\lambda}_1}{\lambda}-1}dv \\
&= \frac{1}{\widetilde{\lambda}_1 k_s(\widetilde{\lambda}_1)\prod_{k=1}^{s}c_k}\frac{i_0^{\widetilde{\lambda}_1/\alpha}}{||n-\sigma||_\beta^{\widetilde{\lambda}_1}}.
\end{aligned}
$$

The lemma is proved. □

3. Main results and operator expressions

Setting $\Phi(m) := ||m-\tau||_\alpha^{p(i_0-\lambda_1)-i_0}(m\in\mathbf{N}^{i_0})$ and $\Psi(n) := ||n-\sigma||_\beta^{q(j_0-\lambda_2)-j_0}$ $(n\in\mathbf{N}^{j_0})$, we have

THEOREM 1. *If* $s\in\mathbf{N}$, $0<\alpha,\ \beta\leqslant 1$, $0<c_1<\cdots<c_s$, $0<\lambda\leqslant s$, $0<\lambda_1\leqslant i_0$, $0<\lambda_2\leqslant j_0$, $\lambda_1+\lambda_2=\lambda$, $\tau\in(0,\frac{1}{2}]^{i_0}$, $\sigma\in(0,\frac{1}{2}]^{j_0}$, *then for* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m,b_n\geqslant 0$, $0<||a||_{p,\Phi}$, $||b||_{q,\Psi}<\infty$, *we have the following inequality*

$$
\begin{aligned}
I &:= \sum_n\sum_m\frac{a_mb_n}{\prod_{k=1}^{s}(||m-\tau||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})} \\
&< K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}||a||_{p,\Phi}||b||_{q,\Psi},
\end{aligned}
\tag{23}
$$

where the constant factor

$$
K_1^{\frac{1}{p}}K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}}k_s(\lambda_1) \tag{24}
$$

is the best possible ($k_s(\lambda_1)$ *is indicated by* (13)).

Proof. By Hölder's inequality (cf. [37]), we have

$$
\begin{aligned}
I = &\sum_n\sum_m\frac{1}{\prod_{k=1}^{s}(||m-\tau||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})} \\
&\times\left[\frac{||m-\tau||_\alpha^{(i_0-\lambda_1)/q}}{||n-\sigma||_\beta^{(j_0-\lambda_2)/p}}a_m\right]\left[\frac{||n-\sigma||_\beta^{(j_0-\lambda_2)/p}}{||m-\tau||_\alpha^{(i_0-\lambda_1)/q}}b_n\right]
\end{aligned}
$$

$$\leqslant \left\{\sum_m W_\lambda(\lambda_1,m)||m-\tau||_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p\right\}^{\frac{1}{p}} \times \left\{\sum_n w_\lambda(\lambda_2,n)||n-\sigma||_\beta^{q(j_0-\lambda_2)-j_0} b_n^q\right\}^{\frac{1}{q}}.$$

Then by (17) and (18), we have (23).

For $0<\varepsilon<p\min\{\lambda_1,1-\lambda_2\}$, $\widetilde{\lambda}_1=\lambda_1-\frac{\varepsilon}{p}, \widetilde{\lambda}_2=\lambda_2+\frac{\varepsilon}{p}$, we set

$$\widetilde{a}_m=||m-\tau||_\alpha^{-i_0+\lambda_1-\frac{\varepsilon}{p}},$$
$$\widetilde{b}_n=||n-\sigma||_\beta^{-j_0+\lambda_2-\frac{\varepsilon}{q}}(m\in\mathbf{N}^{i_0},n\in\mathbf{N}^{j_0}).$$

Then by (10) and (20), we obtain

$$\begin{aligned}
||\widetilde{a}||_{p,\Phi}||\widetilde{b}||_{q,\Psi} &= \left\{\sum_m ||m-\tau||_\alpha^{p(i_0-\lambda_1)-i_0}\widetilde{a}_m^p\right\}^{\frac{1}{p}}\left\{\sum_n ||n-\sigma||_\beta^{q(j_0-\lambda_2)-j_0}\widetilde{b}_n^q\right\}^{\frac{1}{q}} \\
&= \left\{\sum_m ||m-\tau||_\alpha^{-i_0-\varepsilon}\right\}^{\frac{1}{p}}\left\{\sum_n ||n-\sigma||_\beta^{-j_0-\varepsilon}\right\}^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}+\varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}, \quad (25)
\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &:= \sum_n\left[\sum_m \frac{\widetilde{a}_m}{\prod_{k=1}^s(||m-\tau||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})}\right]\widetilde{b}_n \\
&= \sum_n w_\lambda(\widetilde{\lambda}_2,n)||n-\sigma||_\beta^{-j_0-\varepsilon} \\
&> \widetilde{K}_2\sum_n\left(1-O(\frac{1}{||n-\sigma||_\beta^{\widetilde{\lambda}_1}})\right)||n-\sigma||_\beta^{-j_0-\varepsilon} \\
&= \widetilde{K}_2\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\widetilde{O}(1)-O(1)\right]. \quad (26)
\end{aligned}$$

If there exists a constant $K\leqslant K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$, such that (23) is valid as we replace $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$ by K, then we have

$$(K_2+o(1))\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\varepsilon\widetilde{O}(1)-\varepsilon O(1)\right]<\varepsilon\widetilde{I}$$

$$<\varepsilon K||\widetilde{a}||_{p,\varphi}||\widetilde{b}||_{q,\psi}=K\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}+\varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, we find

$$\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k_s(\lambda_1) \leqslant K\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{q}},$$

and then $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}} \leqslant K$. Hence, $K = K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$ is the best possible constant factor of (23). □

THEOREM 2. *With the assumptions of Theorem 1, for* $0 < ||a||_{p,\Phi} < \infty$, *we have the following inequality with the best constant factor* $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$:

$$J := \left\{\sum_n ||n-\sigma||_\beta^{p\lambda_2-j_0}\left(\sum_m \frac{a_m}{\prod_{k=1}^s(||m-\tau||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})}\right)^p\right\}^{\frac{1}{p}} < K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}||a||_{p,\Phi}, \tag{27}$$

which is equivalent to (23).

Proof. We set b_n as follows:

$$b_n := ||n-\sigma||_\beta^{p\lambda_2-j_0}\left(\sum_m \frac{a_m}{\prod_{k=1}^s(||m-\tau||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})}\right)^{p-1}.$$

Then it follows $J^p = ||b||_{q,\Psi}^q$. If $J = 0$, then (27) is trivially valid since $0 < ||a||_{p,\Phi} < \infty$; if $J = \infty$, then it is a contradiction since the right hand side of (27) is finite. Suppose that $0 < J < \infty$. Then by (23), we find

$$||b||_{q,\Psi}^q = J^p = I < K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}||a||_{p,\Phi}||b||_{q,\Psi},$$

namely,

$$||b||_{q,\Psi}^{q-1} = J < K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}||a||_{p,\Phi},$$

and then (27) follows.

On the other hand, assuming that (27) is valid, by Hölder's inequality, we have

$$I = \sum_n (\Psi(n))^{\frac{-1}{q}}\left[\sum_m \frac{a_m}{\prod_{k=1}^s(||m-\tau||_\alpha^{\lambda/s}+c_k||n-\sigma||_\beta^{\lambda/s})}\right][(\Psi(n))^{\frac{1}{q}}b_n] \leqslant J||b||_{q,\Psi}. \tag{28}$$

Then by (27), we have (23). Hence (27) and (23) are equivalent.

By the equivalency, the constant factor $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$ in (27) is the best possible. Otherwise, we would reach a contradiction by (28) that the constant factor $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$ in (23) is not the best possible. □

For $p > 1$, we define two real weight normal discrete spaces $l_{p,\varphi}$ and $l_{q,\psi}$ as follows:

$$l_{p,\varphi} := \left\{ a = \{a_m\}; ||a||_{p,\Phi} = \{\sum_m \Phi(m) a_m^p\}^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b = \{b_n\}; ||b||_{q,\Psi} = \{\sum_n \Psi(n) b_n^q\}^{\frac{1}{q}} < \infty \right\}.$$

With the assumptions of Theorem 1, in view of $J < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi}$, we have the following definition:

DEFINITION 2. Define a multidimensional Hilbert-type operator $T : l_{p,\Phi} \to l_{p,\Psi^{1-p}}$ as follows: For $a \in l_{p,\Phi}$, there exists an unique representation $Ta \in l_{p,\Psi^{1-p}}$, satisfying for $n \in \mathbf{N}^{j_0}$,

$$(Ta)(n) := \sum_m \frac{a_m}{\prod_{k=1}^s (||m-\tau||_\alpha^{\lambda/s} + c_k ||n-\sigma||_\beta^{\lambda/s})}. \tag{29}$$

For $b \in l_{q,\Psi}$, we define the following formal inner product of Ta and b as follows:

$$(Ta, b) := \sum_n \sum_m \frac{a_m b_n}{\prod_{k=1}^s (||m-\tau||_\alpha^{\lambda/s} + c_k ||n-\sigma||_\beta^{\lambda/s})}. \tag{30}$$

Then by Theorem 1 and Theorem 2, for $0 < ||a||_{p,\varphi}, ||b||_{q,\psi} < \infty$, we have the following equivalent inequalities:

$$(Ta, b) < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi} ||b||_{q,\Psi}, \tag{31}$$

$$||Ta||_{p,\Psi^{1-p}} < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi}. \tag{32}$$

It follows that T is bounded since

$$||T|| := \sup_{a(\neq\theta) \in l_{p,\Phi}} \frac{||Ta||_{p,\Psi^{1-p}}}{||a||_{p,\Phi}} \leqslant K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}. \tag{33}$$

Since the constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ in (32) is the best possible, we have

COROLLARY 1. *With the assumptions of Theorem 2, T is defined by Definition 2, it follows*

$$||T|| = K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k_s(\lambda_1). \tag{34}$$

REMARK 1. (i) Setting $\Phi_0(m) := ||m||_\alpha^{p(i_0-\lambda_1)-i_0} (m \in \mathbf{N}^{i_0})$ and

$$\Psi_0(n) := ||n||_\beta^{q(j_0-\lambda_2)-j_0} (n \in \mathbf{N}^{j_0}),$$

then putting $\tau = \sigma = 0$ in (23) and (27), we have the following equivalent inequalities with the best constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$:

$$\sum_n \sum_m \frac{a_m b_n}{\prod_{k=1}^{s}(||m||_\alpha^{\lambda/s} + c_k ||n||_\beta^{\lambda/s})} < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi_0} ||b||_{q,\Psi_0}, \tag{35}$$

$$\left\{ \sum_n ||n||_\beta^{p\lambda_2 - j_0} \left(\sum_m \frac{a_m}{\prod_{k=1}^{s}(||m||_\alpha^{\lambda/s} + c_k ||n||_\beta^{\lambda/s})} \right)^p \right\}^{\frac{1}{p}} < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi_0}. \tag{36}$$

Hence, (23) and (27) are more accurate inequalities of (35) and (36).

(ii) Putting $i_0 = j_0 = 1$ in (32), we have inequality

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{\prod_{k=1}^{s}(m^{\lambda/s} + c_k n^{\lambda/s})} < k_s(\lambda_1) ||a||_{p,\phi} ||b||_{q,\psi}. \tag{37}$$

Hence, (35) is an extension of (4) for

$$k_\lambda(m,n) = \frac{1}{\prod_{k=1}^{s}(m^{\lambda/s} + c_k n^{\lambda/s})}.$$

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