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LYAPUNOV–TYPE INEQUALITIES FOR A FRACTIONAL DIFFERENTIAL EQUATION WITH MIXED BOUNDARY CONDITIONS

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Abstract. Lyapunov-type inequalities are established for a fractional differential equation under mixed boundary conditions. Using such inequalities, we obtain intervals where certain Mittag-Leffler functions have no real zeros.

1. Introduction

Lyapunov's inequality [5] has proved useful in the study of spectral properties of ordinary differential equations. The Lyapunov's result can be stated as follows. If $q:[a,b]\to\mathbb{R}$ is a continuous function, then a necessary condition for the boundary value problem

$$\begin{aligned} &u''(t)+q(t)u(t)=0, \quad a<t<b\\ &u(a)=u(b)=0 \end{aligned}$$

to have nontrivial solutions is that

$$\int_a^b |q(t)|\,dt > \frac{4}{b-a}. \tag{1}$$

Since then many improvements of (1) have been developed and similar inequalities have been obtained for other types of differential equations, as the Pachpatte monograph on inequalities (see [8]) shows with detail.

The search for Lyapunov-type inequalities in which the differential equation depends on a fractional differential operator has begun very recently (see [2, 3]). In [2], a Lyapunov-type inequality was obtained for when the differential equation depends on the Riemann-Liouville fractional derivative, i.e. for the boundary value problem

$$\begin{aligned} ({}_aD^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2,\\ u(a)=u(b)=0, \end{aligned} \tag{2}$$

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where ${}_aD^\alpha$ denotes the Riemann-Liouville fractional derivative of order α and $q : [a,b] \to \mathbb{R}$ is a continuous function. In this case, the author in [2] proved that if (2) has a nontrivial solution, then we have

$$\int_a^b |q(t)|\,dt > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{3}$$

Clearly, if we let $\alpha = 2$ in (3), one obtains Lyapunov's classical inequality (1). Very recently, in [3], the same author considered a differential equation that depends on the Caputo fractional derivative. More precisely, he considered the fractional boundary value problem

$$\begin{gathered} ({}_a^CD^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\, 1 < \alpha \leqslant 2, \\ u(a) = u(b) = 0, \end{gathered} \tag{4}$$

where ${}_a^CD^\alpha$ is the Caputo fractional derivative of order α and $q : [a,b] \to \mathbb{R}$ is a continuous function. The author proved that if (4) has a nontrivial solution, then we have

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{5}$$

Similarly, if we let $\alpha = 2$ in (5), one obtains Lyapunov's classical inequality (1). In both works [2, 3], some interesting applications to the localization of real zeros of certain Mittag-Leffler functions were presented.

Motivated by the above cited works, we consider in this paper a fractional differential equation involving the Caputo fractional derivative of order α, $\alpha \in (0,2]$, under two types of mixed boundary conditions. More precisely, we consider the fractional differential equation

$$({}_a^CD^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\, 1 < \alpha \leqslant 2, \tag{6}$$

under the boundary conditions

$$u(a) = 0,\, u'(b) = 0 \tag{7}$$

or

$$u'(a) = 0,\, u(b) = 0. \tag{8}$$

For each type of boundary conditions, a Lyapunov-type inequality is established. The obtained results will be applied to obtain intervals where certain Mittag-Leffler functions have no real zeros.

2. Preliminaries

In this section, we recall the concepts of the Riemann-Liouville fractional integral and the Caputo fractional derivative of order $\alpha \geqslant 0$. For more details, we refer to [4].

DEFINITION 1. Let $\alpha \geqslant 0$ and f be a real function defined on $[a,b]$. The Riemann-Liouville fractional integral of order α is defined by $({}_aI^0 f) \equiv f$ and

$$({}_aI^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,ds, \quad \alpha > 0, t \in [a,b].$$

DEFINITION 2. The Caputo fractional derivative of order $\alpha \geqslant 0$ is defined by $({}_a^C D^0 f) \equiv f$ and $({}_a^C D^\alpha f)(t) = ({}_aI^{m-\alpha} D^m f)(t)$ for $\alpha > 0$, where m is the smallest integer greater or equal to α.

The following result is standard within the fractional calculus theory involving the Caputo differential operator (see [10]).

LEMMA 1. *$u \in C[a,b]$ is a solution to (6) if and only if*

$$u(t) = c_0 + c_1(t-a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)u(s)\,ds,$$

where c_0 and c_1 are some real constants.

3. Main results

3.1. A Lyapunov-type inequality for (6)–(7)

We start by writing problem (6)–(7) in its equivalent integral form.

LEMMA 2. *$u \in C[a,b]$ is a solution to (6)–(7) if and only if u satisfies the integral equation*

$$u(t) = \int_a^b G(t,s)q(s)u(s)\,ds,$$

where $G(t,s) = H(t,s)/\Gamma(\alpha)(b-s)^{2-\alpha}$ and $H(t,s)$ is given by

$$H(t,s) = \begin{cases} (\alpha-1)(t-a) - (t-s)^{\alpha-1}(b-s)^{2-\alpha}, & a \leqslant s \leqslant t \leqslant b, \\ (\alpha-1)(t-a), & a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{9}$$

Proof. From Lemma 1, we have

$$u(t) = c_0 + c_1(t-a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)u(s)\,ds,$$

where c_0 and c_1 are some real constants. Since $u(a)=0$, we get immediately that $c_0=0$. On the other hand, we have

$$u'(t)=c_1-\frac{1}{\Gamma(\alpha)}\int_a^t(\alpha-1)(t-s)^{\alpha-2}q(s)u(s)\,ds.$$

Since $u'(b)=0$, we obtain that

$$c_1=\frac{1}{\Gamma(\alpha)}\int_a^b(\alpha-1)(b-s)^{\alpha-2}q(s)u(s)\,ds.$$

Then, we get that

$$u(t)=\frac{(\alpha-1)(t-a)}{\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-2}q(s)u(s)\,ds-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}q(s)u(s)\,ds,$$

which concludes the proof. □

LEMMA 3. *For all* $(t,s)\in[a,b]\times[a,b]$, *we have the following properties:*

(i) *If* $a<t<s<b$ *then* $|H(t,s)|\leqslant(\alpha-1)(s-a)$;

(ii) *If* $a<s<t\leqslant b$ *then* $|H(t,s)|\leqslant\max\{(2-\alpha)(b-s),(\alpha-1)(s-a)\}$.

Proof. The proof of (i) follows immediately from the definition (9) of $H(t,s)$. Now, for a fixed $s\in(a,b)$, let us define the function

$$\varphi_s(t)=H(t,s)=(\alpha-1)(t-a)-(t-s)^{\alpha-1}(b-s)^{2-\alpha},\quad t\in(s,b].$$

An easy computation shows us that

$$\varphi_s'(t)=(\alpha-1)\left[1-\left(\frac{b-s}{t-s}\right)^{2-\alpha}\right]<0,\quad t\in(s,b).$$

On the other hand, we have

$$\lim_{t\to s^+}\varphi_s(t)=(\alpha-1)(s-a)\quad\text{and}\quad\varphi_s(b)=(\alpha-1)(b-a)-b+s.$$

Hence, for $a<s<t\leqslant b$, we have

$$|H(t,s)|\leqslant\max\{|\varphi_s(s)|,|\varphi_s(b)|\}.$$

However,

$$|\varphi_s(b))|=|(\alpha-2)(b-s)+(\alpha-1)(s-a)|\leqslant\max\{(2-\alpha)(b-s),(\alpha-1)(s-a)\},$$

since the two terms $(\alpha-2)(b-s)$, $(\alpha-1)(s-a)$ are of opposite sign. this gives (ii). □

From Lemma 3, we obtain immediately the following estimate.

LEMMA 4. *For all* $(t,s) \in [a,b] \times [a,b]$, *we have the following property:*

$$|H(t,s)| \leqslant \max\{\alpha - 1, 2 - \alpha\}(b-a).$$

Now, we are ready to prove our first Lyapunov-type inequality.

THEOREM 1. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned} &({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\, 1 < \alpha \leqslant 2, \\ &u(a) = u'(b) = 0, \end{aligned}$$

exists, where q *is a real and continuous function in* $[a,b]$, *then*

$$\int_a^b (b-s)^{\alpha-2}|q(s)|\,ds \geqslant \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}(b-a)}. \tag{10}$$

Proof. Let $B = C[a,b]$ be the Banach space endowed with norm

$$\|x\|_\infty = \max_{a \leqslant t \leqslant b} |x(t)|, \quad x \in B.$$

From Lemma 2, for all $t \in [a,b]$, we have

$$u(t) = \int_a^b G(t,s)q(s)u(s)\,ds.$$

From (9), we can write that

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-2} H(t,s) q(s) u(s)\,ds, \quad t \in [a,b].$$

Now, an application of Lemma 4 yields

$$\|u\|_\infty \leqslant \frac{\max\{\alpha-1, 2-\alpha\}(b-a)\|u\|_\infty}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-2}|q(s)|\,ds,$$

from which inequality in (10) follows. $\square$

The case $\alpha = 2$ can be deduced immediately form Theorem 1.

COROLLARY 1. *If a nontrivial continuous solution of the boundary value problem*

$$\begin{aligned} &u''(t) + q(t)u(t) = 0, \quad a < t < b, \\ &u(a) = u'(b) = 0, \end{aligned}$$

exists, where q *is a real and continuous function in* $[a,b]$, *then*

$$\int_a^b |q(s)|\,ds \geqslant \frac{1}{(b-a)}.$$

3.2. A Lyapunov-type inequality for (6)–(8)

The following Lemma gives us an equivalent integral form of (6)–(8). The proof is very similar to the proof of Lemma 2, so we omit it.

LEMMA 5. *$u \in C[a,b]$ is a solution to (6)–(8) if and only if u satisfies the integral equation*

$$u(t) = \int_a^b G(t,s)q(s)u(s)\,ds,$$

where $G(t,s)$ is given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \leqslant s \leqslant t \leqslant b, \\ (b-s)^{\alpha-1}, & a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{11}$$

The following estimate can be deduced immediately from the definition (11) of $G(t,s)$.

LEMMA 6. *For all $(t,s) \in [a,b] \times [a,b]$, we have the following property:*

$$0 \leqslant G(t,s) \leqslant \frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-1}.$$

Now, we got the necessary tools to prove our second Lyapunov-type inequality.

THEOREM 2. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned} &({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b, 1 < \alpha \leqslant 2, \\ &u'(a) = u(b) = 0, \end{aligned}$$

exists, where q is a real and continuous function in $[a,b]$, then

$$\int_a^b (b-s)^{\alpha-1}|q(s)|\,ds \geqslant \Gamma(\alpha)\cdot \tag{12}$$

Proof. From Lemma 5, for all $t \in [a,b]$, we have

$$u(t) = \int_a^b G(t,s)q(s)u(s)\,ds.$$

Now, an application of Lemma 6 yields

$$\|u\|_\infty \leqslant \|u\|_\infty \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1}|q(s)|\,ds,$$

from which inequality in (12) follows. □

The case $\alpha = 2$ can be deduced immediately form Theorem 2.

COROLLARY 2. *If a nontrivial continuous solution of the boundary value problem*

$$\begin{aligned} &u''(t)+q(t)u(t)=0, \quad a<t<b,\\ &u'(a)=u(b)=0, \end{aligned}$$

exists, where q *is a real and continuous function in* $[a,b]$, *then*

$$\int_a^b (b-s)|q(s)|\,ds \geqslant 1\cdot$$

4. Applications: Non-existence of real zeros of certain Mittag-Leffler functions

The Mittag-Leffler function with two parameters [6, 7, 9] is defined by the series expansion

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha>0\beta>0, \text{ and } z\in\mathbb{C},$$

which is analytic on the whole complex plane.

In this section, using our Lyapunov-type inequalities, we obtain intervals where certain Mittag-Leffler functions have no real zeros. We confine $z\in\mathbb{R}$ and consider the real zeros of the Mittag-Leffler functions $E_{\alpha,\beta}(z)$. Obviously $E_{\alpha,\beta}(z)>0$ for all $z\geqslant 0$. Thus, the real zeroes of $E_{\alpha,\beta}(z)$, if any, must be negative real numbers.

THEOREM 3. *Let* $1<\alpha\leqslant 2$. *Then, the Mittag-Leffler function* $E_{\alpha,1}(x)$ *has no real zeros for*

$$x\in\left(-\Gamma(\alpha)\frac{(\alpha-1)}{\max\{\alpha-1,2-\alpha\}},0\right]$$

Proof. Let $a=0$ and $b=1$, and consider the following fractional Sturm-Liouville eigenvalue problem

$$({}_0^C D^{\alpha}u)(t)+\lambda u(t)=0, \quad 0<t<1, \tag{13}$$

$$u(0)=u'(1)=0. \tag{14}$$

From [1], we know that the eigenvalues $\lambda\in\mathbb{R}$ of (13)–(14) are the solutions to

$$E_{\alpha,1}(-\lambda)=0, \tag{15}$$

and the corresponding eigenfunctions are given by

$$u(t)=tE_{\alpha,2}(-\lambda t^{\alpha}), \quad t\in[0,1].$$

By Theorem 1, if a real eigenvalue λ of (13)–(14) exists, i.e. $-\lambda$ is a zero of (15), then

$$\lambda\int_0^1(1-s)^{\alpha-2}\,ds\geqslant\frac{\Gamma(\alpha)}{\max\{\alpha-1,2-\alpha\}},$$

that is,

$$\lambda\geqslant\Gamma(\alpha)\frac{(\alpha-1)}{\max\{\alpha-1,2-\alpha\}},$$

which concludes the proof. □

THEOREM 4. *Let* $1 < \alpha \leqslant 2$. *Then, the Mittag-Leffler function* $E_{\alpha,1}(x)$ *has no real zeros for*

$$x \in (-\alpha\Gamma(\alpha), 0].$$

Proof. Let $a = 0$ and $b = 1$, and consider the following fractional Sturm-Liouville eigenvalue problem

$$({}_0^C D^{\alpha} u)(t) + \lambda u(t) = 0, \quad 0 < t < 1, \tag{16}$$

$$u'(0) = u(1) = 0. \tag{17}$$

From [1], we know that the eigenvalues $\lambda \in \mathbb{R}$ of (16)–(17) are the solutions to

$$E_{\alpha,1}(-\lambda) = 0, \tag{18}$$

and the corresponding eigenfunctions are given by

$$u(t) = E_{\alpha,1}(-\lambda t^{\alpha}), \quad t \in [0,1].$$

By Theorem 2, if a real eigenvalue λ of (16)–(17) exists, i.e. $-\lambda$ is a zero of (18), then

$$\lambda \int_0^1 (1-s)^{\alpha-1}\, ds \geqslant \Gamma(\alpha),$$

that is,

$$\lambda \geqslant \alpha\Gamma(\alpha),$$

which concludes the proof. □

Observe now that for $1 < \alpha \leqslant 2$, we have

$$\alpha \geqslant \frac{\alpha - 1}{\max\{\alpha - 1, 2 - \alpha\}},$$

which implies that

$$\left(-\Gamma(\alpha) \frac{(\alpha - 1)}{\max\{\alpha - 1, 2 - \alpha\}}, 0 \right] \subset (-\alpha\Gamma(\alpha), 0].$$

Then Theorem 4 is more general than Theorem 3.

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