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TRIGONOMETRIC INEQUALITIES IN THE MVBV CONDITION

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Abstract. This paper establishes some important trigonometric inequalities under the new *MVBV* real sense frame for future applications.

1. Introduction

A real sequence $A = \{a_n\}$ is said to satisfy the *mean value bounded variation condition* (in real sense) if there is a $\lambda \geqslant 2$ and a positive constant M depending upon the sequence A and λ only such that for all n we have

$$\sum_{k=n}^{2n} |\Delta a_k| := \sum_{k=n}^{2n} |a_k - a_{k+1}| \leqslant \frac{M}{n} \sum_{k=n/\lambda}^{\lambda n} |a_k|, \tag{1}$$

where $\sum\limits_{k=n/\lambda}^{\lambda n}$ means $\sum\limits_{n/\lambda \leqslant k \leqslant \lambda n}$.

We denote the set of real sequences satisfying (1) as MVBVS (Mean Value Bounded Variation Sequences)

The MVBV concept is generalized from positivity sense (see [9]) to real sense in [1]. It is not only an ultimate generalization to monotonicity (see [9]), but also a replacement of positivity in uniform convergence ([1]) and L^1-convergence ([2]). To reconsider and redesign the basic frame of some classical results in Fourier analysis, we need some fundamental inequalities and tools. The object of this paper is to establish some important inequalities in the new frame.

Throughout the paper, we always use M to stand for the positive constant appearing in (1), and M_1 denotes a positive constant that may not be necessarily the same at each occurrence. Sometimes, to avoid confusion, we also use $M_1, M_2, \cdots$ to denote different constants.

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2. The trigonometric inequality

We know, one important tool in Fourier analysis is the following well-known trigonometric inequality (see, e.g., [7])

$$\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| = O(1),$$

where and in the sequel, $O(1)$ always indicates a bound independent of n.

To generalize this inequality, we first establish some preliminary lemmas. For potential applications in future, we give generalized and standardized forms to these preliminaries.

If condition (1) is true for a λ then it is true for any larger λ, therefore we may assume that $\lambda > 8$ is an integer.

LEMMA 2.1. *Let a real sequence* $\{a_n\} \in \mathrm{MVBVS}$*, then*

$$n|a_n| \leqslant M_1 \sum_{k=n/\lambda}^{\lambda n} |a_k|.$$

See [1: Lemma 2.2].

Given a positive decreasing function $M(x)$ defined in $[1,\infty)$ satisfying $M(x)/M(2x) = O(1)$, define, for convenience, that

$$T_j^{(n)} = M(\lambda^{j+1}n)\sum_{l=\lambda^j n}^{\lambda^{j+2}n} |a_l|, \quad j,n \in \mathbb{N},$$

where $n \in \mathbb{N}$ means that n is a natural number.

LEMMA 2.2. *Let a real sequence* $\{a_n\}$ *satisfy* $a_n = O(1)$ *and*

$$\limsup_{n\to\infty} M(n)\sum_{k=n/\lambda}^{\lambda n} |a_k| = S,\ 0 < S \leqslant \infty,$$

then for any given $n_0 \in \mathbb{N}$*, there exists a subsequence of natural numbers* $\{j_k\}$ *such that*

$$T_{j_k}^{(n_0)} + T_{j_k+2}^{(n_0)} = O(T_{j_k+1}^{(n_0)})$$

and

$$\lim_{k\to\infty} T_{j_k+1}^{(n_0)} = S$$

hold.

Lemma 2.2 is proved in [2] for $M(x) = \log x/x$. The proof can be copied from the above reference with slight modifications.

LEMMA 2.3. *Under the conditions and symbols of Lemma* 2.2*, set* $n_k = \lambda^{j_k+2} n_0$*, and define*

$$A_{n_k}^{\alpha} = \left\{ l : |a_l| \geqslant \frac{1}{\alpha l} \sum_{j=n_k/\lambda}^{\lambda n_k} |a_j|,\ \ n_k/\lambda \leqslant l \leqslant \lambda n_k \right\}.$$

Then, taking a sufficiently large α_0*, there is a constant* $M_0 > 0$ *such that* $|A_{n_k}^{\alpha_0}| \geqslant M_0 n_k$*, where* $|A_{n_k}^{\alpha}|$ *indicates the number of the elements in* $A_{n_k}^{\alpha}$*.*

It is proved in [2].

THEOREM 2.4. *Let a real sequence* $\{a_n\} \in$ MVBVS*. Then, for all* n *and* $x \in [0,\pi]$*,*

$$\left| \sum_{k=1}^{n} a_k \sin kx \right| = O(1) \tag{2}$$

holds if and only if

$$n a_n = O(1). \tag{3}$$

Proof. Sufficiency. The technique for the proof is quite standard. The cases $x = 0$ or $x = \pi$ are trivial. Let $x \in (0,\pi)$, set $N = [1/x]$. Write

$$\sum_{k=1}^{n} a_k \sin kx = \sum_{k=1}^{N-1} a_k \sin kx + \sum_{k=N}^{n} a_k \sin kx =: I_1(x) + I_2(x).$$

For the first part, we have, by (3),

$$|I_1(x)| \leqslant x \sum_{k=1}^{N-1} k|a_k| \leqslant M_1 x(N-1) = O(1).$$

On the other hand, by using Abel's transformation we see that

$$|I_2(x)| \leqslant M_1 x^{-1} \left(|a_N| + |a_n| + \sum_{k=N}^{n-1} |\Delta a_k| \right).$$

Take an natural number m such that $2^{m-1}N < n \leqslant 2^m N$, we calculate that, by MVBV condition (1) and (3),

$$\sum_{k=N}^{n-1} |\Delta a_k| \leqslant M_1 \sum_{j=0}^{m} \frac{1}{2^j N} \sum_{l=2^j N/\lambda}^{\lambda 2^j N} |a_l| \leqslant M_1 N^{-1} \sum_{j=0}^{m} 2^{-j} \sum_{l=2^j N/\lambda}^{\lambda 2^j N} \frac{1}{l} \leqslant M_1 N^{-1}.$$

Thus

$$|I_2(x)| \leqslant x^{-1} N^{-1} (N|a_N| + n|a_n|) + M_1 \leqslant M_1,$$

that is the required result.

Necessity. Now note that $a_n = O(1)$ from (2). Take $M(x) \equiv 1$ in Lemma 2.2, and suppose that

$$\limsup_{n\to\infty} \sum_{k=n/\lambda}^{\lambda n} |a_k| = \infty. \tag{4}$$

With the same symbols of Lemmas 2.2 and 2.3 (e.g., $n_k = \lambda^{j_k+2} n_0$), applying Lemma 2.3, we have, there is a natural subsequence $\{n_k\}$ and a sufficiently large α_0 as well as a constant $M_0 > 0$ such that $|A_{n_k}^{\alpha_0}| \geqslant M_0 n_k$.

Set $m_1^{(n_k)} = \min A_{n_k}^{\alpha_0}$, then select ψ_1 according to the following procedure:

(i) If for $j = 0, 1, \cdots, j_0$, $n_k/\lambda \leqslant m_1^{(n_k)} + j \leqslant \lambda n_k$, the numbers $a_{m_1^{(n_k)}+j}$ have the same sign, and for $j = 0, 1, \cdots, j_0 - 1$, $|a_{m_1^{(n_k)}+j}| \geqslant \frac{T_{j_k+1}^{(n_0)}}{2\alpha_0(m_1^{(n_k)}+j)}$ while $|a_{m_1^{(n_k)}+j_0}| < \frac{T_{j_k+1}^{(n_0)}}{2\alpha_0(m_1^{(n_k)}+j_0)}$. Then let $\psi_1 = j_0$.

(ii) If Case (i) is not satisfied for any j_0, then let $\psi_1 = k_0$ for which $a_{m_1^{(n_k)}+k_0}$ is the first element with $m_1^{(n_k)} + k_0 \in [n_k/\lambda, \lambda n_k]$ to become zero or of opposite sign than $a_{m_1^{(n_k)}}$.

(iii) If neither (i) and (ii) happen, then simply let $\psi_1 = l_0$ for which $m_1^{(n_k)} + l_0$ is the first number greater than λn_k. Define

$$U_1 = \left\{ m_1^{(n_k)},\ m_1^{(n_k)} + 1, \cdots, m_1^{(n_k)} + \psi_1 - 1 \right\}.$$

Next set $m_2^{(n_k)} = \min(A_{n_k}^{\alpha_0} \setminus U_1)$ if the latter is not empty, by using the same procedure we select ψ_2 and define

$$U_2 = \left\{ m_2^{(n_k)},\ m_2^{(n_k)} + 1, \cdots, m_2^{(n_k)} + \psi_2 - 1 \right\}.$$

We continuing this procedure until reach a $U_{\kappa_{n_k}}$ for which $A_{n_k}^{\alpha_0} \setminus (U_1 \cup \cdots \cup U_{\kappa_{n_k}}) = \emptyset$.

Now we estimate κ_{n_k}, i.e. the number of these U_j's. Since for all $1 \leqslant j < \kappa_{n_k}$,

$$\sum_{l\in U_j} |\Delta a_l| \geqslant |a_{m_j^{(n_k)}} - a_{m_j^{(n_k)}+\psi_j}| \geqslant \frac{T_{j_k+1}^{(n_0)}}{2\alpha_0(m_j^{(n_k)} + \psi_j)}$$

and

$$\sum_{l=n/\lambda}^{\lambda n} |\Delta a_l| \leqslant \frac{M_1}{n} \sum_{l=n/\lambda^2}^{\lambda^2 n} |a_l|,$$

applying Lemma 2.2 we get

$$\frac{M_1}{n_k} T_{j_k+1}^{(n_0)} \geqslant \sum_{l=n_k/\lambda}^{\lambda n_k} |\Delta a_l| \geqslant \sum_{j=1}^{\kappa_{n_k}-1} \sum_{l\in U_j} |\Delta a_l| \geqslant \sum_{j=1}^{\kappa_{n_k}-1} \frac{T_{j_k+1}^{(n_0)}}{2\alpha_0(m_j^{(n_k)} + \psi_j)} \geqslant (\kappa_{n_k} - 1) \frac{M_2 T_{j_k+1}^{(n_0)}}{\alpha_0 n_k},$$

so that

$$\kappa_{n_k} \leqslant M_3\alpha_0. \tag{5}$$

By applying (2), we get, for $x_n = \pi/(4\lambda n)$,

$$M_1 \geqslant \left|\sum_{l\in U_j} a_l \sin l x_{n_k}\right| \geqslant M_4 \sum_{l\in U_j} |a_l| \geqslant \sum_{l\in U_j} \frac{M_5}{2\alpha_0 n_k} \sum_{i=n_k/\lambda}^{\lambda n_k} |a_i| \geqslant \frac{M_6\psi_j}{2\alpha_0 n_k} \sum_{i=n_k/\lambda}^{\lambda n_k} |a_i|,$$

hence

$$\psi_j \leqslant \frac{M_7\alpha_0 n_k}{\sum\limits_{i=n_k/\lambda}^{\lambda n_k} |a_i|},$$

consequently,

$$M_0 n_k \leqslant |A_{n_k}^{\alpha_0}| \leqslant \sum_{j=1}^{\kappa_{n_k}} |U_j| = \sum_{j=1}^{\kappa_{n_k}} \psi_j \leqslant \frac{M_8\alpha_0 n_k}{\sum\limits_{l=n_k/\lambda}^{\lambda n_k} |a_l|}.$$

Therefore,

$$\sum_{l=n_k/\lambda}^{\lambda n_k} |a_l| \leqslant \alpha_0 M_9.$$

But by applying Lemma 2.2 to (4) we already assume that

$$\lim_{k\to\infty} \sum_{j=n_k/\lambda}^{\lambda n_k} |a_j| = \infty,$$

thus a contradiction is made.

Finally, if $\limsup\limits_{n\to\infty} \sum\limits_{k=n/\lambda}^{\lambda n} |a_k| = O(1)$, by Lemma 2.1, we immediately have $na_n = O(1)$. □

EXAMPLE 1. Let $\{a_n\}$ be a real sequence satisfying (i) $\{|a_n|\}$ decreases; (ii) For any $2^n \leqslant k < 2^{n+1}$, the sign changes of $\{a_k\}$ are bounded independent of n; (iii) $na_n = O(1)$. Then

$$\left|\sum_{k=1}^{n} a_k \sin kx\right| = O(1).$$

EXAMPLE 2. Set

$$a_n = \begin{cases} \frac{(-1)^n}{n\log(n+1)}, & 2^k \leqslant n \leqslant 2^k + k, \\ \frac{1}{n}, & 2^k + k < n < 2^{k+1}, \end{cases} \quad k = 0, 1, \cdots,$$

Then $\{a_n\} \in \mathrm{MVBVS}$. Thus

$$\left|\sum_{k=1}^{n} a_k \sin kx\right| = O(1).$$

Theorems 2.4 can be considered as the ultimate generalization of the classical trigonometric inequality. As pointed in [6], even MVBV condition in positive sense cannot be weakened generally.

For the history of generalizations of the inequality, readers could check references [3–6].

3. The weighted trigonometric inequality

In this section, we investigate the weighted trigonometric inequality.

THEOREM 3.1. *Let a real sequence* $\{a_n\} \in \mathrm{MVBVS},\ 0<\gamma<1$. *Then, for any* n, *and* $x \in (0,\pi]$

$$\left|\sum_{k=1}^{n} a_k \sin kx\right| = O(x^{-\gamma})$$

holds if and only if

$$n^{1-\gamma} a_n = O(1).$$

REMARK. The statement of Theorem 3.1 also holds for cosine series.

Proof. In a similar manner to the proof of Theorem 2.4, we can derive the sufficiency. The only difference is to deal with the estimates as follows (keeping the same symbols as in the proof of Theorem 2.4):

$$|I_1(x)| \leqslant \sum_{k=1}^{N-1} k^{\gamma-1} k^{1-\gamma} |a_k| = O(N^{\gamma}) = O(x^{-\gamma}),$$

while

$$\begin{aligned}|I_2(x)| &\leqslant M_1 x^{-1}\left(N^{\gamma-1} + n^{\gamma-1} + N^{-1}\sum_{j=0}^{m} 2^{-j} \sum_{l=2^jN/\lambda}^{\lambda 2^j N} \frac{1}{l^{1-\gamma}}\right)\\ &= O\left(N^{\gamma}\sum_{j=0}^{\infty} 2^{-j(1-\gamma)}\right)\\ &= O(x^{-\gamma}).\end{aligned}$$

As to the necessity, we only need to set (keeping the same symbols as in the proof of Theorem 2.4, taking $M(x) = x^{-\gamma}$)

$$T_j^{(n)} = \frac{1}{(\lambda^{j+1}n)^{\gamma}} \sum_{l=\lambda^j n}^{\lambda^{j+2}n} |a_l|, \quad j,n \in \mathbb{N},$$

and note that

$$n_k^{-\gamma} \sum_{l\in U_j} |a_l| \leqslant M_1 |x_{n_k}|^{\gamma} \left|\sum_{l\in U_j} a_l \sin l x_{n_k}\right| = O(1)$$

for $x_{n_k} = \pi/(4\lambda n_k)$, and then make the necessary modifications to the proof of Theorem 2.4, following the same way, we can thus complete the proof of the necessity. We omit the details here. $\square$

This inequality generalizes the corresponding inequality in [8], and has some important applications which we shall publish in separate papers.

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