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SHARP CONSTANT FOR $L^p - L^\infty$ TYPE SOBOLEV'S INEQUALITY

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Abstract. Sharp constants for $L^p - L^\infty$ type Sobolev's inequalities $\|y\|_\infty \leqslant C\|\triangle^k y\|_p$ and $\|y\|_\infty \leqslant C\|\nabla(\triangle^k y)\|_p$ are studied. The problems are solved by using Green's function and rearrangement theory.

1. Introduction

Let $p>1$ be a real number and m be a positive integer. In this paper, we always denote by $q=\dfrac{p}{p-1}$ the conjugate number of p. Let $W(-1,1;m,p)$ be the Sobolev space defined by

$$W(-1,1;m,p):=\Big\{y\Big|y,\, y^{(m)}\in L^p(-1,1),\, y^{(2j)}(\pm 1)=0\ (0\leqslant 2j\leqslant m-1)\Big\}. \tag{1.1}$$

In [6], Y. Oshime, Y. Kametaka and H. Yamagishi considered the following problem:

PROBLEM (O). Find the best constant $C(m,p)$ such that the following Sobolev's inequality holds:

$$\sup_{x\in[-1,1]}|y(x)|\leqslant C(m,p)\Big(\int_{-1}^{1}|y^{(m)}(x)|^p\,dx\Big)^{\frac{1}{p}},\qquad \forall y(\cdot)\in W(-1,1;m,p). \tag{1.2}$$

They solved the problem for the even case $m=2k$ $(k=1,2,\ldots)$. Y. Oshime [4] considered high dimensional case for $m=1$ and got the corresponding result for Problem (O). Later, the best constant for the case $m=3$ was obtained by Y. Oshime and K. Watanabe [7]. For $m=1$ and high dimensional case, early in 1976, G. Talenti studied in [8] the best constant for

$$\Big(\int_{\mathbb{R}^n}|y(x)|^r\,dx\Big)^{\frac{1}{r}}\leqslant C\Big(\int_{\mathbb{R}^n}|\nabla y(x)|^p\,dx\Big)^{\frac{1}{p}} \tag{1.3}$$

with $r=\frac{np}{n-p}$ and $1<p<n$.

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In this paper, we will consider analogue problems in $\mathbb{R}^n$ for $n \geqslant 1$ by using Green's function and rearrangement theory. Problem (O) will be solved as a part of our results.

Let B_n be the unit ball in $\mathbb{R}^n$. We consider the following Poisson equation

$$\begin{cases} -\triangle u(x) = f(x),\ x \in B_n, \\ u|_{\partial B_n} = 0. \end{cases} \tag{1.4}$$

For $p > 1$ and $f \in L^p(B_n)$, the weak solution of (1.4) will be denoted as $(-\triangle)^{-1} f$. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with nonnegative integers $\alpha_1, \alpha_2, \dots, \alpha_n$, we denote

$$|\alpha| = \sum_{k=1}^{n} \alpha_k, \quad \partial^\alpha = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}. \tag{1.5}$$

Define

$$W(B_n; m, p) := \left\{ y \middle| \partial^\alpha y \in L^p(B_n), \triangle^j y|_{\partial B_n} = 0,\ 0 \leqslant |\alpha| \leqslant m,\ 0 \leqslant j \leqslant \frac{m-1}{2} \right\} \tag{1.6}$$

and

$$\|y\|_{m,p} = \begin{cases} \left(\int_{B_n} |\triangle^k y(x)|^p \, dx \right)^{\frac{1}{p}}, & \text{if } m = 2k, \\ \left(\int_{B_n} |\nabla(\triangle^k y(x))|^p \, dx \right)^{\frac{1}{p}}, & \text{if } m = 2k+1. \end{cases} \tag{1.7}$$

We mention that $W(B_n; 1, p)$ is just the Sobolev space $W_0^{1,p}(B_n)$.

Fix $X \in B_n$, by Sobolev's embedding theorem, we can consider the following problems:

PROBLEM (C_X). Let $p > \max(1, \frac{n}{m})$. Find the sharp constant $C_X(m, p, n)$ such that the following $L^p - L^\infty$ Sobolev's inequality holds:

$$|y(X)| \leqslant C_X(m, p, n) \|y\|_{m,p}, \qquad \forall y \in W(B_n; m, p). \tag{1.8}$$

PROBLEM (C). Let $p > \max(1, \frac{n}{m})$. Find the sharp constant $C(m, p, n)$ such that the following $L^p - L^\infty$ Sobolev's inequality holds:

$$\sup_{x \in B_n} |y(x)| \leqslant C(m, p, n) \|y\|_{m,p}, \qquad \forall y \in W(B_n; m, p). \tag{1.9}$$

Using Green's function, we can give a solution for Problem (C_X). Using Talenti's rearrangement theorem, we can easily show that

$$C(m, p, n) = C_0(m, p, n). \tag{1.10}$$

Then, we can solve Problem (C) completely.

2. Green's function and solution to Problem (C_X)

It is well known that $(-\triangle)^{-1}f$ can be expressed by Green's function

$$(-\triangle)^{-1}f = \int_{B_n} G(x,s;n)f(s)\,ds, \qquad x \in B_n, \tag{2.1}$$

where the Green's function is

$$G(x,s;n) = \begin{cases} \frac{1-sx-|x-s|}{2}, & \text{if } n=1, \\ \frac{1}{2\pi}\ln\frac{|s|\,|x-\frac{s}{|s|^2}|}{|x-s|}, & \text{if } n=2, \\ \frac{|x-s|^{2-n}-\left(|s|\,|x-\frac{s}{|s|^2}|\right)^{2-n}}{(n-2)n\omega_n}, & \text{if } n \geqslant 3, \end{cases} \quad x,s \in B_n \tag{2.2}$$

and ω_n denotes the volume of the unit ball B_n. In (2.2), we naturally read $|0|\left|x - \frac{0}{|0|^2}\right| = 1$.

Let $G_1(\cdot,\cdot;n) = G(\cdot,\cdot;n)$. Define

$$G_{j+1}(x,s;n) = \int_{B_n} G_1(x,\tau;n)G_j(\tau,s;n)\,d\tau, \qquad x,s \in B_n, \quad j=1,2,\ldots. \tag{2.3}$$

We have

$$G_j(x,s;n) = G_j(s,x;n) > 0, \qquad x,s \in B_n, \quad j=1,2,\ldots \tag{2.4}$$

and

$$G_{i+j}(x,s;n) = \int_{B_n} G_i(x,\tau;n)G_j(\tau,s;n)\,d\tau, \qquad x,s \in B_n, \quad i,j=1,2,\ldots. \tag{2.5}$$

For notation simplicity, we set

$$G_0(x,s;n) = \delta(x-s), \tag{2.6}$$

where $\delta(\cdot)$ is the Dirac distribution. We give our results for Problem (C_X).

PROPOSITION 2.1. *Let $n \geqslant 1$, $k \geqslant 1$, $m = 2k$, $\max(1, \frac{n}{2k}) < p < +\infty$ and $X \in B_n$. Then the inequality (1.8) holds with*

$$C_X(2k,p,n) = \|G_k(X,\cdot;n)\|_{L^q(B_n)}. \tag{2.7}$$

Moreover, for $y \in W(B_n;2k,p)$, the equality

$$|y(X)| = C_X(2k,p,n)\|y\|_{2k,p} \tag{2.8}$$

holds if and only if

$$(-\triangle)^k y(x) = \alpha G_k^{q-1}(X,x;n), \qquad x \in B_n \tag{2.9}$$

for some $\alpha \in \mathbb{R}$.

Proof. For $y \in W(B_n;2k,p)$, we have

$$y(x) = \int_{B_n} G_k(x,s;n)(-\triangle)^k y(s)\,ds. \tag{2.10}$$

If $n \leqslant 2$, it holds that $G(x,\cdot;n) \in L^\theta(B_n)$ for any $\theta > 1$. Therefore $G_k(x,\cdot;n) \in W^{2k-2,\theta}(B_n) \subset L^\theta(B_n)$. Then $G_k(x,\cdot;n) \in L^q(B_n)$, $G_k(x,\cdot;n)^{q-1} \in L^p(B_n)$.

If $n \geqslant 3$, then $G(x,\cdot;n) \in L^\theta(B_n)$ for any $1 < \theta < \frac{n}{n-2}$. Thus, $G_k(x,\cdot;n) \in W^{2k-2,\theta}(B_n)$. Noting that

$$\frac{1}{q} > \frac{n-2}{n} - \frac{2k-2}{n},$$

we get that $G_k(x,\cdot;n) \in L^q(B_n)$ by Sobolev's embedding theorem. Consequently, $G_k(x,\cdot;n)^{q-1} \in L^p(B_n)$.

Thus

$$y(X) = \int_{B_n} G_k(X,s;n)(-\triangle)^k y(s)\,ds \leqslant \|G_k(X,\cdot;n)\|_{L^q(B_n)}\|y\|_{2k,p} \tag{2.11}$$

and the equality holds if and only if (2.9) holds. □

PROPOSITION 2.2. *Let $n \geqslant 1$, $k \geqslant 0$, $m = 2k+1$, $\max(1,\frac{n}{2k+1}) < p < +\infty$ and $X \in B_n$. Then (1.8) holds with*

$$C_X(2k+1,p,n) = \left(\int_{B_n} \left|\nabla_x Z(x,X)\right|^p dx\right)^{\frac{1}{q}}, \tag{2.12}$$

where $Z(\cdot,X)$ is the weak solution of

$$\begin{cases} -\nabla_x \cdot \left(\left|\nabla_x Z(x,X)\right|^{p-2} \nabla_x Z(x,X)\right) = G_k(x,X;n),\ x \in B_n, \\ Z(x,X)|_{x\in\partial B_n} = 0. \end{cases} \tag{2.13}$$

Moreover, for $y \in W(B_n;2k+1,p)$, the equality

$$|y(X)| = C_X(2k+1,p,n)\|y\|_{2k+1,p} \tag{2.14}$$

holds if and only if

$$(-\triangle)^k y(x) = \alpha Z(x,X), \qquad x \in B_n \tag{2.15}$$

for some $\alpha \in \mathbb{R}$.

Proof. Let $Z(\cdot,X)$ be the weak solution of (2.13).

If $k = 0$, then $p > n$ and $W_0^{1,p}(B_n)$ is embedded in $C(\overline{B_n})$. Moreover, $Z(\cdot,X)$ is the unique solution of the following variational problem:

Minimize $\displaystyle\frac{1}{p}\int_{B_n} \left|\nabla Z(x)\right|^p dx - Z(X)$ over $W_0^{1,p}(B_n)$.

If $k \geqslant 1$, then $G_k(\cdot,X;n) \in L^q(B_n)$, $Z(\cdot,X)$ is the unique solution of the following variational problem:

Minimize $\frac{1}{p}\int_{B_n}\left|\nabla Z(x)\right|^p dx - \int_{B_n} Z(x)G_k(x,X;n)\,dx$ over $W_0^{1,p}(B_n)$.
Thus, $\nabla Z(\cdot,X) \in L^p(B_n)$. For $y \in W(B_n;2k+1,p)$, there is

$$\begin{aligned}
y(X) &= \int_{B_n} G_k(X,s;n)(-\triangle_s)^k y(s)\,ds \\
&= -\int_{B_n} \nabla_s\cdot\Big(\left|\nabla_s Z(s,X)\right|^{p-2}\nabla_s Z(s,X)\Big)(-\triangle_s)^k y(s)\,ds \\
&= \int_{B_n} \left|\nabla_s Z(s,X)\right|^{p-2}\nabla_s Z(s,X)\cdot\nabla_s\Big((-\triangle_s)^k y(s)\Big)\,ds \\
&\leqslant \Big(\int_{B_n}\left|\nabla_s Z(s,X)\right|^p ds\Big)^{\frac{1}{q}}\|y\|_{2k+1,p} \qquad (2.16)
\end{aligned}$$

and the equality holds if and only if

$$\nabla_x\Big((-\triangle_x)^k y(x)\Big) = \alpha\nabla_x Z(x,X), \qquad x\in B_n \tag{2.17}$$

for some $\alpha\in\mathbb{R}$.

Since $Z(\cdot,X),(-\triangle)^k y(\cdot)\in W_0^{1,p}(B_n)$, we get (2.15) by (2.17) and Sobolev's inequality.

On the other hand, if (2.15) holds for some $\alpha\in\mathbb{R}$, then (2.14) follows from (2.16) and (2.17). □

When $p=2$, using (2.5), we can simplify the above results.

COROLLARY 2.3. *Let $n\geqslant 1,\ m\geqslant 1,\ 2m>n,\ p=2$ and $X\in B_n$. Then* (1.8) *holds with*

$$C_X(m,2,n) = \sqrt{G_m(X,X;n)}. \tag{2.18}$$

Moreover, for $y\in W(B_n;m,2)$, the equality

$$|y(X)| = C_X(m,2,n)\|y\|_{m,2} \tag{2.19}$$

holds if and only if

$$y(x) = \alpha G_m(x,X;n), \qquad x\in B_n \tag{2.20}$$

for some $\alpha\in\mathbb{R}$.

3. Rearrangement and solution to Problem (C)

We can use Propositions 2.1–2.2 to calculate $C_X(m,p,n)$ easily in a numerical way. It is also reasonable to guess that

$$C(m,p,n) \equiv \max_{X\in B_n} C_X(m,p,n) = C_0(m,p,n). \tag{3.1}$$

Though (3.1) can be got easily in some special cases, it is not so easy to get directly from (2.6) and (2.12) in a general way. For one dimensional case, in [6], the authors proved that $C(2k,p,1)$ is attained for symmetric functions. Their proof crucially depends on that Green's functions $G_k(t,s;1)$ can be expressed by Bernoulli polynomials. In [7], the authors proved that $C(3,p,1)$ is also attained for symmetric functions by two methods. One is based on some kind of symmetrization of functions, and the another is based on Green's functions. These results shows that (3.1) holds for $n=1$ with $m=1,3$ or $m=2k$ $(k=1,2,\dots)$. However, methods used in [6] and [7] seem not easy to extend to other cases.

Thanks to the rearrangement theory, we can use Talenti's inequality to get (3.1).

Now, let us recall some basic results in rearrangement theory. For simplicity, we consider functions in B_n.

Let f be a real valued measurable function in B_n. The distribution function $\mu_f(\cdot)$ of f is defined by

$$\mu_f(\alpha) = \text{meas}\left\{x \in B_n \big| \, |f(x)| > \alpha\right\}, \qquad \forall \alpha \geqslant 0. \tag{3.2}$$

The spherical symmetric rearrangement of f is a spherical non-negative function f^* with the same distribution function as f.

It is easy to see that

$$\|f\|_{L^p(B_n)} = \|f^*\|_{L^p(B_n)}, \qquad 1 \leqslant p \leqslant +\infty. \tag{3.3}$$

There is Hardy-Littlewood's inequality.

LEMMA 3.1. *Let $f \in L^p(B_n)$ and $g \in L^q(B_n)$. Then,*

$$\int_{B_n} f(x)g(x)\,dx \leqslant \int_{B_n} f^*(x)g^*(x)\,dx. \tag{3.4}$$

One can find a proof of the above lemma in [1]. It is also well known that the following Pólya-Szegö's inequality holds.

LEMMA 3.2. *Let $1 \leqslant p \leqslant +\infty$ and $f \in W_0^{1,p}(B_n)$. Then*

$$\|\nabla f^*\|_{L^p(B_n)} \leqslant \|\nabla f\|_{L^p(B_n)}. \tag{3.5}$$

For a proof of Lemma 3.2, see [8] for example, see also [1]. Moreover, we have the following very important Talenti's inequality.

LEMMA 3.3. *Assume $p>1$. Let $f \in L^p(B_n)$ and f^* be the spherical symmetric rearrangement of f. Let $u=(-\triangle)^{-1}f$ and $v=(-\triangle)^{-1}f^*$. Then*

$$u^*(x) \leqslant v(x), \qquad \text{a.e. } x \in B_n. \tag{3.6}$$

The above lemma is a special case of Theorem 1 in [9]. According to Talenti [9], such a result was proved first by R. O'Neil in private communication. In fact, Theorem 1 of [9] contains results more general than Lemma 3.3. Though the result for $n=1$ was not stated in [9], it is valid and can be proven easily in a direct way.

COROLLARY 3.4. *Assume $p > 1$. Let $u_0 \in L^p(B_n)$ and $v_0 \equiv u_0^*$ be the spherical symmetric rearrangement of u_0. Let*

$$u_{j+1} = (-\triangle)^{-1} u_j, \quad v_{j+1} = (-\triangle)^{-1} v_j, \qquad j = 0, 1, 2, \ldots. \tag{3.7}$$

Then

$$u_k^*(x) \leqslant v_k(x), \qquad \text{a.e. } x \in B_n, k = 1, 2, \ldots. \tag{3.8}$$

Proof. If $k = 1$, the result is just what given by Lemma 3.3.

Now, let $k \geqslant 2$. By Lemma 3.3, we have

$$u_1^*(x) \leqslant v_1(x), \qquad \text{a.e. } x \in B_n. \tag{3.9}$$

Let

$$U_2 = (-\triangle)^{-1} u_1^*. \tag{3.10}$$

By the weak maximum principle for elliptic equations, we have

$$U_2(x) \leqslant v_2(x), \qquad \text{a.e. } x \in B_n. \tag{3.11}$$

Thus, using Lemma 3.3 again, we get

$$u_2^*(x) \leqslant U_2(x) \leqslant v_2(x), \qquad \text{a.e. } x \in B_n. \tag{3.12}$$

Then, by induction, (3.8) follows. □

Now, we turn to solve Problem (C).

By Lemmas 3.1–3.3 and Corollary 3.4, we can easily see that for Problem (C), the equality in (1.9) holds for some spherical symmetric function. Moreover, we will prove later that (3.1) holds.

Let us state the following lemma first. For notation simplicity, we denote by $\{v > \alpha\}$ the set $\{x \in B_n | v(x) > \alpha\}$, etc.

LEMMA 3.5. *Let $X \in B_n$, $X \neq 0$ and $v(\cdot) = G(\cdot, X; n)$. Then*

$$v^*(x) < G(x, 0; n), \qquad \text{a.e. } x \in B_n. \tag{3.13}$$

Proof. Denote by $\mu(\cdot)$ and $\nu(\cdot)$ the distribution functions of $v(\cdot)$ and $G(\cdot, 0; n)$ respectively. When $n = 1$, we get (3.13) since

$$v^*(x) = (1 - X^2) G(x, 0; 1). \tag{3.14}$$

Now, let $n \geqslant 2$. We have

$$\sup_{x \in B_n} v(x) = +\infty. \tag{3.15}$$

To get (3.13), we need only to prove that

$$\mu(\alpha) < \nu(\alpha), \qquad \forall \alpha > 0. \tag{3.16}$$

It is easy to see that

$$\nabla v(x) \neq 0, \qquad \forall x \neq X. \tag{3.17}$$

Consider the level set $\{v > \alpha\}$. Then the unit outer normal $\mathbf{n}$ to the boundary $\{v = \alpha\}$ at a point x is $-\dfrac{\nabla v(x)}{|\nabla v(x)|}$. Noting that

$$-\triangle v(x) = \delta(x - X), \tag{3.18}$$

it is easy to get that

$$\int_{\{v=\alpha\}} |\nabla v(x)|\, d\sigma = -\int_{\{v=\alpha\}} \frac{\partial v(x)}{\partial \mathbf{n}}\, d\sigma = \int_{\{v\geqslant\alpha\}} \delta(x - X)\, dx = 1. \tag{3.19}$$

On the other hand, we have from co-area formula (see [1], for example)

$$\mu(\alpha) = \int_{\{v\geqslant\alpha\}} dx = \int_{\alpha}^{+\infty} d\tau \int_{\{v=\tau\}} \frac{1}{|\nabla v(x)|}\, d\sigma. \tag{3.20}$$

Thus

$$-\mu'(\alpha) = \int_{\{v=\alpha\}} \frac{1}{|\nabla v(x)|}\, d\sigma. \tag{3.21}$$

Combing with (3.19), we get

$$-\mu'(\alpha) \geqslant \Big(\int_{\{v=\alpha\}} d\sigma\Big)^2 \tag{3.22}$$

and the equality holds if and only if

$$|\nabla v(x)| = C, \qquad \text{on } \{v = \alpha\}. \tag{3.23}$$

By isoperimetric inequality, we have

$$\int_{\{v=\alpha\}} d\sigma \geqslant n\omega_n \Big(\frac{1}{\omega_n} \int_{\{v\geqslant\alpha\}} dx\Big)^{\frac{n-1}{n}} \tag{3.24}$$

and the equality holds if and only if $\{v > \alpha\}$ is a ball.

When $X \neq 0$, for any $\alpha > 0$, it is easy to verify that neither (3.23) nor (3.24) holds. Thus,

$$-\mu'(\alpha) > \Big(n\omega_n^{\frac{1}{n}} \mu(\alpha)^{\frac{n-1}{n}}\Big)^2 = n^2 \omega_n^{\frac{2}{n}} \mu(\alpha)^{\frac{2(n-1)}{n}}. \tag{3.25}$$

Combining with $\mu(0) = \omega_n$, we get

$$\mu(\alpha) < \begin{cases} \pi e^{-4\pi\alpha}, & \text{if } n = 2, \\ \omega_n \Big(1 + n(n-2)\alpha\omega_n\Big)^{\frac{n}{2-n}}, & \text{if } n \geqslant 3, \end{cases} \qquad \forall \alpha > 0. \tag{3.26}$$

That is, (3.16) holds since the right hand term in (3.26) is just $v(\alpha)$. We get the proof. □

Similarly but a little more difficult, we have

LEMMA 3.6. *Assume* $p>n$. *For* $Y \in B_n$, *let* $Z(\cdot,Y)$ *be the weak solution of*

$$\begin{cases} -\nabla_x \cdot \left(\left|\nabla_x Z(x,Y)\right|^{p-2} \nabla_x Z(x,Y) \right) = \delta(x-Y),\ x\in B_n, \\ Z(x,Y)|_{x\in\partial B_n} = 0. \end{cases} \tag{3.27}$$

Let $X\in B_n$, $X\neq 0$ *and* $v(\cdot)=Z(\cdot,X)$. *Then*

$$v^*(x) < Z(x,0), \qquad \forall x\in B_n. \tag{3.28}$$

Proof. The proof is based on properties of p-harmonic functions (see [5], for example).

Choose $\gamma\in(0,1-\frac{n}{p})$. By Sobolev's embedding theorem, $W_0^{1,p}(B_n)$ can be embedded to $C^\gamma(\overline{B_n})$. Thus,

$$\left|\int_{B_n} \varphi(x)\delta(x-Y)\,dx\right| = |\varphi(Y)| \leqslant \|\varphi\|_{C(\overline{B_n})} \leqslant C\|\varphi\|_{W_0^{1,p}(B_n)}, \qquad \forall \varphi\in W_0^{1,p}(B_n)$$

for some constant $C>0$. This means that $\delta(\cdot-Y)\in W^{-1,q}(B_n)$. Thus, (3.27) admits a unique weak solution $Z(\cdot,X)\in W_0^{1,p}(B_n)$. We have $Z(\cdot,X)\in C^\gamma(\overline{B_n})$. Moreover, since $Z(\cdot,X)$ is p-harmonic in $B_n\setminus\{X\}$, it holds that $Z(\cdot,X)\in C^{1,\beta}(B_n\setminus\{X\})$ for some $\beta\in(0,1)$ (see [2], for example). By the weak maximum principle, one can easily see that

$$Z(x,X)>0, \qquad \forall x\in B_n. \tag{3.29}$$

Further, we can prove that $Z(\cdot,X)$ attains its maximum M_X only at the point X. Otherwise, if $Z(\cdot,X)$ attains its maximum at some point $x_0\neq X$, since $Z(\cdot,X)$ is p-harmonic in $B_n\setminus\{X\}$, the strong maximum principle implies $Z(\cdot,X)\equiv Z(x_0,X)$ in $B_n\setminus\{X\}$. This leads to a contradiction.

By Sard's theorem, for almost all (one dimensional) $\alpha\in(0,M_X)$, the set $\{v=\alpha\}$ contains no critical point of v. Then a similar discussion to the proof of Lemma 3.5 shows that the distribution function $\mu(\cdot)$ of $v(\cdot)$ satisfies

$$-\mu'(\alpha) \geqslant \left(n\omega_n^{\frac{1}{n}}\mu(\alpha)^{\frac{n-1}{n}}\right)^q, \qquad \text{a.e. } \alpha\in(0,M_X) \tag{3.30}$$

and the equality holds if and only if $\{v>\alpha\}$ is a ball and

$$|\nabla v(x)| = C, \qquad \text{on } \{v=\alpha\}. \tag{3.31}$$

We claim that for any $\alpha_0\in(0,M_X)$,

$$\left\{\alpha\in(0,\alpha_0) \,\middle|\, -\mu'(\alpha) > \left(n\omega_n^{\frac{1}{n}}\mu(\alpha)^{\frac{n-1}{n}}\right)^q\right\} \qquad \text{has positive measure.} \tag{3.32}$$

Otherwise, suppose that (3.32) fails for some $\alpha_0\in(0,M_X)$. Then, by continuity, for any $\alpha\in(0,\alpha_0)$, $\{v>\alpha\}$ is a ball and (3.31) holds. Then, using (3.31) and the strong maximum principle, it must hold that

$$\nabla v(x)\neq 0, \qquad \forall x\in\{0<v<\alpha_0\}. \tag{3.33}$$

Thus v is analytic in $\{0<v<\alpha_0\}$ (see [5], for example). Furthermore, we can see that $\{v=\alpha\}$ is a sphere for any $\alpha\in(0,\alpha_0)$.

Now, we want to prove that balls $\{v>\alpha\}_{\{0<\alpha<\alpha_0\}}$ are concentric. To this aim, let $\alpha\in(0,\alpha_0)$ and choose two points x_0,s_0 on the sphere $\{v=\alpha\}$. Consider

$$\frac{dx(t)}{dt}=-\frac{\nabla v(x(t))}{|\nabla v(x(t))|^2},\quad x(0)=x_0 \tag{3.34}$$

and

$$\frac{ds(t)}{dt}=-\frac{\nabla v(s(t))}{|\nabla v(s(t))|^2},\quad s(0)=s_0. \tag{3.35}$$

Then $x(\cdot)$ and $s(\cdot)$ are well defined on $[0,a)$ for some $a>0$. Moreover, we have

$$\frac{dv(x(t))}{dt}=\frac{dv(s(t))}{dt}=-1. \tag{3.36}$$

Therefore,

$$v(x(t))=v(s(t)),\qquad \forall t\in[0,a) \tag{3.37}$$

and consequently, by (3.31),

$$|\nabla v(x(t))|=|\nabla v(s(t))|,\qquad \forall t\in[0,a). \tag{3.38}$$

Let $P(t)$ be the center of $\{v>v(x(t))\}$. By (3.37) and the fact of $\{v=\alpha\}$ being a sphere, we have

$$|x(t)-P(t)|^2=|s(t)-P(t)|^2, \tag{3.39}$$

Then, it follows from (3.33) and (3.38) that there is an $\ell(t)>0$ such that

$$|x(t)-P(t)|=\ell(t)|\nabla v(x(t))|,\quad |s(t)-P(t)|=\ell(t)|\nabla v(s(t))|,\qquad \forall t\in[0,a). \tag{3.40}$$

One can see that $\nabla v(x(t))$ is perpendicular to the tangent plane to the surface $\{v=v(x(t))\}$. Thus, $\nabla v(x(t))$ is an inner normal to $\{v=v(x(t))\}$ at $x(t)$. Similarly, $\nabla v(s(t))$ is an inner normal to $\{v=v(x(t))\}$ at $s(t)$. Combining this fact with (3.40), we have

$$x(t)-P(t)=-\ell(t)\nabla v(x(t)),\quad s(t)-P(t)=-\ell(t)\nabla v(s(t)),\qquad \forall t\in[0,a). \tag{3.41}$$

Thus

$$P(t)=x(t)+\frac{|x(t)-s(t)|}{|\nabla v(x(t))-\nabla v(s(t))|}\nabla v(x(t)),\qquad \forall t\in[0,a). \tag{3.42}$$

Therefore, $P(t)$ is continuous on $[0,a)$ and continuously differentiable on $(0,a)$. By (3.34), (3.35) and (3.39), we have

$$\begin{aligned}&\left\langle x(t)-P(t),-\frac{\nabla v(x(t))}{|\nabla v(x(t))|^2}-P'(t)\right\rangle\\&=\left\langle s(t)-P(t),-\frac{\nabla v(s(t))}{|\nabla v(s(t))|^2}-P'(t)\right\rangle,\quad t\in(0,a).\end{aligned} \tag{3.43}$$

On the other hand,

$$\left\langle x(t)-P(t),-\frac{\nabla v(x(t))}{|\nabla v(x(t))|^2}\right\rangle=\ell(t)$$
$$=\left\langle s(t)-P(t),-\frac{\nabla v(s(t))}{|\nabla v(s(t))|^2}\right\rangle,\qquad t\in(0,a). \tag{3.44}$$

We get

$$\langle P'(t),x(t)-s(t)\rangle=0,\qquad t\in(0,a). \tag{3.45}$$

When s_0 runs over $\{v=\alpha\}$, $s(t)$ runs over $\{v=v(x(t))\}$. Thus,

$$\langle P'(t),x(t)-z\rangle=0,\quad \forall z\in\{v=v(x(t))\},\, t\in(0,a). \tag{3.46}$$

Therefore

$$P'(t)=0,\qquad \forall t\in(0,a). \tag{3.47}$$

This means that the balls $\{v>\eta\}$ are concentric for $\eta\in(v(x(a)),\alpha)$. Consequently, we get that all balls $\{v>\alpha\}$ $(0<\alpha<\alpha_0)$ are concentric. Moreover, since $\{v>0\}=B_n$ and v is continuous near ∂B_n, we see that the common center of $\{v>\alpha\}$ should be the original point. Thus,

$$v(x)=CZ(x,0),\qquad \forall x\in\{v=\alpha_0\} \tag{3.48}$$

for some constant C. By (3.27), for $\varphi(x)=v(x),CZ(x,0)$, it holds that

$$\begin{cases}-\nabla_x\cdot\left(\left|\nabla\varphi(x)\right|^{p-2}\nabla\varphi(x)\right)=0, & \text{in } \{0<v<\alpha_0\},\\ \varphi(x)|_{x\in\{v=0\}}=0,\quad \varphi(x)|_{x\in\{v=\alpha_0\}}=\alpha_0. & \end{cases} \tag{3.49}$$

Therefore,

$$v(x)=CZ(x,0),\qquad \forall x\in\{0\leqslant v\leqslant\alpha_0\}. \tag{3.50}$$

We claim that

$$v(x)=CZ(x,0),\qquad \forall x\neq X. \tag{3.51}$$

Otherwise, let $\alpha_1\in[\alpha_0,M_X)$ be the biggest α such that

$$v(x)=CZ(x,0),\qquad \forall x\in\{v=\alpha\}. \tag{3.52}$$

Then

$$\nabla v(x)\neq 0,\qquad \forall x\in\{v=\alpha_1\} \tag{3.53}$$

and consequently v is analytic in a neighborhood Ω of $\{v=\alpha_1\}$. This implies

$$v(x)=CZ(x,0),\qquad \forall x\in\Omega. \tag{3.54}$$

Contradicts to that α_1 is the biggest α satisfying (3.52). Therefore (3.51) holds. We get a contradiction since we assume $X\neq 0$.

Thus, (3.32) holds. Let ν be the distribution function of $Z(\cdot,0)$. Then, ν satisfies

$$-\nu'(\alpha)=\Big(N\omega_n^{\frac{1}{N}}\lambda(\alpha)^{\frac{n-1}{N}}\Big)^q,\qquad \nu(0)=\omega_n. \tag{3.55}$$

Combining (3.32) with $\nu(0)=\omega_n$, we have

$$\lambda(\alpha)<\nu(\alpha),\qquad \forall\,\alpha\in(0,M_X]. \tag{3.56}$$

This implies (3.28). □

Now, we can get a solution to Problem (C).

THEOREM 3.7. *Let $n\geqslant 1$, $m=1$, $n<p<+\infty$. Then (1.9) holds with*

$$C(1,p,n)=(n\omega_n)^{-\frac{1}{p}}\Big[(p-n)(q-1)\Big]^{-\frac{1}{q}}. \tag{3.57}$$

Moreover, for $y\in W(B_n;1,p)$, the equality

$$\max_{x\in B_n}|y(x)|=C(1,p,n)\|y\|_{1,p} \tag{3.58}$$

holds if and only if

$$y(x)=\alpha\Big(1-|x|^{(p-n)(q-1)}\Big),\qquad x\in B_n \tag{3.59}$$

for some $\alpha\in\mathbb{R}$.

Proof. By Proposition 2.2, we know that

$$C_X(1,p,n)=\Big(\int_{B_n}\Big|\nabla_x Z(x,X)\Big|^p dx\Big)^{\frac{1}{q}}\equiv\|\nabla Z(\cdot,X)\|^{q-1}_{L^p(B_n)}, \tag{3.60}$$

where $Z(\cdot,X)$ is the weak solution of

$$\begin{cases}-\nabla_x\cdot\Big(\Big|\nabla_x Z(x,X)\Big|^{p-2}\nabla_x Z(x,X)\Big)=\delta(x-X),\ x\in B_n,\\ Z(x,X)|_{x\in\partial B_n}=0.\end{cases} \tag{3.61}$$

By Lemma 3.6, if $X\neq 0$, $v(\cdot)=Z(\cdot,X)$ satisfies

$$v^*(x)<Z(x,0),\qquad \forall x\in B_n. \tag{3.62}$$

This implies

$$v^*(X)<Z(X,0). \tag{3.63}$$

On the other hand, by Lemma 3.2,

$$\|\nabla v^*\|_{L^p(B_n)}\leqslant\|\nabla v\|_{L^p(B_n)}. \tag{3.64}$$

Thus

$$C_X(1,p,n) = \frac{v(X)}{\|\nabla v\|_{L^p(B_n)}} < \frac{v^*(0)}{\|\nabla v^*\|_{L^p(B_n)}} \leqslant C_0(1,p,n). \tag{3.65}$$

Therefore

$$C(1,p,n) = C_0(1,p,n) \tag{3.66}$$

and the equality (3.58) holds if and only if

$$y(x) = \alpha Z(x,0), \qquad x \in B_n \tag{3.67}$$

for some $\alpha \in \mathbb{R}$. Finally, we get the proof by a direct calculation:

$$Z(x,0) = \frac{p-1}{(n-p)\left(n\omega_n\right)^{q-1}}\left(1-|x|^{(p-n)(q-1)}\right), \quad x \in B_n. \quad \square \tag{3.68}$$

THEOREM 3.8. *Let* $n \geqslant 1$, $k \geqslant 1$, $m = 2k$ *and* $\max(1, \frac{n}{2k}) < p < +\infty$. *Then* (1.9) *holds with*

$$C(2k,p,n) = \|G_k(\cdot,0;n)\|_{L^q(B_n)}. \tag{3.69}$$

Moreover, for $y \in W(B_n;2k,p)$, *the equality*

$$\max_{x\in B_n}|y(x)| = C(2k,p,n)\|y\|_{2k,p} \tag{3.70}$$

holds if and only if

$$(-\triangle)^k y(x) = \alpha G_k^{q-1}(x,0;n), \qquad x \in B_n \tag{3.71}$$

for some $\alpha \in \mathbb{R}$.

Proof. Let $X \in B_n$, $X \neq 0$ and

$$u(x) = G_k(x,X;n), \quad v(x) = G(x,X;n) \qquad x \in B_n. \tag{3.72}$$

By Lemma 3.5, we have

$$v^*(x) < G(x,0;n), \qquad x \in B_n. \tag{3.73}$$

Generally, we can prove that

$$u^*(x) < G_k(x,0;n), \qquad x \in B_n. \tag{3.74}$$

In fact, if $k = 1$, then (3.74) is just (3.73). If $k \geqslant 2$, by Corollary 3.4,

$$\begin{aligned} u^*(x) &\leqslant \int_{B_n} G_{k-1}(x,s;n)v^*(s)\,ds \\ &< \int_{B_n} G_{k-1}(x,s;n)G(s,0;n)\,ds = G_k(x,0;n), \qquad x \in B_n. \end{aligned} \tag{3.75}$$

Therefore, by Proposition 2.1,

$$C_X(2k,p,n) = \|u\|_{L^q(B_n)} = \|u^*\|_{L^q(B_n)} < \|G_k(\cdot,0;n)\|_{L^q(B_n)} = C_0(2k,p,n). \tag{3.76}$$

And (3.69)–(3.71) follows immediately from Proposition 2.1. $\square$

THEOREM 3.9. *Let* $n \geqslant 1$, $k \geqslant 1$, $m = 2k+1$ *and* $\max(1, \frac{n}{2k+1}) < p < +\infty$. *Then* (1.9) *holds with*

$$C(2k+1,p,n) = \Big(\int_{B_n} \Big|\nabla Y(x)\Big|^p dx\Big)^{\frac{1}{q}}, \tag{3.77}$$

where $Y(\cdot)$ *is the weak solution of*

$$\begin{cases} -\nabla\cdot\Big(\Big|\nabla Y(x)\Big|^{p-2}\nabla Y(x)\Big) = G_k(x,0;n),\ x \in B_n, \\ Y(x)|_{\partial B_n} = 0. \end{cases} \tag{3.78}$$

Moreover, for $y \in W(B_n;2k+1,p)$, *the equality*

$$\max_{x\in B_n}|y(x)| = C(2k+1,p,n)\|y\|_{2k+1,p} \tag{3.79}$$

holds if and only if

$$(-\triangle)^k y(x) = \alpha Y(x), \qquad x \in B_n \tag{3.80}$$

for some $\alpha \in \mathbb{R}$.

Proof. Let $X \in B_n$, $X \neq 0$ and $Z(\cdot,X)$ be defined by (2.13). Let

$$v(x) = Z(x,X),\ \hat{y}(x) = [(-\triangle)^{-1}]^{k-1}v(x),\ y(x) = [(-\triangle)^{-1}]^k v(x),\ x \in B_n. \tag{3.81}$$

By Corollary 3.4,

$$\hat{y}^*(x) \leqslant \int_{B_n} G_{k-1}(x,s;n)v^*(s)\,ds, \qquad x \in B_n. \tag{3.82}$$

Thus, by Lemmas 3.1 and 3.5,

$$\begin{aligned} y(x) &= \int_{B_n} G(x,s;n)\hat{y}(s)\,ds < \int_{B_n} G(0,s;n)\hat{y}^*(s)\,ds \\ &\leqslant \int_{B_n} ds \int_{B_n} G(0,s;n)G_{k-1}(s,\tau;n)v^*(s)\,d\tau \\ &= \int_{B_n} G_k(0,s;n)v^*(s)\,ds, \qquad x \in B_n. \end{aligned} \tag{3.83}$$

Therefore, by Lemma 3.2,

$$\begin{aligned} C_X(2k+1,p,n) &= \frac{y(X)}{\|\nabla v\|_{L^p(B_n)}} \\ &< \frac{\int_{B_n} G_k(0,s;n)v^*(s)\,ds}{\|\nabla v^*\|_{L^p(B_n)}} \leqslant C_0(2k+1,p,n). \end{aligned} \tag{3.84}$$

The remains follows easily from Proposition 2.2. □

When $p = 2$, we can simplify the above results.

COROLLARY 3.10. *Let $n \geqslant 1$, $m \geqslant 1$, $2m > n$ and $p = 2$. Then (1.9) holds with*

$$C(m,2,n) = \sqrt{G_m(0,0;n)}. \tag{3.85}$$

Moreover, for $y \in W(B_n;m,2)$, the equality

$$\max_{x\in B_n} |y(x)| = C(m,2,n)\|y\|_{m,2} \tag{3.86}$$

holds if and only if

$$y(x) = \alpha G_m(x,0;n), \qquad x \in B_n \tag{3.87}$$

for some $\alpha \in \mathbb{R}$.

4. Further discussions

In this section, we give some discussions on the calculation of $G_k(x,s;n)$. Since $G_k(\cdot,0;n)$ is spherical symmetric, we have

$$G_k(x,0;n) = g_k(|x|;n), \qquad \forall x \in B_n, k \geqslant 1, \tag{4.1}$$

where

$$g_1(r;n) = \begin{cases} \frac{1-r}{2}, & \text{if } n = 1, \\ \frac{1}{2\pi}\ln\frac{1}{r}, & \text{if } n = 2, \qquad r \in (0,1], \\ \frac{1}{(n-2)n\omega_n}\left(r^{2-n}-1\right), & \text{if } n \geqslant 3, \end{cases} \tag{4.2}$$

$$g_{k+1}(r;n) = \int_0^1 h(r,s)g_k(s;n)\,ds, \qquad r \in (0,1], \quad k = 1,2,\ldots, \tag{4.3}$$

$$h(r,s) = \begin{cases} \frac{1-\max(r,s)}{2}, & \text{if } n = 1, \\ s\ln\frac{1}{\max(r,s)}, & \text{if } n = 2, \qquad r,s \in (0,1]. \\ \frac{1}{n-2}s^{n-1}\left(\left(\max(r,s)\right)^{2-n}-1\right), & \text{if } n \geqslant 3, \end{cases} \tag{4.4}$$

If we set

$$F(x,s,\lambda;n) = \sum_{k=1}^{\infty}(-1)^{k-1}G_k(x,s;n)\lambda^{2(k-1)}, \qquad x,s \in B_n,\ \lambda \geqslant 0, \tag{4.5}$$

then for λ small enough, $F(x,s,\lambda;n)$ is well defined and

$$-\triangle_x F(x,s,\lambda;n) = \delta(x-s) - \lambda^2 F(x,s,\lambda;n), \qquad x,s \in B_n,\ \lambda \geqslant 0. \tag{4.6}$$

For $n = 1$, we can get easily from (4.6) that

$$F(x,s,\lambda;1) = \frac{\operatorname{sh}\lambda(x+1)}{\lambda\operatorname{sh}2\lambda}\int_{-1}^{1}\operatorname{sh}\lambda(x-t)\cdot\delta(t-s)\,dt - \frac{1}{\lambda}\int_{-1}^{x}\operatorname{sh}\lambda(x-t)\cdot\delta(t-s)\,dt$$

$$= \begin{cases} \frac{\operatorname{sh}\lambda(x+1)}{\lambda\operatorname{sh}2\lambda}\operatorname{sh}\lambda(1-s) - \frac{\operatorname{sh}\lambda(x-s)}{\lambda}, & \text{if } x > s, \\ \frac{\operatorname{sh}\lambda(x+1)}{\lambda\operatorname{sh}2\lambda}\operatorname{sh}\lambda(1-s), & \text{if } x \leqslant s, \end{cases}$$

$$= \begin{cases} \frac{\operatorname{sh}\lambda(s+1)}{\lambda\operatorname{sh}2\lambda}\operatorname{sh}\lambda(1-x), & \text{if } x > s, \\ \frac{\operatorname{sh}\lambda(x+1)}{\lambda\operatorname{sh}2\lambda}\operatorname{sh}\lambda(1-s), & \text{if } x \leqslant s. \end{cases} \tag{4.7}$$

Let b_k be the Bernoulli numbers and $B_k(x)$ be the Bernoulli polynomials, that is

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_j(x)\frac{t^k}{k!} \tag{4.8}$$

and $b_k = B_k(0)$. The following is a table of Bernoulli numbers and Bernoulli polynomials.

k	b_k	$B_k(x)$
0	1	1
1	$-\frac{1}{2}$	$x-\frac{1}{2}$
2	$\frac{1}{6}$	$-x+x^2+\frac{1}{6}$
3	0	$\frac{1}{2}x-\frac{3}{2}x^2+x^3$
4	$-\frac{1}{30}$	$x^2-2x^3+x^4-\frac{1}{30}$
5	0	$-\frac{1}{6}x+\frac{5}{3}x^3-\frac{5}{2}x^4+x^5$
6	$\frac{1}{42}$	$-\frac{1}{2}x^2+\frac{5}{2}x^4-3x^5+x^6+\frac{1}{42}$
7	0	$\frac{1}{6}x-\frac{7}{6}x^3+\frac{7}{2}x^5-\frac{7}{2}x^6+x^7$
8	$-\frac{1}{30}$	$\frac{2}{3}x^2-\frac{7}{3}x^4+\frac{14}{3}x^6-4x^7+x^8-\frac{1}{30}$
9	0	$-\frac{3}{10}x+2x^3-\frac{21}{5}x^5+6x^7-\frac{9}{2}x^8+x^9$
10	$\frac{5}{66}$	$-\frac{3}{2}x^2+5x^4-7x^6+\frac{15}{2}x^8-5x^9+x^{10}+\frac{5}{66}$
…	…	……

We have

$$\begin{aligned} &\frac{\operatorname{sh}\lambda(x+1)}{\lambda\operatorname{sh}2\lambda}\operatorname{sh}\lambda(1-s) \\ &= \frac{1}{8\lambda^2}\Big[\Big(\frac{4\lambda e^{\lambda(s-x)}}{e^{4\lambda}-1} + \frac{-4\lambda e^{-\lambda(s-x)}}{e^{-4\lambda}-1}\Big) - \Big(\frac{4\lambda e^{\lambda(s+x+2)}}{e^{4\lambda}-1} + \frac{-4\lambda e^{-\lambda(s+x+2)}}{e^{-4\lambda}-1}\Big)\Big] \\ &= \frac{1}{4\lambda^2}\Big[\sum_{k=0}^{\infty} B_{2k}\Big(\frac{s-x}{4}\Big)\frac{(4\lambda)^{2k}}{(2k)!} - \sum_{k=0}^{\infty} B_{2k}\Big(\frac{s+x+2}{4}\Big)\frac{(4\lambda)^{2k}}{(2k)!}\Big] \\ &= \sum_{k=1}^{\infty}\frac{4^{2k-1}\lambda^{2(k-1)}}{(2k)!}\Big(B_{2k}\Big(\frac{s-x}{4}\Big) - B_{2k}\Big(\frac{s+x+2}{4}\Big)\Big). \end{aligned} \tag{4.9}$$

Therefore,

$$G_k(x,s;1) = \frac{(-1)^k 4^{2k-1}}{(2k)!}\Big[B_{2k}\Big(\frac{s+x+2}{4}\Big) - B_{2k}\Big(\frac{|s-x|}{4}\Big)\Big], \qquad k=1,2,\dots. \tag{4.10}$$

We can find the above equality in Theorem 3.1 of [3]. It is interesting and easy to prove by (4.8) that $B_k(x)$ satisfies the following equalities:

$$B_k(1-x) = (-1)^k B_k(x), \qquad k=0,1,2,\dots, \tag{4.11}$$

$$B_k\Big(\frac{1}{2}-x\Big) = \frac{1}{2^{k-1}}B_k(-2x) - B_k(-x)$$
$$= (-1)^k\Big(\frac{1}{2^{k-1}}B_k(2x) - B_k(x)\Big), \qquad k=0,1,2,\dots, \tag{4.12}$$

$$B_k(-x) = (-1)^k B_k(x) + (-1)^k k x^{k-1}, \qquad k=1,2,\dots. \tag{4.13}$$

$$\int_0^x B_k(s)\,ds = \frac{B_{k+1}(x) - b_{k+1}}{k+1}, \qquad k=0,1,2,\dots. \tag{4.14}$$

Therefore

$$G_k(x,0;1) = \frac{(-1)^k 4^{2k-1}}{(2k)!}\Big[B_{2k}\Big(\frac{x+2}{4}\Big) - B_{2k}\Big(\frac{|x|}{4}\Big)\Big]$$
$$= \frac{(-1)^k 2^{2k-1}}{(2k)!}\Big[B_{2k}\Big(\frac{|x|}{2}\Big) - 4^k B_{2k}\Big(\frac{|x|}{4}\Big)\Big], \quad k=1,2,\dots, \tag{4.15}$$

$$G_k(0,0;1) = \frac{(-1)^{k-1}2^{2k-1}(4^k-1)b_{2k}}{(2k)!}, \quad k=1,2,\dots, \tag{4.16}$$

$$\int_0^x G_k(s,0;1)\,ds = \frac{(-1)^k 2^{2k}}{(2k+1)!}\Big[B_{2k+1}\Big(\frac{|x|}{2}\Big) - 2^{2k+1}B_{2k+1}\Big(\frac{|x|}{4}\Big) + (2^{2k+1}-1)b_{2k+1}\Big]\operatorname{sgn} x,$$
$$k=1,2,\dots. \tag{4.17}$$

Using the above results, we have

$$C_X(2k,p,1) = \frac{4^{2k-1}}{(2k)!}\Big[\int_{-1}^1 \Big|B_{2k}\Big(\frac{x+X+2}{4}\Big) - B_{2k}\Big(\frac{|x-X|}{4}\Big)\Big|^q dx\Big]^{\frac{1}{q}}, \tag{4.18}$$

$$C_X(m,2,1) = 2^{2m-1}\sqrt{\frac{1}{(2m)!}\Big[B_{2m}\Big(\frac{X+1}{2}\Big) - b_{2m}\Big]} \tag{4.19}$$

$$= 2^{2m-1}\sqrt{\frac{1}{(2m)!}\Big[\frac{1}{2^{2m-1}}B_{2m}(X) - B_{2m}\Big(\frac{X}{2}\Big) - b_{2m}\Big]}. \tag{4.20}$$

Concerning Problem (O), we have

COROLLARY 4.1. *Let $k \geqslant 1$, $m = 2k$ and $1 < p < +\infty$. Then*

$$\max_{x\in(-1,1)} |y(x)| \leqslant C\Big(\int_{-1}^1 |y^{(2k)}(x)|^p\,dx\Big)^{\frac{1}{p}}, \qquad \forall y(\cdot) \in W(-1,1;2k,p) \tag{4.21}$$

with

$$C = \frac{4^{2k-1}}{(2k)!}\left[2\int_0^1 \left|B_{2k}\left(\frac{x+2}{4}\right) - B_{2k}\left(\frac{x}{4}\right)\right|^q dx\right]^{\frac{1}{q}} \tag{4.22}$$

$$= \frac{2^{2k-1}}{(2k)!}\left[2\int_0^1 \left|B_{2k}\left(\frac{x}{2}\right) - 4^k B_{2k}\left(\frac{x}{4}\right)\right|^q dx\right]^{\frac{1}{q}} \tag{4.23}$$

and the equality holds if and only if

$$y^{(2k)}(x) = \alpha\left[B_{2k}\left(\frac{|x|}{2}\right) - 4^k B_{2k}\left(\frac{|x|}{4}\right)\right]^{q-1}, \qquad x \in (-1,1) \tag{4.24}$$

for some $\alpha \in \mathbb{R}$.

COROLLARY 4.2. *Let* $k \geqslant 1$, $m = 2k+1$ *and* $1 < p < +\infty$. *Then*

$$\max_{x\in(-1,1)} |y(x)| \leqslant C\left(\int_{-1}^1 |y^{(2k+1)}(x)|^p\, dx\right)^{\frac{1}{p}}, \qquad \forall y(\cdot) \in W(-1,1;2k+1,p) \tag{4.25}$$

with

$$C = \frac{2^{2k}}{(2k+1)!}\left(2\int_0^1 \left|B_{2k+1}\left(\frac{x}{2}\right) - 2^{2k+1}B_{2k+1}\left(\frac{x}{4}\right) + (2^{2k+1}-1)b_{2k+1}\right|^q dx\right)^{\frac{1}{q}} \tag{4.26}$$

and the equality holds if and only if

$$y^{(2k)}(x) = \alpha\left\{(-1)^k\left[B_{2k+1}\left(\frac{|x|}{2}\right) - 2^{2k+1}B_{2k+1}\left(\frac{|x|}{4}\right) + (2^{2k+1}-1)b_{2k+1}\right]\right\}^{q-1} \operatorname{sgn} x,$$
$$x \in (-1,1)$$

for some $\alpha \in \mathbb{R}$.

COROLLARY 4.3. *Let* $m \geqslant 1$. *Then for any* $y \in W(-1,1;m,2)$,

$$\max_{x\in(-1,1)} |y(x)| \leqslant \sqrt{\frac{(-1)^{m-1}2^{2m-1}(4^m-1)b_{2m}}{(2m)!}}\left(\int_{-1}^1 |y^{(m)}(x)|^2\, dx\right)^{\frac{1}{2}}. \tag{4.27}$$

The equality holds if and only if

$$y(x) = \alpha\left[B_{2m}\left(\frac{|x|}{2}\right) - 4^m B_{2m}\left(\frac{|x|}{4}\right)\right], \qquad x \in (-1,1) \tag{4.28}$$

for some $\alpha \in \mathbb{R}$.

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