Mathematical
Inequalities
& Applications
Volume 18, Number 2 (2015), 735–750 doi:10.7153/mia-18-54

POINTWISE STRONG APPROXIMATION OF ALMOST PERIODIC FUNCTIONS IN S^1

WŁODZIMIERZ ŁENSKI AND BOGDAN SZAL

(*Communicated by L. Leindler*)

Abstract. We consider the Fourier series of S^1 almost periodic functions and construct the matrix means of partial sums of such series by the class $GM({}_2\beta)$. In two approximation theorems using these means we give the estimates of pointwise strong deviation of such means from the functions in terms of moduli of continuity defined by the Gabisoniya points, and the best approximation of functions by entire functions.

1. Introduction

Let S^p $(1 \leqslant p \leqslant \infty)$ be the class of all almost periodic functions in the sense of Stepanov with the norm

$$\|f\|_{S^p} := \begin{cases} \sup\limits_u \left\{ \frac{1}{\pi} \int_u^{u+\pi} |f(t)|^p \, dt \right\}^{1/p} & \text{when} \quad 1 \leqslant p < \infty \\ \sup\limits_u |f(u)| & \text{when} \quad p = \infty. \end{cases}$$

Suppose that the Fourier series of $f \in S^p$ has the form

$$Sf(x) = \sum_{\nu=-\infty}^{\infty} A_\nu(f) e^{i\lambda_\nu x}, \quad \text{where } A_\nu(f) = \lim_{L\to\infty} \frac{1}{L} \int_0^L f(t) e^{-i\lambda_\nu t} dt,$$

with the partial sums

$$S_{\gamma_k} f(x) = \sum_{|\lambda_\nu| \leqslant \gamma_k} A_\nu(f) e^{i\lambda_\nu x}$$

and that $0 = \lambda_0 < \lambda_\nu < \lambda_{\nu+1}$ if $\nu \in \mathbb{N} = \{1,2,3...\}$, $\lim\limits_{\nu\to\infty} \lambda_\nu = \infty$, $\lambda_{-\nu} = -\lambda_\nu$, $|A_\nu| + |A_{-\nu}| > 0$. Let $\Omega_{\alpha,p}$, with some fixed positive α, be the set of functions of class S^p bounded on $\mathbb{R} = (-\infty, \infty)$ whose Fourier exponents satisfy the condition

$$\lambda_{\nu+1} - \lambda_\nu \geqslant \alpha \quad (\nu \in \mathbb{N}).$$

Mathematics subject classification (2010): 42A247.
Keywords and phrases: Almost periodic functions, rate of strong approximation, summability of Fourier series.

Paper MIA-18-54

In case $f \in \Omega_{\alpha,p}$

$$S_{\lambda_k} f(x) = \int_0^\infty \{f(x+t) + f(x-t)\} \Psi_{\lambda_k,\lambda_k+\alpha}(t)\, dt,$$

where

$$\Psi_{\lambda,\eta}(t) = \frac{2\sin\frac{(\eta-\lambda)t}{2}\sin\frac{(\eta+\lambda)t}{2}}{\pi(\eta-\lambda)t^2} \quad (0<\lambda<\eta, \ \ |t|>0).$$

Let $A := (a_{n,k})$ be an infinite matrix of real nonnegative numbers such that

$$\sum_{k=0}^{\infty} a_{n,k} = 1, \text{ where } n = 0,1,2,\ldots . \tag{1}$$

Let us consider the strong mean

$$H^q_{n,A,\gamma} f(x) = \left\{ \sum_{k=0}^{\infty} a_{n,k} \left| S_{\gamma_k} f(x) - f(x) \right|^q \right\}^{1/q} \quad (q>0). \tag{2}$$

As measures of approximation by the quantity (2), we use the best approximation of f by entire functions g_σ of exponential type σ bounded on the real axis, shortly $g_\sigma \in B_\sigma$ and the moduli of continuity of f defined by the formulas

$$E_\sigma(f)_{S^p} = \inf_{g_\sigma} \|f - g_\sigma\|_{S^p},$$

$$\omega f(\delta)_{S^p} = \sup_{|t|\leqslant\delta} \|f(\cdot+t) - f(\cdot)\|_{S^p},$$

and

$$w_x f(\delta)_p := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p\, dt \right\}^{1/p},$$

$$G_x f(\delta)_{s,p} := \left\{ \sum_{k=0}^{[\pi/(\alpha\delta)]} \left(\frac{1}{(k+1)\delta} \int_{k\delta}^{(k+1)\delta} |\varphi_x(t)|^p\, dt \right)^{s/p} \right\}^{1/s}, \quad s>1,$$

where $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$, respectively.

Recently, L. Leindler [6] defined a new class of sequences named as sequences of rest bounded variation, briefly denoted by $RBVS$, i.e.

$$RBVS = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leqslant K(a)\,|a_m| \text{ for all } m \in \mathbb{N} \right\}, \tag{3}$$

where here and throughout the paper $K(a)$ always indicates a constant depending only on a.

Denote by MS the class of nonnegative and nonincreasing sequences. The class of general monotone coefficients, GM, will be defined as follows (see [13]):

$$GM = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leqslant K(a)\,|a_m| \text{ for all } m \in \mathbb{N} \right\}. \tag{4}$$

It is obvious that

$$MS \subset RBVS \subset GM.$$

In [7, 13, 14, 15] was defined the class of $\beta-$general monotone sequences (see also [5]) as follows:

DEFINITION 1. Let $\beta := (\beta_n)$ be a nonnegative sequence. The sequence of complex numbers $a := (a_n)$ is said to be $\beta-$general monotone, or $a \in GM(\beta)$, if the relation

$$\sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leqslant K(a)\beta_m \tag{5}$$

holds for all m.

In the paper [15] Tikhonov considered, among others, the following examples of the sequences β_n:

(1) ${}_1\beta_n = |a_n|$,

(2) ${}_2\beta_n = \sum_{k=[n/c]}^{[cn]} \frac{|a_k|}{k}$ for some $c > 1$.

It is clear that $GM({}_1\beta) = GM$ and (see [15, Remark 2.1])

$$GM({}_1\beta + {}_2\beta) \equiv GM({}_2\beta).$$

Moreover, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leqslant K(\alpha_n) \leqslant K$$

holds for all n, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (3)-(5) for the sequences $\alpha_n := (a_{n,k})_{k=0}^{\infty}$.

Now we can give the conditions to be used later on. We assume that for all n

$$\sum_{k=m}^{2m-1} |a_{n,k} - a_{n,k+1}| \leqslant K \sum_{k=[m/c]}^{[cm]} \frac{a_{n,k}}{k} \tag{6}$$

holds if $\alpha_n = (a_{n,k})_{k=0}^{\infty}$ belongs to $GM({}_2\beta)$, for $n = 1, 2, \ldots$

We have shown in [9] the following theorem:

THEOREM 1. *If* $f \in \Omega_{\alpha,p}$ $(p > 1)$, $p \geqslant q$, $\alpha > 0$, $(a_{n,k})_{k=0}^{\infty} \in GM({}_2\beta)$ *for all* n, (1) *and* $\lim_{n\to\infty} a_{n,0} = 0$ *hold, then*

$$\left\| H^q_{n,A,\gamma} f \right\|_{S^p} \ll \left\{ \sum_{k=0}^{\infty} a_{n,k}\, \omega^q f\left(\frac{\pi}{k+1}\right)_{S^p} \right\}^{1/q},$$

for $n = 0, 1, 2, \ldots$, *where* $\gamma = (\gamma_k)$ *is a sequence with* $\gamma_k = \frac{\alpha k}{2}$.

In this paper we consider the class $GM({}_2\beta)$ in pointwise estimate of the quantity $H^q_{n,A,\gamma}f$ for $f\in S^1$. Our theorems have generalized the following result of P. Pych-Taberska (see [12, Theorem 5]):

THEOREM 2. *If $f\in\Omega_{\alpha,\infty}$ and $q\geqslant 2$, then*

$$\left\|H^q_{n,A,\gamma}f\right\|_{S^\infty}\ll\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left[\omega f\left(\frac{\pi}{k+1}\right)_{S^\infty}\right]^q\right\}^{1/q}+\frac{\|f\|_{S^\infty}}{(n+1)^{1/q}},$$

for $n=0,1,2,\ldots$, where $\gamma=(\gamma_k)$ is a sequence with $\gamma_k=\frac{\alpha k}{2}$, $a_{n,k}=\frac{1}{n+1}$ when $k\leqslant n$ and $a_{n,k}=0$ otherwise.

For the function $f\in S^p$ $(p>1)$ such pointwise study was prepared for publication in [4].

We shall write $I_1\ll I_2$ if there exists a positive constant K, sometimes depended on some parameters, such that $I_1\leqslant KI_2$.

2. Statement of the results

Let us consider a function w_x of modulus of continuity type on the interval $[0,+\infty)$, i.e. a nondecreasing continuous function having the following properties: $w_x(0)=0$, $w_x(\delta_1+\delta_2)\leqslant w_x(\delta_1)+w_x(\delta_2)$ for any $\delta_1,\delta_2\geqslant 0$ with x such that the set

$$\Omega_{\alpha,p,s}(w_x)=\left\{f\in\Omega_{\alpha,p}:\left[\frac{1}{\delta}\int_0^\delta|\varphi_x(t)-\varphi_x(t\pm\gamma)|^p\,dt\right]^{1/p}\ll w_x(\gamma)\right.$$
$$\left.\text{and }\ G_xf(\delta)_{s,p}\ll w_x(\delta)\ \ ,\ \text{where }\ \gamma,\delta>0\right\}$$

is nonempty.

We start with proposition

PROPOSITION 1. *If $f\in\Omega_{\alpha,1,2}(w_x)$, $\alpha>0$ and $0<q\leqslant 2$, then*

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|S_{\frac{\alpha k}{2}}f(x)-f(x)\right|^q\right\}^{1/q}\ll w_x\left(\frac{\pi}{n+1}\right)+E_{\alpha n/2}(f)_{S^1},$$

for $n=0,1,2,\ldots$

Our main results are following

THEOREM 3. *If $f\in\Omega_{\alpha,1,2}(w_x)$, $\alpha>0$, $0<q\leqslant 2$, $(a_{n,k})_{k=0}^\infty\in GM({}_2\beta)$ for all n, (1) and $\lim_{n\to\infty}a_{n,0}=0$ hold, then*

$$H^q_{n,A,\gamma}f(x)\ll\left\{\sum_{k=0}^{\infty}a_{n,k}\left[w_x\left(\frac{\pi}{k+1}\right)+E_{\frac{\alpha k}{2^{1+[c]}}}(f)_{S^1}\right]^q\right\}^{1/q}$$

for some $c>1$ and $n=0,1,2,\ldots$, where $\gamma=(\gamma_k)$ is a sequence with $\gamma_k=\frac{\alpha k}{2}$.

THEOREM 4. *If* $f \in \Omega_{\alpha,1,2}(w_x)$, $\alpha > 0$, $0 < q \leqslant 2$, $(a_{n,k})_{k=0}^{\infty} \in MS$ *for all* n, *(1) and* $\lim_{n\to\infty} a_{n,0} = 0$ *hold, then*

$$H^q_{n,A,\gamma} f(x) \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x\left(\frac{\pi}{k+1}\right) + E_{\frac{\alpha k}{2}}(f)_{S^1} \right]^q \right\}^{1/q}$$

for $n = 0, 1, 2, \ldots$, *where* $\gamma = (\gamma_k)$ *is a sequence with* $\gamma_k = \frac{\alpha k}{2}$.

REMARK 1. Since, by the Jackson type theorem

$$E_\sigma(f)_{S^p} \ll \omega f\left(\frac{1}{\sigma}\right)_{S^p}$$

and

$$\left\| \left[\frac{1}{\delta} \int_0^\delta |\varphi_\cdot(t) - \varphi_\cdot(t \pm \gamma)|\, dt \right] \right\|_{S^p} \leqslant \omega f(\gamma)_{S^p},$$

$$\left\| G_\cdot f(\delta)_{2,p} \right\|_{S^p} \leqslant \omega f(\delta)_{S^p},$$

the analysis of the proof of Proposition 1 shows that, the estimate from Theorem 3 implies the estimate from Theorem 1 with $p \geqslant 2$ and $0 < q \leqslant 2$. Thus, taking $a_{n,k} = \frac{1}{n+1}$ when $k \leqslant n$ and $a_{n,k} = 0$ otherwise, in the case $p \in [2, \infty]$ we obtain the better estimate than this one from Theorem 2 with $q = 2$ [12].

3. Proofs of the results

3.1. Proof of Proposition 1

In the proof we will use the following function $\Phi_x f(\delta, \nu) = \frac{1}{\delta} \int_\nu^{\nu+\delta} \varphi_x(u)\, du$, with $\delta = \delta_n = \frac{\pi}{n+1}$ and its estimate from [8, Lemma 1, p. 218]

$$|\Phi_x f(\zeta_1, \zeta_2)| \leqslant w_x(\zeta_1) + w_x(\zeta_2) \tag{7}$$

for $f \in \Omega_{\alpha,1,2}(w_x)$ and any $\zeta_1, \zeta_2 > 0$.

We can also note that by monotonicity in $q \in (0, 2]$

$$\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^q \right\}^{1/q} \leqslant \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^2 \right\}^{1/2}.$$

Moreover, for $n = 0$ our estimate is evident. Therefore, we give the estimate of the quantity $H^q_{n,A,\gamma} f(x)$ with $q = 2$ and $n > 0$, only.

Denote by $S^*_k f$ the sums of the form

$$S_{\frac{\alpha k}{2}} f(x) = \sum_{|\lambda_\nu| \leqslant \frac{\alpha k}{2}} A_\nu(f)\, e^{i\lambda_\nu x}$$

such that the interval $\left(\frac{\alpha k}{2},\frac{\alpha(k+1)}{2}\right)$ does not contain any λ_ν. Applying Lemma 1.10.2 of [10] we easily verify that

$$S_k^* f(x) - f(x) = \int_0^\infty \varphi_x(t)\Psi_k(t)\,dt,$$

where $\Psi_k(t) = \Psi_{\frac{\alpha k}{2},\frac{\alpha(k+1)}{2}}(t)$, i.e.

$$\Psi_k(t) = \frac{4\sin\frac{\alpha t}{4}\sin\frac{\alpha(2k+1)t}{4}}{\alpha\pi t^2}$$

(see also [2], p.41). Evidently, if the interval $\left(\frac{\alpha k}{2},\frac{\alpha(k+1)}{2}\right)$ contains a Fourier exponent λ_ν, then

$$S_{\frac{\alpha k}{2}} f(x) = S_{k+1}^* f(x) - \left(A_\nu(f)e^{i\lambda_\nu x} + A_{-\nu}(f)e^{-i\lambda_\nu x}\right).$$

Analyzing the proof of Proposition 1.2.2 from [1, p. 8] we can write

$$\begin{aligned}
|A_{\pm\nu}(f)| &= \left|A_{\pm\nu}\left(f - g_{\alpha\mu/2}\right)\right| \\
&= \left|\lim_{L\to\infty}\frac{1}{L}\int_0^L \left(f(t) - g_{\alpha\mu/2}(t)\right)e^{-i\lambda_\nu t}dt\right| \\
&\leqslant \left|\lim_{L\to\infty}\sup_{T\geqslant L}\frac{1}{T}\int_0^T \left|\left(f(t) - g_{\alpha\mu/2}(t)\right)e^{-i\lambda_\nu t}\right|dt\right| \\
&\leqslant \left|\lim_{L\to\infty}\sup_{T\geqslant L}\frac{1}{T}\int_0^T \left|f(t) - g_{\alpha\mu/2}(t)\right|dt\right| \\
&\leqslant \left|\lim_{L\to\infty}\sup_{T\geqslant L}\sup_{U\in\mathbb{R}}\frac{1}{T}\int_U^{U+T} \left|f(t) - g_{\alpha\mu/2}(t)\right|dt\right| \\
&= \left\|f - g_{\alpha\mu/2}\right\|_W \leqslant \left\|f - g_{\alpha\mu/2}\right\|_{S^1} = E_{\alpha\mu/2}(f)_{S^1},
\end{aligned}$$

for some $g_{\alpha\mu/2} \in B_{\alpha\mu/2}$, with $\alpha k/2 < \alpha\mu/2 < \lambda_\nu$, where $\|\cdot\|_W$ is the Weyl norm.

Therefore, the deviation

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|S_{\frac{\alpha k}{2}} f(x) - f(x)\right|^2\right\}^{\frac{1}{2}}$$

can be estimated from above by

$$\begin{aligned}
&\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\int_0^\infty \varphi_x(t)\Psi_{k+\kappa}(t)\,dt\right|^2\right\}^{1/2} + \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(E_{\alpha k/2}(f)_{S^1}\right)^2\right\}^{1/2} \\
\leqslant &\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\int_0^\infty \varphi_x(t)\Psi_{k+\kappa}(t)\,dt\right|^2\right\}^{1/2} + E_{\alpha n/2}(f)_{S^1},
\end{aligned}$$

where κ equals 0 or 1. Applying the Minkowski inequality we obtain

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\int_0^{\infty}\varphi_x(t)\,\Psi_{k+\kappa}(t)\,dt\right|^2\right\}^{1/2}$$
$$=\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\left(\int_0^{\pi/\alpha}+\int_{\pi/\alpha}^{\infty}\right)\varphi_x(t)\,\Psi_{k+\kappa}(t)\,dt\right|^2\right\}^{1/2}$$
$$\leqslant\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_1(k)|^2\right\}^{1/2}+\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_2(k)|^2\right\}^{1/2}.$$

So, for the first term we have

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_1(k)|^2\right\}^{1/2}\leqslant\left\{\frac{2^{\kappa}e^2}{n+1}\sum_{k=n}^{2n}\left(1-\frac{1}{n+1}\right)^{2n+\kappa}|I_1(k)|^2\right\}^{1/2}$$
$$\leqslant\left\{\frac{2^{\kappa}e^2}{n+1}\sum_{k=n}^{2n}\left(1-\frac{1}{n+1}\right)^{k+\kappa}|I_1(k)|^2\right\}^{1/2}$$
$$\leqslant\left\{\frac{2^{\kappa}e^2}{n+1}\sum_{k=0}^{\infty}\left(1-\frac{1}{n+1}\right)^{k+\kappa}|I_1(k)|^2\right\}^{1/2}$$
$$=\left\{\frac{2^{\kappa}e^2}{n+1}\sum_{k=0}^{\infty}\left(1-\frac{1}{n+1}\right)^{k+\kappa}\left|\int_0^{\pi/\alpha}\varphi_x(t)\,\Psi_{k+\kappa}(t)\,dt\right|^2\right\}^{1/2}$$
$$=\left\{\frac{2^{\kappa}e^2}{n+1}\int_0^{\pi/\alpha}\int_0^{\pi/\alpha}\varphi_x(u)\,\overline{\varphi_x(v)}\sum_{k=0}^{\infty}\left(1-\frac{1}{n+1}\right)^{k+\kappa}\Psi_{k+\kappa}(u)\,\Psi_{k+\kappa}(v)\,dudv\right\}^{1/2}$$
$$\ll\left\{\frac{2^{\kappa}e^2}{n+1}\int_0^{\pi/\alpha}\int_0^{u}\varphi_x(u)\,\overline{\varphi_x(v)}\sum_{k=0}^{\infty}\left(1-\frac{1}{n+1}\right)^{k+\kappa}\Psi_{k+\kappa}(u)\,\Psi_{k+\kappa}(v)\,dudv\right\}^{1/2}$$
$$=\left\{\frac{2^{\kappa}e^2}{n+1}\left(\frac{4}{\alpha\pi}\right)^2\int_0^{\pi/\alpha}\int_0^{u}\frac{\varphi_x(u)\,\overline{\varphi_x(v)}\sin\frac{\alpha u}{4}\sin\frac{\alpha v}{4}}{u^2v^2}\times\right.$$
$$\left.\times\sum_{k=0}^{\infty}\left(1-\frac{1}{n+1}\right)^{k+\kappa}\sin\frac{\alpha u(2\,(k+\kappa)+1)}{4}\sin\frac{\alpha v(2\,(k+\kappa)+1)}{4}dudv\right\}^{1/2}$$
$$\leqslant\left\{\frac{2^{\kappa}e^2}{n+1}\left(\frac{4}{\alpha\pi}\right)^2\int_0^{\pi/\alpha}\int_0^{u}\frac{\varphi_x(u)\,\overline{\varphi_x(v)}\sin\frac{\alpha u}{4}\sin\frac{\alpha v}{4}}{u^2v^2}\times\right.$$
$$\left.\times\sum_{k=0}^{\infty}\left(1-\frac{1}{n+1}\right)^{k}\sin\frac{\alpha u(2k+1)}{4}\sin\frac{\alpha v(2k+1)}{4}dudv\right\}^{1/2}$$

Taking $y=\frac{\alpha u}{2}$, $z=\frac{\alpha v}{2}$ and $r=1-\frac{1}{n+1}$ in the relation (see [3] and [11])

$$\sum_{k=0}^{\infty} r^k \sin\frac{y(2k+1)}{2}\sin\frac{z(2k+1)}{2}$$
$$=\frac{\sin\frac{y}{2}\sin\frac{z}{2}(1-r)\left[(1+r)^2+2r(\cos y+\cos z)\right]}{\left[(1-r)^2+4r\sin^2\frac{y+z}{2}\right]\left[(1-r)^2+4r\sin^2\frac{y-z}{2}\right]}$$

and using the inequality $\sin\frac{(y+z)}{2}\geqslant\frac{y+z}{\pi}$ $(y+z\leqslant\pi)$, we obtain

$$\left|\sum_{k=0}^{\infty}\left(1-\frac{1}{n+1}\right)^k\sin\frac{\alpha u(2k+1)}{4}\sin\frac{\alpha v(2k+1)}{4}\right|$$
$$\ll\frac{1}{n+1}\frac{uv}{\left[(1-r)^2+(u+v)^2\right]\left[(1-r)^2+(u-v)^2\right]}.$$

Hence, taking $u-v=t$, by the Gabisoniya idea [3]

$$\frac{1}{n+1}\sum_{k=n}^{2n}|I_1(k)|^2$$
$$\ll\frac{1}{(n+1)^2}\int_0^{\pi/\alpha}\int_0^u\frac{|\varphi_x(u)\,\varphi_x(v)|\,dudv}{\left[(n+1)^{-2}+(u+v)^2\right]\left[(n+1)^{-2}+(u-v)^2\right]}$$
$$\leqslant\frac{1}{(n+1)^2}\int_0^{\pi/\alpha}\int_0^u\frac{|\varphi_x(u)\,\varphi_x(v)|\,dudv}{\left[(n+1)^{-2}+u^2\right]\left[(n+1)^{-2}+(u-v)^2\right]}$$
$$=\frac{1}{(n+1)^2}\int_0^{\pi/\alpha}\int_0^u\frac{|\varphi_x(u)\,\varphi_x(u-t)|\,dudt}{\left[(n+1)^{-2}+u^2\right]\left[(n+1)^{-2}+t^2\right]}$$
$$\leqslant\frac{1}{(n+1)^2}\sum_{i=0}^{[\pi(n+1)/\alpha]}\sum_{j=0}^{i}\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}\frac{|\varphi_x(u)\,\varphi_x(u-t)|\,dudt}{(n+1)^{-2}(1+i^2)\left[(n+1)^{-2}+t^2\right]}$$
$$\leqslant\sum_{i=0}^{[\pi(n+1)/\alpha]}\sum_{j=0}^{i}\frac{(n+1)^2}{(1+i^2)(1+j^2)}\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}}|\varphi_x(u)|\,du\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}|\varphi_x(u-t)|\,dt$$
$$\leqslant\sum_{i=0}^{[\pi(n+1)/\alpha]}\sum_{j=0}^{i}\frac{(n+1)^2}{(1+i^2)(1+j^2)}\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}}|\varphi_x(u)|\,du\int_{\frac{i}{n+1}-\frac{j+1}{n+1}}^{\frac{i+1}{n+1}-\frac{j}{n+1}}|\varphi_x(v)|\,dv$$
$$\ll\sum_{i=0}^{[\pi(n+1)/\alpha]}\sum_{j=0}^{i}\frac{(n+1)^2}{(1+i^2)(1+j^2)}\times$$
$$\times\left[\left(\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}}|\varphi_x(u)|\,du\right)^2+\left(\int_{\frac{i}{n+1}-\frac{j+1}{n+1}}^{\frac{i+1}{n+1}-\frac{j}{n+1}}|\varphi_x(v)|\,dv\right)^2\right]$$

$$\ll \sum_{i=0}^{[\pi(n+1)/\alpha]}\left(\frac{n+1}{1+i}\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}}|\varphi_x(u)|\,du\right)^2$$
$$+\sum_{i=0}^{[\pi(n+1)/\alpha]}\sum_{j=0}^{i}\frac{1}{(1+j)^2}\left(\frac{n+1}{1+i}\int_{\frac{i}{n+1}-\frac{j+1}{n+1}}^{\frac{i+1}{n+1}-\frac{j}{n+1}}|\varphi_x(v)|\,dv\right)^2$$
$$\ll \sum_{i=0}^{[\pi(n+1)/\alpha]}\left(\frac{n+1}{1+i}\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}}|\varphi_x(u)|\,du\right)^2$$
$$+\sum_{j=0}^{[\pi(n+1)/\alpha]}\frac{1}{(1+j)^2}\sum_{i=j}^{[\pi(n+1)/\alpha]}\left(\frac{n+1}{1+i}\int_{\frac{i}{n+1}-\frac{j+1}{n+1}}^{\frac{i+1}{n+1}-\frac{j}{n+1}}|\varphi_x(v)|\,dv\right)^2$$
$$\ll \sum_{i=0}^{[\pi(n+1)/\alpha]}\left(\frac{n+1}{1+i}\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}}|\varphi_x(u)|\,du\right)^2+\sum_{\nu=0}^{[\pi(n+1)/\alpha]}\left(\frac{n+1}{1+\nu}\int_{\frac{\nu-1}{n+1}}^{\frac{\nu+1}{n+1}}|\varphi_x(v)|\,dv\right)^2$$
$$\ll \sum_{i=0}^{[\pi(n+1)/\alpha]}\left(\frac{n+1}{1+i}\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}}|\varphi_x(u)|\,du\right)^2=\left[G_xf\left(\frac{1}{n+1}\right)_{2,1}\right]^2$$
$$\ll \left[w_x\left(\frac{\pi}{n+1}\right)\right]^2.$$

For the second term, using the Łenski method [8], we obtain

$$\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_2(k)|^2\right\}^{1/2}$$
$$\leqslant \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\sum_{\mu=1}^{\infty}\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\left[\varphi_x(t)-\Phi_xf(\delta_k,t)\right]\Psi_{k+\kappa}(t)\,dt\right|^2\right\}^{1/2}$$
$$+\left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left|\sum_{\mu=1}^{\infty}\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\Phi_xf(\delta_k,t)\,\Psi_{k+\kappa}(t)\,dt\right|^2\right\}^{1/2}$$
$$=\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_{21}(k)|^2\right\}^{1/2}+\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_{22}(k)|^2\right\}^{1/2}$$

and

$$|I_{21}(k)| \leqslant \frac{4}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}|\varphi_x(t)-\Phi_xf(\delta_k,t)|\,t^{-2}dt$$
$$\leqslant \frac{4}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\left[\frac{1}{\delta_k t^2}\int_0^{\delta_k}|\varphi_x(t)-\varphi_x(t+u)|\,du\right]dt$$
$$=\frac{4}{\alpha\pi}\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\left\{\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\frac{1}{t^2}|\varphi_x(t)-\varphi_x(t+u)|\,dt\right\}du$$

$$
\begin{aligned}
&= \frac{4}{\alpha\pi}\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\left\{\left[\frac{1}{t^2}\int_0^t |\varphi_x(s)-\varphi_x(s+u)|\,ds\right]_{t=\mu\pi/\alpha}^{t=(\mu+1)\pi/\alpha}\right.\\
&\quad \left.+2\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\left[\frac{1}{t^3}\int_0^t |\varphi_x(s)-\varphi_x(s+u)|\,ds\right]dt\right\}du\\
&\ll \left|\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\left\{\frac{1}{[(\mu+1)\pi/\alpha]^2}\int_0^{(\mu+1)\pi/\alpha}|\varphi_x(s)-\varphi_x(s+u)|\,ds\right.\right.\\
&\quad \left.\left.-\frac{1}{[\mu\pi/\alpha]^2}\int_0^{\mu\pi/\alpha}|\varphi_x(s)-\varphi_x(s+u)|\,ds\right\}du\right|\\
&\quad +\frac{1}{\delta_k}\int_0^{\delta_k}\sum_{\mu=1}^{\infty}\left\{\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\left[\frac{1}{t^3}\int_0^t|\varphi_x(s)-\varphi_x(s+u)|\,ds\right]dt\right\}du.
\end{aligned}
$$

Since $f\in\Omega_{\alpha,1,2}(w_x)$, for any x

$$
\begin{aligned}
\lim_{\zeta\to\infty}\frac{1}{\zeta^2}\int_0^{\zeta}|\varphi_x(s)-\varphi_x(s+u)|\,ds &\leqslant \lim_{\zeta\to\infty}\frac{1}{\zeta}w_x(u)\leqslant \lim_{\zeta\to\infty}\frac{1}{\zeta}w_x(\delta_k)\\
&\leqslant \lim_{\zeta\to\infty}\frac{1}{\zeta}w_x(\pi)=0,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
|I_{21}(k)| &\leqslant \frac{1}{\delta_k}\int_0^{\delta_k}\frac{\alpha}{\pi}\left[\frac{\alpha}{\pi}\int_0^{\pi/\alpha}|\varphi_x(s)-\varphi_x(s+u)|\,ds\right]du\\
&\quad +\frac{1}{\delta_k}\int_0^{\delta_k}w_x(u)\,du\sum_{\mu=1}^{\infty}\left\{\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\frac{1}{t^2}dt\right\}\\
&\ll \frac{1}{\delta_k}\int_0^{\delta_k}w_x(u)\,du+w_x(\delta_k)\sum_{\mu=1}^{\infty}\frac{\alpha}{\pi\mu^2}\\
&\ll w_x(\delta_k).
\end{aligned}
$$

Next, we will estimate the term $|I_{22}(k)|$. So,

$$
\begin{aligned}
I_{22}(k) &= \frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\frac{\Phi_x f(\delta_k,t)}{t^2}\frac{d}{dt}\left(-\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\right)dt\\
&= \frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\left[\frac{\Phi_x f(\delta_k,t)}{t^2}\left(-\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\right)\right]_{t=\mu\pi/\alpha}^{t=(\mu+1)\pi/\alpha}\\
&\quad +\frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\frac{d}{dt}\left(\frac{\Phi_x f(\delta_k,t)}{t^2}\right)\left(\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}-\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\right)dt\\
&= I_{221}(k)+I_{222}(k)
\end{aligned}
$$

Since $f \in \Omega_{\alpha,1,2}(w_x)$, for any x (using (7))

$$\lim_{\zeta\to\infty}\left|\frac{\Phi_x f\left(\delta_k,\frac{\pi}{\alpha}\zeta\right)}{\left[\frac{\pi}{\alpha}\zeta\right]^2}\left(-\frac{\cos\left[\frac{\pi\zeta}{2}(k+\kappa)\right]}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\left[\frac{\pi\zeta}{2}(k+\kappa+1)\right]}{\frac{\alpha(k+\kappa+1)}{2}}\right)\right|$$

$$\ll \lim_{\zeta\to\infty}\frac{w_x(\delta_k)+w_x\left(\frac{\pi}{\alpha}\zeta\right)}{\zeta^2 k} \ll \lim_{\zeta\to\infty}\frac{w_x(\delta_k)+\zeta w_x\left(\frac{\pi}{\alpha}\right)}{\zeta^2 k} \ll w_x(\pi)\lim_{\zeta\to\infty}\frac{1+\zeta}{\zeta^2}=0,$$

and therefore

$$\begin{aligned} I_{221}(k) =& \frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\left[\frac{\Phi_x f\left(\delta_k,\frac{\pi}{\alpha}(\mu+1)\right)}{\left[\frac{\pi}{\alpha}(\mu+1)\right]^2}\left(-\frac{\cos\left[\frac{\pi}{2}(\mu+1)(k+\kappa)\right]}{\frac{\alpha(k+\kappa)}{2}}\right.\right. \\ &\left.+\frac{\cos\left[\frac{\pi}{2}(\mu+1)(k+\kappa+1)\right]}{\frac{\alpha(k+\kappa+1)}{2}}\right) \\ &\left.-\frac{\Phi_x f\left(\delta_k,\frac{\pi}{\alpha}\mu\right)}{\left[\frac{\pi}{\alpha}\mu\right]^2}\left(-\frac{\cos\left[\frac{\pi}{2}\mu(k+\kappa)\right]}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\left[\frac{\pi}{2}\mu(k+\kappa+1)\right]}{\frac{\alpha(k+\kappa+1)}{2}}\right)\right] \\ =& -\frac{2}{\alpha\pi}\frac{\Phi_x f(\delta_k,\pi/\alpha)}{[\pi/\alpha]^2}\left(-\frac{\cos\left[\frac{\pi}{2}(k+\kappa)\right]}{\frac{\alpha(k+\kappa)}{2}}+\frac{\cos\left[\frac{\pi}{2}(k+\kappa+1)\right]}{\frac{\alpha(k+\kappa+1)}{2}}\right) \\ =& -\frac{4}{\pi^3}\Phi_x f(\delta_k,\pi/\alpha)\left(\frac{\cos\left[\frac{\pi}{2}(k+\kappa+1)\right]}{k+\kappa+1}-\frac{\cos\left[\frac{\pi}{2}(k+\kappa)\right]}{k+\kappa}\right). \end{aligned}$$

Using (7), we get

$$|I_{221}(k)| \ll \frac{1}{k+1}|\Phi_x f(\delta_k,\pi/\alpha)| \leqslant \frac{1}{(k+1)}(w_x(\delta_k)+w_x(\pi/\alpha)).$$

Similarly

$$\begin{aligned} I_{222}(k) =& \frac{2}{\alpha\pi}\sum_{\mu=1}^{\infty}\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\left(\frac{\frac{d}{dt}\Phi_x f(\delta_k,t)}{t^2}-\frac{2\Phi_x f(\delta_k,t)}{t^3}\right) \\ &\cdot\left(\frac{\cos\frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}}-\frac{\cos\frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}}\right)dt \end{aligned}$$

and

$$\begin{aligned} |I_{222}(k)| \ll& \frac{8}{\alpha^2(k+1)\pi}\sum_{\mu=1}^{\infty}\left[\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\frac{|\varphi_x(t+\delta_k)-\varphi_x(t)|}{\delta_k t^2}dt\right. \\ &\left.+2\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\frac{|\Phi_x f(\delta_k,t)|}{t^3}dt\right] \\ \leqslant& \frac{8}{\alpha^2(k+1)\pi\delta_k}\sum_{\mu=1}^{\infty}\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\frac{|\varphi_x(t+\delta_k)-\varphi_x(t)|}{t^2}dt \\ &+\frac{16}{\alpha^2(k+1)\pi}\sum_{\mu=1}^{\infty}\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha}\frac{w_x(\delta_k)+w_x(t)}{t^3}dt \end{aligned}$$

$$
\begin{aligned}
&\ll \frac{1}{(k+1)\delta_k} w_x(\delta_k) + \frac{1}{k+1}\sum_{\mu=1}^{\infty}\left[\left(w_x(\delta_k) + w_x\left(\frac{\pi(\mu+1)}{\alpha}\right)\right)\frac{\alpha^2}{\pi^2\mu^3}\right] \\
&\ll w_x(\delta_k) + \frac{1}{k+1}\left[w_x(\delta_k)\sum_{\mu=1}^{\infty}\frac{1}{\mu^3} + \sum_{\mu=1}^{\infty}\frac{w_x\left(\frac{\pi(\mu+1)}{\alpha}\right)}{\mu^3}\right] \\
&\ll w_x(\delta_k) + \frac{1}{k+1}\left(w_x(\delta_k) + w_x\left(\frac{2\pi}{\alpha}\right)\sum_{\mu=1}^{\infty}\frac{\mu+1}{\mu^3}\right) \\
&\ll w_x(\delta_k) + \frac{1}{k+1}\left(w_x(\delta_k) + w_x\left(\frac{2\pi}{\alpha}\right)\right).
\end{aligned}
$$

Summing up

$$
|I_2(k)| \ll w_x(\delta_k) + \frac{1}{k+1}\left(w_x(\delta_k) + w_x\left(\frac{\pi}{\alpha}\right) + w_x\left(\frac{2\pi}{\alpha}\right)\right),
$$

whence

$$
\begin{aligned}
\left\{\frac{1}{n+1}\sum_{k=n}^{2n}|I_2(k)|^2\right\}^{1/2} &\ll \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(w_x\left(\frac{\pi}{k+1}\right) + \frac{1}{k+1}w_x\left(\frac{\pi}{\alpha}\right)\right)^2\right\}^{1/2} \\
&\ll \left\{\frac{1}{n+1}\sum_{k=n}^{2n}\left(w_x\left(\frac{\pi}{k+1}\right)\right)^2\right\}^{1/2} \leqslant w_x\left(\frac{\pi}{n+1}\right)
\end{aligned}
$$

and thus the desired result follows. □

3.2. Proof of Theorem 3

For some $c > 1$

$$
\begin{aligned}
H^q_{n,A,\gamma}f(x) &= \left\{\sum_{k=0}^{2^{[c]}-1} a_{n,k}\left|S_{\frac{\alpha k}{2}}f(x) - f(x)\right|^q + \sum_{k=2^{[c]}}^{\infty} a_{n,k}\left|S_{\frac{\alpha k}{2}}f(x) - f(x)\right|^q\right\}^{1/q} \\
&\ll \left\{\sum_{k=0}^{2^{[c]}-1} a_{n,k}\left|S_{\frac{\alpha k}{2}}f(x) - f(x)\right|^q\right\}^{1/q} \\
&\quad + \left\{\sum_{m=[c]}^{\infty}\sum_{k=2^m}^{2^{m+1}-1} a_{n,k}\left|S_{\frac{\alpha k}{2}}f(x) - f(x)\right|^q\right\}^{1/q} \\
&= I_1(x) + I_2(x).
\end{aligned}
$$

Using Proposition 1 and denoting $F_n = w_x\left(\frac{\pi}{n+1}\right) + E_{\alpha n/2}(f)_{S^1}$, we get

$$\begin{aligned} I_1(x) &\leqslant \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} \frac{k/2+1}{k/2+1} \sum_{l=k/2}^{k} \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^q \right\}^{1/q} \\ &\leqslant \left\{ 2^{[c]} \sum_{k=0}^{2^{[c]}-1} a_{n,k} \frac{1}{k/2+1} \sum_{l=k/2}^{k} \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^q \right\}^{1/q} \\ &\ll \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} F_{k/2}^q \right\}^{1/q}. \end{aligned}$$

By partial summation, our Proposition 1 gives

$$\begin{aligned} I_2^q(x) &= \sum_{m=[c]}^{\infty} \left[\sum_{k=2^m}^{2^{m+1}-2} \left(a_{n,k} - a_{n,k+1}\right) \sum_{l=2^m}^{k} \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^q \right. \\ &\quad \left. + a_{n,2^{m+1}-1} \sum_{l=2^m}^{2^{m+1}-1} \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^q \right] \\ &\ll \sum_{m=[c]}^{\infty} \left[2^m \sum_{k=2^m}^{2^{m+1}-2} \left| a_{n,k} - a_{n,k+1} \right| F_{\alpha 2^m/2}^q \right. \\ &\quad \left. + 2^m a_{n,2^{m+1}-1} F_{\alpha 2^m/2}^q \right] \\ &= \sum_{m=[c]}^{\infty} 2^m F_{\alpha 2^m/2}^q \left[\sum_{k=2^m}^{2^{m+1}-2} \left| a_{n,k} - a_{n,k+1} \right| + a_{n,2^{m+1}-1} \right]. \end{aligned}$$

Since (6) holds, we have

$$\begin{aligned} &a_{n,s+1} - a_{n,r} \\ &\leqslant \left| a_{n,r} - a_{n,s+1} \right| \leqslant \sum_{k=r}^{s} \left| a_{n,k} - a_{n,k+1} \right| \\ &\leqslant \sum_{k=2^m}^{2^{m+1}-2} \left| a_{n,k} - a_{n,k+1} \right| \ll \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \quad \left(2 \leqslant 2^m \leqslant r \leqslant s \leqslant 2^{m+1}-2\right), \end{aligned}$$

whence

$$a_{n,s+1} \ll a_{n,r} + \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \quad \left(2 \leqslant 2^m \leqslant r \leqslant s \leqslant 2^{m+1}-2\right)$$

and

$$\begin{aligned} 2^m a_{n,2^{m+1}-1} &= \frac{2^m}{2^m-1} \sum_{r=2^m}^{2^{m+1}-2} a_{n,2^{m+1}-1} \\ &\ll \sum_{r=2^m}^{2^{m+1}-2} \left(a_{n,r} + \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \right) \end{aligned}$$

$$\ll \sum_{r=2^m}^{2^{m+1}-1} a_{n,r} + 2^m \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k}.$$

Thus

$$I_2^q(x) \ll \sum_{m=[c]}^{\infty} \left\{ 2^m F^q_{\alpha 2^m/2} \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} + F^q_{\alpha 2^m/2} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \right\}.$$

Finally, by elementary calculations we get

$$\begin{aligned}
I_2^q(x) &\ll \sum_{m=[c]}^{\infty} \left\{ 2^m F^q_{\alpha 2^m/2} \sum_{k=2^{m-[c]}}^{2^{m+[c]}} \frac{a_{n,k}}{k} + F^q_{\alpha 2^m/2} \sum_{k=2^m}^{2^{m+1}} a_{n,k} \right\} \\
&\ll \sum_{m=[c]}^{\infty} F^q_{\alpha 2^m/2} \sum_{k=2^{m-[c]}}^{2^{m+[c]}} a_{n,k} \\
&= \sum_{m=[c]}^{\infty} F^q_{\alpha 2^m/2} \sum_{k=2^{m-[c]}}^{2^m-1} a_{n,k} + \sum_{m=[c]}^{\infty} F^q_{\alpha 2^m/2} \sum_{k=2^m}^{2^{m+[c]}} a_{n,k} \\
&\ll \sum_{m=[c]}^{\infty} \sum_{k=2^{m-[c]}}^{2^m-1} a_{n,k} F^q_{\alpha k/2} + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+[c]}} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}} \\
&= \sum_{m=[c]}^{\infty} \sum_{k=2^{m-[c]}}^{2^m-1} a_{n,k} F^q_{\alpha k/2} + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+[c]}-1} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}} + \sum_{m=[c]}^{\infty} F^q_{\frac{\alpha 2^m}{2}} a_{n,2^{m+[c]}} \\
&= \sum_{m=[c]}^{\infty} \sum_{r=1}^{[c]} \sum_{k=2^{m-r}}^{2^{m-r+1}-1} a_{n,k} F^q_{\alpha k/2} + \sum_{m=[c]}^{\infty} \sum_{r=0}^{[c]-1} \sum_{k=2^{m+r}}^{2^{m+r+1}-1} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}} \\
&\quad + \sum_{m=[c]}^{\infty} F^q_{\frac{\alpha 2^m}{2}} a_{n,2^{m+[c]}} \\
&\leqslant \sum_{r=1}^{[c]} \sum_{k=2^{[c]-r}}^{\infty} a_{n,k} F^q_{\alpha k/2} + \sum_{r=0}^{[c]-1} \sum_{k=2^{[c]+r}}^{\infty} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}} + \sum_{k=2^{2[c]}}^{\infty} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}} \\
&\ll \sum_{k=0}^{\infty} a_{n,k} F^q_{\frac{\alpha k}{2^{1+[c]}}}.
\end{aligned}$$

Thus we obtain the desired result. □

3.3. Proof of Theorem 4

If $(a_{n,k})_{k=0}^{\infty} \in MS$ then $(a_{n,k})_{k=0}^{\infty} \in GM({}_2\beta)$ and using Theorem 3 we obtain

$$H^q_{n,A,\gamma} f(x) \leqslant \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x\left(\frac{\pi}{k+1}\right) \right]^q \right\}^{1/q}$$

$$+\left\{\sum_{k=0}^{\infty}\sum_{m=k2^{[c]}}^{(k+1)2^{[c]}-1} a_{n,m}\left[E_{\frac{\alpha m}{2^{1+[c]}}}(f)_{S^p}\right]^q\right\}^{1/q}$$

$$\leqslant\left\{\sum_{k=0}^{\infty} a_{n,k}\left[w_x\Big(\frac{\pi}{k+1}\Big)\right]^q\right\}^{1/q}+\left\{\sum_{k=0}^{\infty}\sum_{m=k2^{[c]}}^{(k+1)2^{[c]}-1} a_{n,m}\left[E_{\frac{\alpha k}{2}}(f)_{S^p}\right]^q\right\}^{1/q}$$

$$\leqslant\left\{\sum_{k=0}^{\infty} a_{n,k}\left[w_x\Big(\frac{\pi}{k+1}\Big)\right]^q\right\}^{1/q}+\left\{\sum_{k=0}^{\infty}2^{[c]}a_{n,k2^{[c]}}\left[E_{\frac{\alpha k}{2}}(f)_{S^p}\right]^q\right\}^{1/q}$$

$$\ll\left\{\sum_{k=0}^{\infty} a_{n,k}\left[w_x\Big(\frac{\pi}{k+1}\Big)+E_{\frac{\alpha k}{2}}(f)_{S^p}\right]^q\right\}^{1/q}$$

This ends our proof. □

REFERENCES

[1] A. D. BAILEY, *Almost Everywhere Convergence of Dyadic Partial Sums of Fourier Series for Almost Periodic Functions, Master of Philosophy*, A thesis submitted to School of Mathematics of The University of Birmingham for the degree of Master of Philosophy, September 2008.

[2] A. S. BESICOVITCH, *Almost periodic functions*, Cambridge, 1932.

[3] O. D. GABISONIYA, *Points of strong summability of Fourier series*, (Translated from Matematicheskie Zametki, Vol. 14, No. 5, pp. 615–626) Math. Notes 14 (1973), 913–918 (1974).

[4] R. KRANZ, W. ŁENSKI AND B. SZAL, *Pointwise strong approximation of almost periodic functions*, (`http://arxiv.org/pdf/1204.2953.pdf`), submitted.

[5] R. J. LE AND S. P. ZHOU, *A new condition for the uniform convergence of certain trigonometric series*, Acta Math. Hung., 108 (1-2) (2005), 161–169.

[6] L. LEINDLER, *On the uniform convergence and boundedness of a certain class of sine series*, Analysis Math., 27 (2001), 279–285.

[7] L. LEINDLER, *A new extension of monotone sequence and its application*, J. Inequal. Pure and Appl. Math., 7 (1) (2006), Art. 39, 7 pp.

[8] W. ŁENSKI, *Pointwise strong and very strong approximation of Fourier series*, Acta Math. Hung., 115 (3), 207, p. 215–233.

[9] W. ŁENSKI AND B. SZAL, *Strong approximation of almost periodic functions*, Math. Inequal. Appl. **17**, 4 (2014). 1353–1364.

[10] B. L. LEVITAN, *Almost periodic functions*, Gos. Izdat. Tekh-Teoret. Liter., Moscov 1953 (in Russian).

[11] J. MARCINKIEWICZ, *Sur la sommabilite forte de series de Fourier*, J. London Math. Soc. 14 (1939), pp. 162–168.

[12] P. PYCH-TABERSKA, *Approximation properties of the partial sums of Fourier series of almost periodic functions*, Studia Math. XCVI (1990), 91–103.

[13] S. TIKHONOV, *Trigonometric series with general monotone coefficients*, J. Math. Anal. Appl., 326 (1) (2007), 721–735.

[14] S. TIKHONOV, *On uniform convergence of trigonometric series*, Mat. Zametki, 81 (2) (2007), 304–310, translation in Math. Notes, 81 (2) (2007), 268–274.

[15] S. TIKHONOV, *Best approximation and moduli of smoothness: Computation and equivalence theorems*, J. Approx. Theory, 153 (2008), 19–39.

(Received February 12, 2014)

Włodzimierz Łenski
University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
65-516 Zielona Góra, ul. Szafrana 4a, Poland
e-mail: W.Lenski@wmie.uz.zgora.pl

Bogdan Szal
University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
65-516 Zielona Góra, ul. Szafrana 4a, Poland
e-mail: B.Szal@wmie.uz.zgora.pl