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ON THE ORDER OF MAGNITUDE OF FOURIER TRANSFORM

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Abstract. For a Lebesgue integrable complex-valued function f defined on $\mathbb{R}$, let $\hat{f}$ be its Fourier transform. The Riemann-Lebesgue lemma says that $\hat{f}(t)\to 0$ as $|t|\to\infty$. But in general, there is no definite rate at which the Fourier transform tends to zero. In fact, the Fourier transform of an integrable function can tend to zero as slowly as we wish. Therefore, it is interesting to know for functions of which subclasses of $L^1(\mathbb{R})$ there is a definite rate at which the Fourier transform tends to zero. In this paper, we determine this rate for functions of bounded variation on $\mathbb{R}$. We also determine such rate of Fourier transform for functions of bounded variation in the sense of Vitali defined on $\mathbb{R}^N$ $(N\in\mathbb{N})$.

1. Introduction

For a Lebesgue integrable complex-valued function f defined on $\mathbb{R}$, let $\hat{f}$ be its Fourier transform. The Riemann-Lebesgue lemma says that $\hat{f}(t)\to 0$ as $|t|\to\infty$. But in general, there is no definite rate at which the Fourier transform tends to zero. In fact, the Fourier transform of an integrable function can tend to zero as slowly as we wish (see, e.g., [9, 32.47 (b)]). Therefore, it is interesting to know for functions of which subclasses of $L^1(\mathbb{R})$ there is a definite rate at which the Fourier transform tends to zero. Looking to the periodic case, that is, for functions on one-dimensional torus $\mathbb{T}:=[0,2\pi)$, the study of order of magnitude of Fourier coefficients is done extensively (see, e.g., [11], [16], see also [3, Section 2.3, p. 30] and [17, Section 4, p. 45]). This study in periodic case is done even for more general cases, that is, in the case of functions on two-dimensional torus, or more generally, on the N-dimensional torus $\mathbb{T}^N:=[0,2\pi)^N$ $(N\in\mathbb{N})$ (see, e.g., [12], [4], [5]). But it appears that such a study for the Fourier transform has not yet been done. In this paper we carry out this study and determine the rate of decay of Fourier transform for functions of bounded variation on $\mathbb{R}$. We also determine such rate of Fourier transform for functions of bounded variation in the sense of Vitali defined on $\mathbb{R}^N$ $(N\in\mathbb{N})$.

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2. One-dimensional case

We recall that a function $f:\mathbb{R}\to\mathbb{C}$ is said to be of bounded variation over $\mathbb{R}$, in symbol: $f\in BV(\mathbb{R})$, if

$$\sup_{\mathscr{S}}\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|<\infty, \tag{1}$$

where the supremum is extended over all finite sequences

$$\mathscr{S}: -\infty<x_0<x_1<x_2<\ldots<x_n<\infty \quad \text{and} \quad n=1,2,\ldots.$$

The supremum in (1), denoted by $V(f)$, is called the total variation of f over $\mathbb{R}$.

It is clear that the above definition of bounded variation over $\mathbb{R}$ can be reformulated equivalently as follows. A function f is of bounded variation over $\mathbb{R}$ if and only if f is of bounded variation over any finite interval $[a,b]$ in the ordinary sense and the set of the total variations $V(f,[a,b])$ of f over all finite intervals $[a,b]$ is bounded. Furthermore, if this is the case, then the supremum of the total variations over all finite intervals is equal to $V(f)$ defined above (see, e.g., [14, p. 238]).

In a similar way, one can define the notion of bounded variation over the intervals of the form $(-\infty,a]$ and $[a,\infty)$, where $a\in\mathbb{R}$ is arbitrary.

Given $f\in BV(\mathbb{R})$, let $V(f,x):=V(f,(-\infty,x])$ denote the total variation of f over the interval $(-\infty,x]$. Then it is evident that

$$\lim_{x\to-\infty}V(f,x)=0 \tag{2}$$

and

$$\lim_{x\to\infty}V(f,x)=V(f). \tag{3}$$

We note that the variation of f over $[x,\infty)$ is given by $V(f,[x,\infty))=V(f)-V(f,x)$ (see (9) in Lemma 1) and hence from (3) it follows that

$$\lim_{x\to\infty}V(f,[x,\infty))=0. \tag{4}$$

We recall that for a complex-valued Lebesgue integrable function f on $\mathbb{R}$, in symbol: $f\in L^1(\mathbb{R})$, the Fourier transform of f is defined as

$$\hat{f}(t):=\frac{1}{2\pi}\int_{\mathbb{R}}f(x)e^{-itx}dx, \quad t\in\mathbb{R}. \tag{5}$$

By the Riemann-Lebesgue lemma (see, e.g., [7, p. 7]), we know that $\hat{f}(t)\to 0$ as $|t|\to\infty$. But in general, there is no definite rate at which the Fourier transform tends to zero. In fact, the Fourier transform of an integrable function can tend to zero as slowly as we wish (see, e.g., [9, 32.47 (b)]). Therefore, it is interesting to know for functions of which subclasses of $L^1(\mathbb{R})$ there is a definite rate at which the Fourier transform tends to zero. In this section we carry out this study for functions of bounded variation on $\mathbb{R}$. Our main theorem of this section is as follows.

THEOREM 1. *If* $f \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ *then* $\hat{f}(t) = O(1/|t|)$, $t \neq 0$.

We need the following lemma.

LEMMA 1. *If* $f \in BV(\mathbb{R})$ *and if* $\{a_n : n \in \mathbb{Z}\}$ *is a doubly infinite increasing sequence of real numbers with*

$$\lim_{n\to\infty} a_n = \infty \tag{6}$$

and

$$\lim_{n\to\infty} a_{-n} = -\infty, \tag{7}$$

then the series $\sum_{n\in\mathbb{Z}} V(f,[a_{n-1},a_n])$ *converges and*

$$V(f) = \sum_{n\in\mathbb{Z}} V(f,[a_{n-1},a_n]). \tag{8}$$

Proof. First, we show that for any $a \in \mathbb{R}$, we have

$$V(f) = V(f,(-\infty,a]) + V(f,[a,\infty)). \tag{9}$$

Note that if $\mathscr{S}_1 : -\infty < x_0 < x_1 < x_2 < \ldots < x_m = a$ and $\mathscr{S}_2 : a = y_0 < y_1 < y_2 < \ldots < y_n < \infty$ are two finite sequences in $(-\infty,a]$ and $[a,\infty)$ respectively, then clearly $\mathscr{S} := \mathscr{S}_1 \cup \mathscr{S}_2 : -\infty < x_0 < x_1 < x_2 < \ldots < x_m = a = y_0 < y_1 < y_2 < \ldots < y_n < \infty$, is a finite sequence in $\mathbb{R}$, and hence

$$\sum_{j=1}^{m} |f(x_j) - f(x_{j-1})| + \sum_{k=1}^{n} |f(y_k) - f(y_{k-1})| \leqslant V(f).$$

Taking supremum over all finite sequences $\mathscr{S}_1$ contained in $(-\infty,a]$ (keeping the sequence $\mathscr{S}_2$ fixed) we get

$$V(f,(-\infty,a]) + \sum_{k=1}^{n} |f(y_k) - f(y_{k-1})| \leqslant V(f).$$

Now, taking supremum over all finite sequences $\mathscr{S}_2$ contained in $[a,\infty)$ we get

$$V(f,(-\infty,a]) + V(f,[a,\infty)) \leqslant V(f). \tag{10}$$

Next, we prove the reverse inequality to (10). Consider a finite sequence

$$\mathscr{S} : -\infty < x_0 < x_1 < x_2 < \ldots < x_n < \infty.$$

Since $a \in \mathbb{R}$, there are three possibilities: (i) $a < x_0$, or (ii) $a \geqslant x_n$, or (iii) $x_{k-1} \leqslant a < x_k$ for some $k \in \{1,2,\ldots,n\}$. In case (i) the sequence $a < x_0 < x_1 < x_2 < \ldots < x_n < \infty$ is a finite sequence in $[a,\infty)$ and hence

$$\begin{aligned}\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| &\leqslant |f(x_0) - f(a)| + \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \\ &\leqslant V(f,[a,\infty)) \\ &\leqslant V(f,(-\infty,a]) + V(f,[a,\infty)).\end{aligned} \tag{11}$$

Similarly, in case (ii) the sequence $-\infty < x_0 < x_1 < x_2 < \ldots < x_n \leqslant a$ is clearly contained in $(-\infty, a]$ and hence as above it follows that

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leqslant V(f, (-\infty, a]) + V(f, [a, \infty)). \tag{12}$$

In case (iii) we have

$$\begin{aligned}
&\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \\
&\quad = \sum_{j=1}^{k-1} |f(x_j) - f(x_{j-1})| + |f(x_k) - f(x_{k-1})| + \sum_{j=k+1}^{n} |f(x_j) - f(x_{j-1})| \\
&\quad \leqslant \sum_{j=1}^{k-1} |f(x_j) - f(x_{j-1})| + |f(a) - f(x_{k-1})| + |f(x_k) - f(a)| \\
&\qquad\qquad + \sum_{j=k+1}^{n} |f(x_j) - f(x_{j-1})| \\
&\quad \leqslant V(f, (-\infty, a]) + V(f, [a, \infty)),
\end{aligned} \tag{13}$$

because $-\infty < x_0 < x_1 < x_2 < \ldots < x_{k-1} \leqslant a$ is a finite sequence contained in $(-\infty, a]$ and $a < x_k < x_{k+1} < \ldots < x_n < \infty$ is a finite sequence contained in $[a, \infty)$. From (11)–(13), it follows that in any case

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leqslant V(f, (-\infty, a]) + V(f, [a, \infty)).$$

Now, taking supremum over all finite sequences $\mathscr{S}$, we get

$$V(f) \leqslant V(f, (-\infty, a]) + V(f, [a, \infty)). \tag{14}$$

From (10) and (14), we get (9).

Second, we note that if $-\infty < a < b < \infty$, then by appropriately modifying the above proof we have

$$V(f, [a, \infty)) = V(f, [a, b]) + V(f, [b, \infty))$$

so from (9) we have

$$V(f) = V(f, (-\infty, a]) + V(f, [a, b]) + V(f, [b, \infty)). \tag{15}$$

Third, repeatedly using (15), or using, induction principle, we get

$$V(f) = V(f, (-\infty, a_{-n}]) + \sum_{k=-n+1}^{n} V(f, [a_{k-1}, a_k]) + V(f, [a_n, \infty)). \tag{16}$$

for all $n \in \mathbb{N}$. From (16) it follows that if $s_n := \sum_{k=-n+1}^{n} V(f,[a_{k-1},a_k])$ is the nth symmetric partial sum of the series $\sum_{k\in\mathbb{Z}} V(f,[a_{k-1},a_k])$, then $\{s_n\}$ is non-decreasing and bounded above by $V(f)$, so converges. Thus the series on the right-hand side of (8) converges. Further, in view of (2), (4), (6), and (7) it follows that

$$\lim_{n\to\infty} V(f,(-\infty,a_{-n}]) = 0 \quad \text{and} \quad \lim_{n\to\infty} V(f,[a_n,\infty)) = 0. \tag{17}$$

So, taking the limit as $n \to \infty$ in (16), in view of (17), we get (8) to be proved. □

Proof of Theorem 1. We present a proof using Taibleson [16] like technique developed for $\mathbb{R}$. Fix $0 \neq t \in \mathbb{R}$, and put $a_k := 2k\pi/|t|$ for $k \in \mathbb{Z}$. Since the function e^{-itx} is periodic with a period $2\pi/|t|$, it follows that

$$\int_{a_{k-1}}^{a_k} e^{-itx} dx = 0, \quad k \in \mathbb{Z}. \tag{18}$$

Define a step function g on $\mathbb{R}$ by $g(x) := f(a_{k-1})$ on $[a_{k-1},\, a_k)$, $k \in \mathbb{Z}$. Then in view of (18) for each $k \in \mathbb{Z}$ we have

$$\int_{a_{k-1}}^{a_k} g(x)e^{-itx} dx = f(a_{k-1}) \int_{a_{k-1}}^{a_k} e^{-itx} dx = 0. \tag{19}$$

Therefore, by definition (5) and in view of (19), we have

$$\begin{aligned}
2\pi|\hat{f}(t)| &= \left| \int_{\mathbb{R}} f(x)e^{-itx} dx \right| \\
&= \left| \sum_{k\in\mathbb{Z}} \int_{a_{k-1}}^{a_k} f(x)e^{-itx} dx \right| \\
&= \left| \sum_{k\in\mathbb{Z}} \int_{a_{k-1}}^{a_k} [f(x)-g(x)]e^{-itx} dx \right| \\
&\leqslant \sum_{k\in\mathbb{Z}} \int_{a_{k-1}}^{a_k} |f(x)-g(x)| dx \\
&= \sum_{k\in\mathbb{Z}} \int_{a_{k-1}}^{a_k} |f(x)-f(a_{k-1})| dx \\
&= \sum_{k\in\mathbb{Z}} V(f,[a_{k-1},a_k])(a_k - a_{k-1}) \\
&= \sum_{k\in\mathbb{Z}} V(f,[a_{k-1},a_k]) \frac{2\pi}{|t|} \\
&= \frac{2\pi}{|t|} V(f).
\end{aligned}$$

Note that we have used Lemma 1 in the last step. Therefore we have

$$|\hat{f}(t)| \leqslant \frac{V(f)}{|t|}. \tag{20}$$

This completes the proof of the Theorem 1. □

PROBLEM 1. *What can be said about the exactness of the constant in (20)?*

3. Two-dimensional case

There is a number of definitions extending the concept of bounded variation for functions of two variables defined on closed and bounded rectangle (see, e.g., [1], [2], [8]). We recall one of them.

Let $R := [a_1,b_1] \times [a_2,b_2]$ be a closed and bounded rectangle on the real plane $\mathbb{R}^2$. We recall that (see, e.g., [6, p. 21]) a collection of points (x_0,y_0), (x_0,y_1), $\ldots$, (x_m,y_n) in R, where $m,n \in \mathbb{N}$, satisfying

$$a_1 = x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_m = b_1 \text{ and } a_2 = y_0 \leqslant y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n = b_2$$

is called a collection of grid points of R. If P is any such collection of grid points of R and $f : R \to \mathbb{C}$ is any function we put

$$S(P,f) = \sum_{j=1}^{m} \sum_{k=1}^{n} |f(x_j,y_k) - f(x_{j-1},y_k) - f(x_j,y_{k-1}) + f(x_{j-1},y_{k-1})|. \tag{21}$$

Now, such a function $f : R \to \mathbb{C}$ is said to be of bounded variation over the rectangle R in the sense of Vitali (-Lebesgue, -Fréchet, -de la Vallée Poussin, as indicated in [2]), in symbol: $f \in BV_V(R)$, if

$$V(f) = V(f,R) := \sup S(P,f) < \infty, \tag{22}$$

where the supremum is extended over all collections P of grid points of R, while $V(f)$ defined in (22) is called the total variation of f over R.

Next, we recall the concept of bounded variation for functions on $\mathbb{R}^2$ which is defined as follows (see, e.g., [13, Section 2]). To do this, analogous to the grid points of a rectangle as defined above, we say that a collection of points $(x_0,y_0), (x_0,y_1), \ldots, (x_m,y_n)$ in $\mathbb{R}^2$, where $m,n \in \mathbb{N}$, satisfying

$$-\infty < x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_m < \infty$$

and

$$-\infty < y_0 \leqslant y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n < \infty$$

is called a collection of grid points of $\mathbb{R}^2$. If P is any such collection of grid points of $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{C}$ is any function we define $S(P,f)$ as in (21).

Now, such a function $f : \mathbb{R}^2 \to \mathbb{C}$ is said to be of bounded variation over the real plane $\mathbb{R}^2$ in the sense of Vitali, in symbol: $f \in BV_V\left(\mathbb{R}^2\right)$, if

$$V(f) = V\left(f,\mathbb{R}^2\right) := \sup S(P,f) < \infty, \tag{23}$$

where the supremum is extended over all collections P of grid points of $\mathbb{R}^2$, while $V(f)$ defined in (23) is called the total variation of f over $\mathbb{R}^2$.

Similarly to the case of functions $f \in BV(\mathbb{R})$, the above definition can also be equivalently reformulated as follows. A function $f : \mathbb{R}^2 \to \mathbb{C}$ is of bounded variation over $\mathbb{R}^2$ in the sense of Vitali if and only if f is of bounded variation over all finite rectangles

$$[a_1,b_1]\times[a_2,b_2], \quad -\infty < a_1 < b_1 < \infty \text{ and } -\infty < a_2 < b_2 < \infty;$$

in the sense of Vitali, and in addition, the set of the total variations of f over all finite rectangles $[a_1,b_1]\times[a_2,b_2]$ is bounded. Furthermore, if this is the case, then the supremum of the set of these total variations over all closed and bounded rectangles $[a_1,b_1]\times[a_2,b_2]$ is equal to $V(f)$ defined in (23).

In a similar way, one can define the notion of bounded variation in the sense of Vitali over the closed half planes of the form $[a_1,\infty)\times\mathbb{R}$, or $(-\infty,b_1]\times\mathbb{R}$, or $\mathbb{R}\times[a_2,\infty)$, or $\mathbb{R}\times(-\infty,b_2]$; the closed infinite strips of the form $[a_1,b_1]\times\mathbb{R}$ or $\mathbb{R}\times[a_2,b_2]$; or closed infinite corners of the form $[a_1,\infty)\times[a_2,\infty)$, or $[a_1,\infty)\times(-\infty,b_2]$, $(-\infty,b_1]\times(-\infty,b_2]$, or $(-\infty,b_1]\times[a_2,\infty)$, where a_1,b_1,a_2,b_2 are any real numbers.

Next, we recall that for a complex-valued Lebesgue integrable function f on $\mathbb{R}^2$, in symbol $f \in L^1(\mathbb{R}^2)$, the Fourier transform of f is defined as

$$\hat{f}(\xi,\eta) := \frac{1}{4\pi^2}\iint_{\mathbb{R}^2} f(x,y)e^{-i(\xi x+\eta y)}dxdy, \quad (\xi,\eta)\in\mathbb{R}^2. \tag{24}$$

In view of the Riemann-Lebesgue lemma (see, e.g., [15, Theorem 1.2]), we know that $\hat{f}(\xi,\eta)| \to 0$ as $|(\xi,\eta)| := \sqrt{\xi^2+\eta^2} \to \infty$. But in general, there is no definite rate at which the Fourier transform tends to zero. In fact, the Fourier transform of an integrable function can tend to zero as slowly as we wish (see, e.g., [9, 32.47 (b)]). Therefore, as in the one-dimensional case, it is interesting to know for functions of which subclasses of $L^1(\mathbb{R}^2)$ there is a definite rate at which the Fourier transform tends to zero. In this section, we carry out this study for functions of bounded variation on $\mathbb{R}^2$ in the sense of Vitali. Our main theorem of this section is as follows.

THEOREM 2. *If $f \in L^1(\mathbb{R}^2)\cap BV_V(\mathbb{R}^2)$ and $(\xi,\eta)\in\mathbb{R}^2$ is such that $\xi\eta \neq 0$ then*

$$\hat{f}(\xi,\eta) = O\left(\frac{1}{|\xi\eta|}\right).$$

We need the following lemma.

LEMMA 2. *If $f \in BV_V(\mathbb{R}^2)$ and if $\{a_n : n\in\mathbb{Z}\}$ and $\{b_n : n\in\mathbb{Z}\}$ are two doubly infinite increasing sequences of real numbers with*

$$\lim_{n\to\infty} a_n = \infty = \lim_{n\to\infty} b_n \tag{25}$$

and

$$\lim_{n\to\infty} a_{-n} = -\infty = \lim_{n\to\infty} b_{-n}, \tag{26}$$

then

$$V(f) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} V(f,[a_{m-1},a_m]\times[b_{n-1},b_n]), \tag{27}$$

the series on right-hand side being convergent in the Pringsheim's sense.

Proof. First, we show that for any $(a,b) \in \mathbb{R}^2$, we have

$$V(f) = V(f,(-\infty,a] \times (-\infty,b]) + V(f,(-\infty,a] \times [b,\infty)) + V(f,[a,\infty) \times (-\infty,b]) + V(f,[a,\infty) \times [b,\infty)). \tag{28}$$

We shall prove (28) in the lines of the proof given by Ghorpade and Limaye [6, Lemma 1.16]. We note that if P_1, P_2, P_3, and P_4 are (finite) collections of grid points of the infinite corners $(-\infty,a] \times (-\infty,b]$, $(-\infty,a] \times [b,\infty)$, $[a,\infty) \times (-\infty,b]$, and $[a,\infty) \times [b,\infty)$, respectively, then by collating the grid points of P_1, P_2, P_3, and P_4, we obtain a set P of grid points of $\mathbb{R}^2$ with $S(P,f) = \sum_{i=1}^{4} S(P_i,f)$. In particular, $0 \leqslant S(P_i,f) \leqslant S(P,f) \leqslant V(f,\mathbb{R}^2)$ for $i = 1,2,3,4$. This shows that f is of bounded variation in the sense of Vitali over the infinite corners $(-\infty,a] \times (-\infty,b]$, $(-\infty,a] \times [b,\infty)$, $[a,\infty) \times (-\infty,b]$, and $[a,\infty) \times [b,\infty)$. Moreover,

$$S(P,f) \leqslant V(f,(-\infty,a] \times (-\infty,b]) + V(f,(-\infty,a] \times [b,\infty)) + V(f,[a,\infty) \times (-\infty,b]) + V(f,[a,\infty) \times [b,\infty)).$$

Now, let Q be any (finite) collection of grid points of $\mathbb{R}^2$. Let P be obtained by adjoining (a,b) to Q. Then clearly, $S(Q,f) \leqslant S(P,f)$. Further, P can be regarded as a collection of grid points of $\mathbb{R}^2$ obtained by collating certain grid points of the corners $(-\infty,a] \times (-\infty,b]$, $(-\infty,a] \times [b,\infty)$, $[a,\infty) \times (-\infty,b]$, and $[a,\infty) \times [b,\infty)$, and hence

$$S(Q,f) \leqslant S(P,f) \leqslant V(f,(-\infty,a] \times (-\infty,b]) + V(f,(-\infty,a] \times [b,\infty)) + V(f,[a,\infty) \times (-\infty,b]) + V(f,[a,\infty) \times [b,\infty)).$$

Since Q is any arbitrary collection of grid points of $\mathbb{R}^2$, it follows that

$$V(f,\mathbb{R}^2) \leqslant V(f,(-\infty,a] \times (-\infty,b]) + V(f,(-\infty,a] \times [b,\infty)) + V(f,[a,\infty) \times (-\infty,b]) + V(f,[a,\infty) \times [b,\infty)). \tag{29}$$

To prove the reverse inequality to (29), let $\varepsilon > 0$ be given. Then, by the definition of total variation, there exist collections P_1, P_2, P_3, and P_4 of grid points of the corners $(-\infty,a] \times (-\infty,b]$, $(-\infty,a] \times [b,\infty)$, $[a,\infty) \times (-\infty,b]$, and $[a,\infty) \times [b,\infty)$ respectively, such that

$$\begin{aligned} V(f,(-\infty,a] \times (-\infty,b]) - \frac{\varepsilon}{4} &< S(P_1,f); \\ V(f,(-\infty,a] \times [b,\infty)) - \frac{\varepsilon}{4} &< S(P_2,f); \\ V(f,[a,\infty) \times (-\infty,b]) - \frac{\varepsilon}{4} &< S(P_3,f); \\ V(f,[a,\infty) \times [b,\infty)) - \frac{\varepsilon}{4} &< S(P_4,f). \end{aligned}$$

Let P denote the collection of grid points of $\mathbb{R}^2$ obtained by collating the grid points of P_1, P_2, P_3, and P_4. Then by above inequalities we have

$$\begin{aligned} &V(f,(-\infty,a] \times (-\infty,b]) + V(f,(-\infty,a] \times [b,\infty)) \\ &\quad + V(f,[a,\infty) \times (-\infty,b]) + V(f,[a,\infty) \times [b,\infty)) - \varepsilon \\ &\quad < S(P_1,f) + S(P_2,f) + S(P_3,f) + S(P_4,f) = S(P,f) \leqslant V(f,\mathbb{R}^2). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\begin{aligned} &V(f,(-\infty,a]\times(-\infty,b]) + V(f,(-\infty,a]\times[b,\infty)) \\ &\quad + V(f,[a,\infty)\times(-\infty,b]) + V(f,[a,\infty)\times[b,\infty)) \leqslant V(f,\mathbb{R}^2). \end{aligned} \tag{30}$$

In view of (29) and (30) we get (28).

Second, if (a,b) and (c,d) are points in $\mathbb{R}^2$ with $a \leqslant c$ and $b \leqslant d$, then modifying above appropriately, we can show that

$$\begin{aligned} &V(f,[a,\infty)\times[b,\infty)) = V(f,[a,c]\times[d,\infty)) \\ &\quad + V(f,[c,\infty)\times[d,\infty)) + V(f,[c,\infty)\times[b,d]) + V(f,[a,c]\times[b,d]), \end{aligned}$$

so from (28) we get

$$\begin{aligned} V(f) =& V(f,(-\infty,a]\times(-\infty,b]) + V(f,(-\infty,a]\times[b,\infty)) \\ &+ V(f,[a,\infty)\times(-\infty,b]) + V(f,[a,c]\times[d,\infty)) \\ &+ V(f,[c,\infty)\times[d,\infty)) + V(f,[c,\infty)\times[b,d]) + V(f,[a,c]\times[b,d]). \end{aligned} \tag{31}$$

Third, repeatedly using (31), or using, induction principle, we get

$$\begin{aligned} V(f) = &\big\{V(f,(-\infty,a_{-m}]\times(-\infty,b_{-n}]) + V(f,(-\infty,a_{-m}]\times[b_{-n},\infty)) \\ &+ V(f,[a_{-m},\infty)\times(-\infty,b_{-n}])\big\} + \big\{V(f,[a_{-m},a_m]\times[b_n,\infty)) \\ &+ V(f,[a_m,\infty)\times[b_n,\infty)) + V(f,[a_m,\infty)\times[b_{-n},b_n])\big\} \\ &+ \sum_{j=-m+1}^{m}\sum_{k=-n+1}^{n} V(f,[a_{j-1},a_j]\times[b_{k-1},b_k]) \end{aligned} \tag{32}$$

We claim that the sum of the terms in the braces in the right-hand side in above equality tend to zero as $m,n\to\infty$. Since

$$\begin{aligned} &V(f,[a_{-m},a_m]\times[b_n,\infty)) + V(f,[a_m,\infty)\times[b_n,\infty)) + V(f,[a_m,\infty)\times[b_{-n},b_n]) \\ &\leqslant V(f,(-\infty,a_m]\times[b_n,\infty)) + V(f,[a_m,\infty)\times[b_n,\infty)) + V(f,[a_m,\infty)\times(-\infty,b_n]), \end{aligned}$$

in view of (25) and (26), it is enough to prove the following identities:

$$\begin{aligned} \lim_{x,y\to-\infty} &\big\{V(f,(-\infty,x]\times(-\infty,y]) + V(f,(-\infty,x]\times[y,\infty)) \\ &\qquad + V(f,[x,\infty)\times(-\infty,y])\big\} = 0 \end{aligned} \tag{33}$$

and

$$\begin{aligned} \lim_{x,y\to\infty} &\big\{V(f,(-\infty,x]\times[y,\infty)) + V(f,[x,\infty)\times[y,\infty)) \\ &\qquad + V(f,[x,\infty)\times[-\infty,y)\big\} = 0. \end{aligned} \tag{34}$$

To prove (33), let $\varepsilon > 0$ be given. Then there exists a set P of grid points $(x_0,y_0),\dots,$ (x_m,y_n) $(m,n\in\mathbb{N})$ of $\mathbb{R}^2$ such that $V(f)-\varepsilon < S(P,f)$. Let $x < x_0$ and $y < y_0$. Then

$$\begin{aligned} V(f)-\varepsilon < S(P,f) &\leqslant S(P,f) + |f(x_0,y_0) - f(x_0,y) - f(x,y_0) + f(x,y)| \\ &\leqslant V(f,[x,\infty)\times[y,\infty)). \end{aligned}$$

Therefore,

$$V(f) - V(f,[x,\infty) \times [y,\infty)) < \varepsilon,$$

that is,

$$\{V(f,(-\infty,x] \times (-\infty,y]) + V(f,(-\infty,x] \times [y,\infty)) \\ + V(f,[x,\infty) \times (-\infty,y])\} < \varepsilon.$$

This proves (33). Proof of (34) is similar and left to the reader.

Note that if $s_{m,n} := \sum_{j=-m+1}^{m} \sum_{k=-n+1}^{n} V(f,[a_{j-1},a_j] \times [b_{k-1},b_k])$ is the (m,n)th symmetric partial sum of the series $\sum_{j,k\in\mathbb{Z}} V(f,[a_{j-1},a_j] \times [b_{k-1},b_k])$, then $\{s_{m,n}\}$ is a non-decreasing function of both m and n and the double sequence $\{s_{m,n} : m,n \in \mathbb{N}\}$ is bounded above by $V(f)$, so converges in Pringsheim's sense. Thus the series the right-hand side of (27) converges in Pringsheim's sense.

Now, taking limit as $m,n \to \infty$ in (32), in view of (25), (26), (33), and (34), we get (27) to be proved. □

Proof of Theorem 2. As in the proof of Theorem 1, here we present a proof using Taibleson [16] like technique developed in [5] for $\mathbb{R}^2$. Let $(\xi,\eta) \in \mathbb{R}^2$ be such that $\xi \neq 0$, $\eta \neq 0$. Then the functions $e^{-i\xi x}$ and $e^{-i\eta y}$ are periodic functions of periods $\frac{2\pi}{|\xi|}$ and $\frac{2\pi}{|\eta|}$ respectively. Thus by putting

$$a_r = r \cdot \frac{2\pi}{|\xi|}, \quad b_s = s \cdot \frac{2\pi}{|\eta|} \quad (r,s \in \mathbb{Z})$$

we get

$$\int_{a_{r-1}}^{a_r} e^{-i\xi x} dx = 0, \quad \int_{b_{s-1}}^{b_s} e^{-i\eta y} dy = 0 \quad (r,s \in \mathbb{Z}). \tag{35}$$

Define three functions f_1, f_2, f_3 on $\mathbb{R}^2$ by setting

$$f_1(x,y) = f(x,b_{s-1}) \ \ (x \in \mathbb{R};\ b_{s-1} \leqslant y < b_s) \text{ for } s \in \mathbb{Z};$$

$$f_2(x,y) = f(a_{r-1},y) \ \ (a_{r-1} \leqslant x < a_r;\ y \in \mathbb{R}) \text{ for } r \in \mathbb{Z};$$

and

$$f_3(x,y) = f(a_{r-1},b_{s-1}) \ \ (a_{r-1} \leqslant x < a_r;\ b_{s-1} \leqslant y < b_s) \text{ for } r,s \in \mathbb{Z}.$$

Then in view of Fubini's theorem and relations (35), for each $r,s \in \mathbb{Z}$, we have

$$\int_{a_{r-1}}^{a_r} \int_{b_{s-1}}^{b_s} f_1(x,y) e^{-i\xi x} e^{-i\eta y} dx dy = \int_{a_{r-1}}^{a_r} \left[f(x,b_{s-1}) \int_{b_{s-1}}^{b_s} e^{-i\eta y} dy \right] e^{-i\xi x} dx = 0,$$

$$\int_{a_{r-1}}^{a_r} \int_{b_{s-1}}^{b_s} f_2(x,y) e^{-i\xi x} e^{-i\eta y} dx dy = \int_{b_{s-1}}^{b_s} \left[f(a_{r-1},y) \int_{a_{r-1}}^{a_r} e^{-i\xi x} dx \right] e^{-i\eta y} dy = 0$$

and

$$\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} f_3(x,y)e^{-i\xi x}e^{-i\eta y}dxdy = f(a_{r-1},b_{s-1})\left[\int_{a_{r-1}}^{a_r} e^{-i\xi x}dx\right]\left[\int_{b_{s-1}}^{b_s} e^{-i\eta y}dy\right] = 0.$$

Using these equations in the definition (24) of $\hat{f}(\xi,\eta)$ we get

$$\begin{aligned}
(2\pi)^2|\hat{f}(\xi,\eta))| &= \left|\iint_{\mathbb{R}^2} f(x,y)e^{-i\xi x}e^{-i\eta y}dxdy\right| \\
&= \left|\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} f(x,y)e^{-i\xi x}e^{-i\eta y}dxdy\right| \\
&= \left|\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} [f(x,y)-f_1(x,y)-f_2(x,y)+f_3(x,y)]e^{-i\xi x}e^{-i\eta y}dxdy\right| \\
&\leqslant \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} |f(x,y)-f_1(x,y)-f_2(x,y)+f_3(x,y)|dxdy \\
&= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} |f(x,y)-f(x,b_{s-1})-f(a_{r-1},y)+f(a_{r-1},b_{s-1})|dxdy \\
&\leqslant \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} V(f,[a_{r-1},a_r]\times[b_{s-1},b_s])(a_r-a_{r-1})(b_s-b_{s-1}) \\
&= \frac{(2\pi)^2}{|\xi\eta|}V(f,\mathbb{R}^2),
\end{aligned}$$

in view of Lemma 2. Thus we get

$$|\hat{f}(\xi,\eta)| \leqslant \frac{V(f,\mathbb{R}^2)}{|\xi\eta|}. \tag{36}$$

The proof of Theorem 2 is complete. □

PROBLEM 2. *How to estimate* $\hat{f}(\xi,0)$, $\xi \neq 0$ *(respectively,* $\hat{f}(0,\eta)$, $\eta \neq 0$*) in terms of* $|\xi|$ *(respectively,* $|\eta|$*), even assuming that* f *is of bounded variation over* $\mathbb{R}^2$ *in the sense of Hardy (see,* [13] *for definition)?*

PROBLEM 3. *What can be said about the exactness of the constant in* (36)*?*

4. Extension of the result to $\mathbb{R}^N$, $N \in \mathbb{N}$

We start by defining the concept of bounded variation for functions on $\mathbb{R}^N$ $(N \in \mathbb{N})$ in the sense of Vitali.

For a function $f : \mathbb{R}^N \to \mathbb{C}$ and for any rectangle $R = [\alpha_1,\beta_1]\times\ldots\times[\alpha_N,\beta_N]$ in $\mathbb{R}^N$ with $-\infty < \alpha_i < \beta_i < \infty$ for all $i = 1,2,\ldots,N$, we define $\Delta f(R)$ as follows: When $N = 2$ we put

$$\Delta f(R) := \Delta f([\alpha_1,\beta_1]\times[\alpha_2,\beta_2]) = f(\beta_1,\beta_2)-f(\beta_1,\alpha_2)-f(\alpha_1,\beta_2)+f(\alpha_1,\alpha_2);$$

for $N=3$

$$\begin{aligned}\Delta f(R) :&= \Delta f([\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\times[\alpha_3,\beta_3])\\ &= [f(\beta_1,\beta_2,\beta_3)-f(\beta_1,\alpha_2,\beta_3)-f(\alpha_1,\beta_2,\beta_3)+f(\alpha_1,\alpha_2,\beta_3)]\\ &\qquad -[f(\beta_1,\beta_2,\alpha_3)-f(\beta_1,\alpha_2,\alpha_3)-f(\alpha_1,\beta_2,\alpha_3)+f(\alpha_1,\alpha_2,\alpha_3)]\\ &= \Delta_{[\alpha_3,\beta_3]}\Delta f([\alpha_1,\beta_1]\times[\alpha_2,\beta_2]), say;\end{aligned}$$

and successively for any $N \geqslant 3$

$$\Delta f(R) := \Delta f([\alpha_1,\beta_1]\times\ldots\times[\alpha_N,\beta_N]) = \Delta_{[\alpha_N,\beta_N]}\Delta f([\alpha_1,\beta_1]\times\ldots\times[\alpha_{N-1},\beta_{N-1}]).$$

A collection of points $(x_1^0,\ldots,x_N^0)$, $(x_1^0,\ldots,x_{N-1}^0,x_N^1)$, $\ldots$, $(x_1^{s_1},\ldots,x_N^{s_N})$ of $\mathbb{R}^N$ satisfying

$$-\infty < x_j^0 \leqslant x_j^1 \leqslant \ldots \leqslant x_j^{s_j} \leqslant \infty,\ \ s_j \in \mathbb{N};\ \ j=1,2,\ldots,N,$$

is called a collection of grid points of $\mathbb{R}^N$. If P is any such collection of grid points of $\mathbb{R}^N$ and $f:\mathbb{R}^N \to \mathbb{C}$ is any function we put

$$S(P,f) = \sum_{i_1=1}^{s_1}\cdots\sum_{i_N=1}^{s_N}\left|\Delta f\left([x_1^{i_1-1},x_1^{i_1}]\times\ldots\times[x_N^{i_N-1},x_N^{i_N}]\right)\right|.$$

Now, a function $f:\mathbb{R}^N\to\mathbb{C}$ is said to be of bounded variation over the Euclidean space $\mathbb{R}^N$ in the sense of Vitali, in symbol: $f \in BV_V\left(\mathbb{R}^N\right)$, if

$$V(f) = V\left(f,\mathbb{R}^N\right) := \sup S(P,f) < \infty, \tag{37}$$

where the supremum is extended over all collections P of grid points of $\mathbb{R}^N$, while $V(f)$ defined in (37) is called the total variation of f over $\mathbb{R}^N$.

Similarly to the case of functions $f \in BV(\mathbb{R})$ and $f \in BV_V(\mathbb{R}^2)$, the above definition can also be equivalently reformulated as follows. A function $f:\mathbb{R}^N\to\mathbb{C}$ is of bounded variation over $\mathbb{R}^N$ in the sense of Vitali if and only if f is of bounded variation over all closed and bounded N-rectangles

$$[a_1,b_1]\times\ldots\times[a_N,b_N],\ \ -\infty < a_i < b_i < \infty,\ \ i=1,2,\ldots,N;$$

in the sense of Vitali (see, e.g., [10] or [4] for definition), and in addition, the set of the total variations of f over all closed and bounded N-rectangles $[a_1,b_1]\times\ldots\times[a_N,b_N]$ is bounded. Furthermore, if this is the case, then the supremum of the set of these total variations over all closed and bounded N-rectangles $[a_1,b_1]\times\ldots\times[a_N,b_N]$ is equal to $V(f)$ defined in (37).

In a similar way, one can define the notion of bounded variation in the sense of Vitali over the closed half space $[a_1,\infty)\times\mathbb{R}^{N-1}$ or a closed N-dimensional infinite corner $[a_1,\infty)\times\ldots\times[a_N,\infty)$, etc.

Next, we recall that for a complex-valued Lebesgue integrable function f on $\mathbb{R}^N$, in symbol: $f \in L^1(\mathbb{R}^N)$, the Fourier transform of f is defined as

$$\hat{f}(\xi_1,\ldots,\xi_N) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} f(x_1,\ldots,x_N)e^{-i(\xi_1x_1+\ldots+\xi_Nx_N)}dx_1\ldots dx_N,$$

where $(\xi_1,\ldots,\xi_N)\in\mathbb{R}^N$.

In this case also by the Riemann-Lebesgue lemma (see, e.g., [15, Theorem 1.2]), we know that $|\hat{f}(\xi_1,\ldots,\xi_N)|\to 0$ as $|(\xi_1,\ldots,\xi_N)|:=\sqrt{\xi_1^2+\ldots+\xi_N^2}\to\infty$. But in general, there is no definite rate at which the Fourier transform tends to zero. In fact, the Fourier transform of an integrable function can tend to zero as slowly as we wish (see, e.g., [9, 32.47 (b)]). Therefore, as in one and two dimensional cases, it is interesting to know for functions of which subclasses of $L^1(\mathbb{R}^N)$ there is a definite rate at which the Fourier transform tends to zero. In this section, we state the result for functions of bounded variation on $\mathbb{R}^N$ in the sense of Vitali, which is an extension of our theorems in Sections 2 and 3, as follows. The proof of this theorem is similar to that of Theorem 2.

THEOREM 3. *If* $f\in L^1(\mathbb{R}^N)\cap BV_V(\mathbb{R}^N)$ *and* $(\xi_1,\ldots,\xi_N)\in\mathbb{R}^N$ *is such that* $\prod_{i=1}^N\xi_i\neq 0$ *then*

$$\hat{f}(\xi_1,\ldots,\xi_N)=O\left(\frac{1}{|\prod_{i=1}^N\xi_i|}\right). \tag{38}$$

PROBLEM 4. *How to estimate* $\hat{f}(\xi_1,\ldots,\xi_N)$, *if* $(\xi_1,\ldots,\xi_N)\neq(0,\ldots,0)$, *but at least one* $\xi_j=0$ *for some* $j\in\{1,\ldots,N\}$, *even assuming that* f *is of bounded variation over* $\mathbb{R}^N$ *in the sense of Hardy (which can be defined similarly as in the case of* $\mathbb{R}^2$ *)?*

PROBLEM 5. *What can be said about the exactness of the constant in* (38)*?*

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