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CONVERGENCE IN NORM OF LOGARITHMIC MEANS OF MULTIPLE FOURIER SERIES

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Abstract. The maximal Orlicz space such that the mixed logarithmic means of multiple Fourier series for the functions from this space converge in L_1-norm is found.

Let $\mathbb{T}^d := [-\pi,\pi)^d$ denote a cube in the d-dimensional Euclidean space $\mathbb{R}^d$. The elements of $\mathbb{R}^d$ are denoted by $\mathbf{x} := (x_1,\dots,x_d)$.

Let $D := \{1,2,\dots,d\}$, $B := \{l_1,l_2,\dots,l_r\}$, $1 \leqslant r \leqslant d$, $B \subset D$, $l_k < l_{k+1}$, $k = 1,2,\dots,r-1$, $B' := D \backslash B$. For any $\mathbf{x} = (x_1,\dots,x_d)$ and any $B \subset D$, denote $\mathbf{x}_B := (x_{l_1},x_{l_2},\dots,x_{l_r}) \in \mathbb{R}^r$. We assume that $|B|$ is the number of elements of B. If $B \neq \varnothing$, then for any natural numbers n we suppose that $n(B) := (n,n,\dots,n) \in \mathbb{R}^{|B|}$. The notation $a \lesssim b$ in the paper stands for $a \leqslant cb$, where c is an absolute constant.

In the sequel we shall identify the symbols

$$\sum_{i_B=0_B}^{n_B} \text{ and } \sum_{i_{l_1}=0}^{n_{l_1}} \cdots \sum_{i_{l_r}=0}^{n_{l_r}}, d\mathbf{t}_B \text{ and } dt_{l_1}\cdots dt_{l_r}.$$

We denote by $L_p\left(\mathbb{T}^d\right)$ the class of all measurable functions f that are 2π-periodic with respect to all variable and satisfy

$$\|f\|_p := \left(\int_{\mathbb{T}^d} |f|^p\right)^{1/p} < \infty.$$

Let $f \in L_1\left(\mathbb{T}^d\right)$. The Fourier series of f with respect to the trigonometric system is the series

$$S[f] := \sum_{n_1,\dots,n_d=-\infty}^{+\infty} \widehat{f}(n_1,\dots,n_d)\, e^{i(n_1x_1+\cdots+n_dx_d)},$$

where

$$\widehat{f}(n_1,\dots,n_d) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x_1,\dots,x_d)e^{-i(n_1x_1+\cdots+n_dx_d)}dx_1\cdots dx_d$$

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are the Fourier coefficients of the function f. The rectangular partial sums are defined as follows:

$$S_{N_D}(f;\mathbf{x}) := \sum_{n_D=-N_D}^{N_D} \widehat{f}(n_1,\dots,n_d)\, e^{i(n_1x_1+\cdots+n_dx_d)}.$$

In the literature, there is known the notion of the Riesz's logarithmic means of a Fourier series. The n-th Riesz logarithmic mean of the Fourier series of the integrable function f is defined by

$$\frac{1}{l_n}\sum_{k=0}^{n}\frac{S_k(f)}{k+1}, \qquad l_n := \sum_{k=0}^{n}\frac{1}{k+1},$$

where $S_k(f)$ is the partial sum of its Fourier series. This Riesz's logarithmic means with respect to the trigonometric system has been studied by a lot of authors. We mention for instance the papers of Szász, and Yabuta [16, 19]. This mean with respect to the Walsh, Vilenkin system is discussed by Simon, and Gát [15, 2].

Let $\{q_k : k \geqslant 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{\sum_{k=0}^{n} q_k}\sum_{k=0}^{n} q_k S_{n-k}(f).$$

If $q_k = \frac{1}{k+1}$, then we get the (Nörlund) logarithmic means:

$$\frac{1}{l_n}\sum_{k=0}^{n}\frac{S_{n-k}(f)}{k+1}.$$

Although, it is a kind of "reverse" Riesz's logarithmic means. In [3] we proved some convergence and divergence properties of the logarithmic means of Walsh-Fourier series of functions in the class of continuous functions, and in the Lebesgue space L.

The Nörlund logarithmic and Reisz logarithmic means of multiple Fourier series are defined by

$$L_{n_D}(f;\mathbf{x}) := \frac{1}{\prod\limits_{i\in D} l_{n_i}}\sum_{i_D=0_D}^{n_D}\frac{S_{n_D-i_D}(f;\mathbf{x})}{\prod\limits_{j\in D}(i_j+1)},$$

$$R_{n_D}(f;\mathbf{x}) := \frac{1}{\prod\limits_{i\in D} l_{n_i}}\sum_{i_D=0_D}^{n_D}\frac{S_{i_D}(f;\mathbf{x})}{\prod\limits_{j\in D}(i_j+1)}.$$

It is evident that

$$L_{n_D}(f;\mathbf{x}) = \frac{1}{\pi^d}\int\limits_{\mathbb{T}^d} f(\mathbf{t})\, F_{n_D}(\mathbf{x}-\mathbf{t})\, d\mathbf{t}$$

and

$$R_{n_D}(f;\mathbf{x}) = \frac{1}{\pi^d}\int\limits_{\mathbb{T}^d} f(\mathbf{t})\, G_{n_D}(\mathbf{x}-\mathbf{t})\, d\mathbf{t},$$

where

$$F_{n_D}(\mathbf{x}) := \prod_{j\in D} F_{n_j}(x_j), \qquad G_{n_D}(\mathbf{x}) := \prod_{j\in D} G_{n_j}(x_j),$$

$$F_n(u) := \frac{1}{l_n}\sum_{i=0}^{n}\frac{D_{n-i}(u)}{i+1}, \qquad G_n(u) := \frac{1}{l_n}\sum_{i=0}^{n}\frac{D_i(u)}{i+1}.$$

Let $B \subset D$. Then the mixed logarithmic means of multiple Fourier series are defined by

$$\left(L_{n_B}\circ R_{n_{B'}}\right)(f;\mathbf{x}) := \frac{1}{\prod\limits_{i\in D} l_{n_i}}\sum_{i_D=0_D}^{n_D}\frac{S_{n_B-i_B,i_{B'}}(f;\mathbf{x})}{\prod\limits_{j\in D}(i_j+1)},$$

where

$$S_{n_B-i_B,i_{B'}}(f;\mathbf{x}) := \sum_{l_B=-(n_B-i_B)}^{n_B-i_B}\sum_{l_{B'}=-i_{B'}}^{i_{B'}}\widehat{f}(l_1,\ldots,l_d)\,e^{i(l_1x_1+\cdots+l_dx_d)}.$$

It is easy to show that

$$\left(L_{n_B}\circ R_{n_{B'}}\right)(f;\mathbf{x}) = \frac{1}{\pi^d}\int\limits_{\mathbb{T}^d} f(\mathbf{t})\,F_{n_B}(\mathbf{x}_B-\mathbf{t}_B)\,G_{n_{B'}}(\mathbf{x}_{B'}-\mathbf{t}_{B'})\,d\mathbf{t}$$

We denote by $L_0(\mathbb{T}^d)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbb{T}^d$. $\operatorname{mes}(A)$ is the Lebesgue measure of the set $A\subset\mathbb{T}^d$.

Let $L_Q = L_Q(\mathbb{T}^d)$ be the Orlicz space [13] generated by Young function Q, i.e. Q is convex continuous even function such that $Q(0)=0$ and

$$\lim_{u\to+\infty}\frac{Q(u)}{u} = +\infty, \quad \lim_{u\to 0}\frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_Q(\mathbb{T}^d)} = \inf\{k>0 : \int\limits_{\mathbb{T}^d} Q(|f|/k) \leqslant 1\}.$$

In particular, if $Q(u) = u\log^r(1+u)$, $r=1,2,\ldots,\quad u>0$, then the corresponding space will be denoted by $L\log^r L(\mathbb{T}^d)$.

The rectangular partial sums of double Fourier series $S_{n,m}(f;x,y)$ of the function $f\in L_p\left(\mathbb{T}^2\right)$, $\mathbb{T}:=[-\pi,\pi)$, $1<p<\infty$ converge in L_p norm to the function f, as $n\to\infty$ [20]. In the case $L_1\left(\mathbb{T}^2\right)$ this result does not hold. But for $f\in L_1(\mathbb{T})$, the operator $S_n(f;x)$ are of weak type (1,1) [21]. This estimate implies convergence of $S_n(f;x)$ in measure on $\mathbb{T}$ to the function $f\in L_1(\mathbb{T})$. However, for double Fourier series this result does not hold [6, 12]. Moreover, it is proved that quadratical partial sums $S_{n,n}(f;x,y)$ of double Fourier series do not converge in two-dimensional measure on $\mathbb{T}^2$ even for functions from Orlicz spaces wider than Orlicz space $L\log L\left(\mathbb{T}^2\right)$. On the other hand, it is well-known that if the function $f\in L\log L\left(\mathbb{T}^2\right)$, then rectangular partial sums $S_{n,m}(f;x,y)$ converge in measure on $\mathbb{T}^2$.

Classical regular summation methods often improve the convergence of Fourier series. For instance, the Fejér means of the double Fourier series of the function $f \in L_1(\mathbb{T}^2)$ converge in $L_1(\mathbb{T}^2)$ norm to the function f [20]. These means present the particular case of the Nörlund means.

It is well known that the method of Nörlund logarithmic means of double Fourier series, is weaker than the Cesáro method of any positive order. In [18] Tkebuchava proved, that these means of double Fourier series in general do not converge in two-dimensional measure on $\mathbb{T}^d$ even for functions from Orlicz spaces wider than Orlicz space $L\log^{d-1}L(\mathbb{T}^d)$. In particular, the following result is true.

THEOREM T. *Let $L_Q(\mathbb{T}^d)$ be an Orlicz space, such that*

$$L_Q\left(\mathbb{T}^d\right) \nsubseteq L\log^{d-1}L\left(\mathbb{T}^d\right).$$

Then the set of the functions from the Orlicz space $L_Q(\mathbb{T}^d)$ with Nörlund logarithmic means of rectangular partial sums of d-dimensional Fourier series, convergent in measure on $\mathbb{T}^d$, is of first Baire category in $L_Q(\mathbb{T}^d)$.

In [9] we considered the strong logarithmic means of rectangular partial sums double Fourier series

$$\sigma_{n,m}(f;x,y) := \frac{1}{l_n l_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\left|S_{i,j}(f;x,y)\right|}{(n-i+1)(m-j+1)}$$

and prove that these means are acting from space $L\log L(\mathbb{T}^2)$ into space $L_p(\mathbb{T}^2)$, $0<p<1$. This fact implies the convergence of strong logarithmic means of rectangular partial sums of double Fourier series in measure on $\mathbb{T}^2$ to the function $f \in L\log L(\mathbb{T}^2)$. Uniting these results with statement from [17] we obtain, that the rectangular partial sums of double Fourier series converge in measure for all functions from Orlicz space if and only if their strong Nörlund logarithmic means converge in measure. Thus, not all classic regular summation methods can improve the convergence in measure of double Fourier series.

The results for summability of logarithmic means of Walsh-Fourier series can be found in [8, 5, 10, 3, 4, 7, 16, 19].

In this paper we consider the mixed logarithmic means $\left(L_{n_B}\circ R_{n_{B'}}\right)(f)$ of rectangular partial sums multiple Fourier series and prove that these means are acting from space $L\log^{|B|}L(\mathbb{T}^d)$ into space $L_1(\mathbb{T}^d)$ (see Theorem 1). This fact implies the convergence of mixed logarithmic means of rectangular partial sums of multiple Fourier series converge in L_1-norm. We also prove sharpness of this result (see Theorem 3). In particular, the following are true.

THEOREM 1. *Let $B \subset D$ and $f \in L\log^{|B|}L(\mathbb{T}^d)$. Then*

$$\left\|\left(L_{n_B}\circ R_{n_{B'}}\right)(f)\right\|_{L_1(\mathbb{T}^d)} \lesssim 1+\left\|\left|f\right|\log^{|B|}\left|f\right|\right\|_{L_1(\mathbb{T}^d)}.$$

THEOREM 2. *Let* $B\subset D$ *and* $f\in L\log^{|B|}L\left(\mathbb{T}^d\right)$. *Then*

$$\left\|\left(L_{n_B}\circ R_{n_{B'}}\right)(f)-f\right\|_{L_1\left(\mathbb{T}^d\right)}\to 0 \;\; as \;\; n_i\to\infty, \;\; i\in D;$$

THEOREM 3. *Let* $L_Q\left(\mathbb{T}^d\right)$ *be an Orlicz space, such that*

$$L_Q\left(\mathbb{T}^d\right)\nsubseteq L\log^{|B|}L\left(\mathbb{T}^d\right).$$

Then

a)

$$\sup_n\left\|\left(L_{n_B}\circ R_{n_{B'}}\right)\right\|_{L_Q\left(\mathbb{T}^d\right)\to L_1\left(\mathbb{T}^d\right)}=\infty;$$

b) there exists a function $f\in L_Q\left(\mathbb{T}^d\right)$ *such that* $\left(L_{n_B}\circ R_{n_{B'}}\right)(f)$ *does not converge to* f *in* $L_1\left(\mathbb{T}^d\right)$*-norm.*

Thus, the space $L\log^{|B|}L\left(\mathbb{T}^d\right)$ is maximal Orlicz space such that for each function f from this space the means $\left(L_{n_B}\circ R_{n_{B'}}\right)(f)$ converge to f in $L_1\left(\mathbb{T}^d\right)$-norm.

Proof of Theorem 1. We apply the following particular case of the Marcinkiewicz interpolation theorem (see [1, 14]). Let $T:L_1\left(T^1\right)\to L_0\left(T^1\right)$ be a quasilinear operator of weak type $(1,1)$ and of type (α,α) for some $1<\alpha<\infty$ at the same time, i. e.

a)

$$\begin{aligned}&\operatorname{mes}\left\{x\in\mathbb{T}^1:\left|T(f,x)\right|>y\right\}\\ \lesssim\;&\frac{1}{y}\int_{\mathbb{T}^1}\left|f(x)\right|dx;\;\;\forall f\in L^1\left(\mathbb{T}^1\right)\;\forall y>0;\end{aligned}\tag{1}$$

b)

$$\|Tf\|_{L^\alpha\left(\mathbb{T}^1\right)}\lesssim\|f\|_{L^\alpha\left(\mathbb{T}^1\right)},\;\forall f\in L^\alpha\left(T^1\right).\tag{2}$$

Then

$$\begin{aligned}&\int_{\mathbb{T}^1}\left|T(f,x)\right|\ln^\beta\left|T(f,x)\right|dx\\ \lesssim\;&\int_{\mathbb{T}^1}\left|f(x)\right|\ln^{\beta+1}\left|f(x)\right|dx+1,\;\forall\beta\geqslant 0.\end{aligned}\tag{3}$$

In [9] it it proved that for any $f\in L_1\left(T^1\right)$ the operator $f*F_n$ has weak type (1,1), i. e.

$$\begin{aligned}&\operatorname{mes}\left\{x\in\mathbb{T}^1:\left|f*F_n\right|>y\right\}\\ \lesssim\;&\frac{1}{y}\int_{\mathbb{T}^1}\left|f(x)\right|dx;\;\;\forall f\in L^1\left(\mathbb{T}^1\right)\;\forall y>0.\end{aligned}\tag{4}$$

Since

$$\sup_n \|G_n\|_1 < \infty,$$

it is easy to prove that the operator $f * G_n$ has type (1,1), i.e.

$$\|f * G_n\|_{L_1(\mathbb{T}^1)} \lesssim \|f\|_{L_1(\mathbb{T}^1)}. \tag{5}$$

From (1)–(5) we have ($B' := \{s_1, s_2, ..., s_{r'}\}$, $B := \{l_1, ..., l_r\}$)

$$\begin{aligned}
&\left\|\left(L_{n_B} \circ R_{n_{B'}}\right)(f)\right\|_{L_1(\mathbb{T}^d)} \\
&= \left\|\left(R_{n_{s_1}} \circ \cdots \circ R_{n_{s_{r'}}} \circ L_{n_{l_1}} \circ \cdots \circ L_{n_{l_r}}\right)(f)\right\|_{L_1(\mathbb{T}^d)} \\
&\lesssim \cdots \lesssim \left\|\left(L_{n_{l_1}} \circ \cdots \circ L_{n_{l_r}}\right)(f)\right\|_{L_1(\mathbb{T}^d)} \\
&\lesssim 1 + \left\|\left|L_{n_{l_2}} \circ \cdots \circ L_{n_{l_r}}(f)\right| \log\left|L_{n_2} \circ \cdots \circ L_{n_{l_r}}(f)\right|\right\|_{L_1(\mathbb{T}^d)} \\
&\lesssim \cdots \lesssim 1 + \left\|\left|L_{n_{l_r}}(f)\right| \log^{r-1}\left|L_{n_{l_r}}(f)\right|\right\|_{L_1(\mathbb{T}^d)} \\
&\lesssim 1 + \left\||f| \log^r |f|\right\|_{L_1(\mathbb{T}^d)}.
\end{aligned}$$

Theorem 1 is proved. □

The validity of Theorem 2 follows immediately from Theorem 1.

Proof of Theorem 3. a) Set

$$\alpha_{mn} := \frac{\pi(12m+1)}{6(2^{2n}+1/2)}, \qquad \beta_{mn} := \frac{\pi(12m+5)}{6(2^{2n}+1/2)}, \qquad \gamma_n := \frac{\pi}{6(2^{2n}+1/2)},$$

$$J_n := \bigcup_{m=1}^{2^{n-1}} [\alpha_{mn} + \gamma_n, \beta_{mn} - \gamma_n].$$

In order to prove theorem we need the following lemma proved in [11].

LEMMA 1. *Let $0 \leqslant z \leqslant \gamma_n$ and $x \in J_n$. Then*

$$F_{2^{2n}}(x-z) \gtrsim \frac{1}{x}.$$

Let

$$Q\left(2^{2n|B|}\right) \gtrsim 2^{2n|B|} \quad \text{for} \quad n > n_0.$$

By virtue of estimate ([13], Ch. 2)

$$\|f\|_{L_Q} \leqslant 1 + \|Q(|f|)\|_{L_1}.$$

We can write

$$\begin{aligned}
&\left\| L_{2^{2n}(B)} \circ R_{2^{2n}(B')} \left(\frac{\mathbf{1}_{[0,\gamma_n]^{|B|}}}{(2\gamma_n)^{|B|}} \right) \right\|_{L_1(\mathbb{T}^d)} \\
&\leqslant \left\| L_{2^{2n}(B)} \circ R_{2^{2n}(B')} \right\|_{L_Q(\mathbb{T}^d) \to L_1(\mathbb{T}^d)} \left\| \frac{\mathbf{1}_{[0,\gamma_n]^{|B|}}}{(2\gamma_n)^{|B|}} \right\|_{L_Q(\mathbb{T}^d)} \\
&\leqslant \left\| L_{2^{2n}(B)} \circ R_{2^{2n}(B')} \right\|_{L_Q(\mathbb{T}^d) \to L_1(\mathbb{T}^d)} \left(1 + \left\| Q \left(\frac{\mathbf{1}_{[0,\gamma_n]^{|B|}}}{(2\gamma_n)^{|B|}} \right) \right\|_{L_1(\mathbb{T}^d)} \right) \\
&\lesssim \left\| L_{2^{2n}(B)} \circ R_{2^{2n}(B')} \right\|_{L_Q(\mathbb{T}^d) \to L_1(\mathbb{T}^d)} \left(1 + \gamma_n^{|B|} Q \left(\frac{1}{(2\gamma_n)^{|B|}} \right) \right) \\
&\lesssim \left\| L_{2^{2n}(B)} \circ R_{2^{2n}(B')} \right\|_{L_Q(\mathbb{T}^d) \to L_1(\mathbb{T}^d)} \frac{Q\left(2^{2n|B|}\right)}{2^{2n|B|}}.
\end{aligned} \tag{6}$$

From Lemma 1 we get

$$\begin{aligned}
&L_{2^{2n}(B)} \circ R_{2^{2n}(B')} \left(\frac{\mathbf{1}_{[0,\gamma_n]^{|B|}}}{(2\gamma_n)^{|B|}} ; \mathbf{x} \right) \\
&= \frac{1}{(2\gamma_n)^{|B|}} \frac{1}{\pi^d} \int\limits_{[0,\gamma_n]^{|B|}} \prod_{j\in B} F_{2^{2n}}\left(x_j - z_j\right) d\mathbf{z}_B \int\limits_{\mathbb{T}^{|B'|}} \prod_{i\in B'} G_{2^{2n}}\left(x_i - z_i\right) d\mathbf{z}_{B'} \\
&= \frac{1}{(2\gamma_n)^{|B|}} \prod_{j\in B} \frac{1}{\pi} \int\limits_{[0,\gamma_n]} F_{2^{2n}}\left(x_j - z_j\right) dz_j \prod_{i\in B'} \frac{1}{\pi} \int\limits_{\mathbb{T}} G_{2^{2n}}\left(z_i\right) dz_i \\
&= \frac{1}{(2\gamma_n)^{|B|}} \prod_{j\in B} \frac{1}{\pi} \int\limits_{[0,\gamma_n]} F_{2^{2n}}\left(x_j - z_j\right) dz_j \\
&\gtrsim \frac{1}{(2\gamma_n)^{|B|}} \prod_{j\in B} \frac{\gamma_n}{x_j}, x_j \in J_n, j \in B.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left\| L_{2^{2n}(B)} \circ R_{2^{2n}(B')} \left(\frac{\mathbf{1}_{[0,\gamma_n]^{|B|}}}{(2\gamma_n)^{|B|}} \right) \right\|_{L_1(\mathbb{T}^d)} \\
&\gtrsim \prod_{j\in B} \int\limits_{J_n} \frac{dx_j}{x_j} \gtrsim c n^{|B|}.
\end{aligned} \tag{7}$$

Combining (6) and (7) we obtain

$$\left\| L_{2^{2n}(B)} \circ R_{2^{2n}(B')} \right\|_{L_Q(\mathbb{T}^d) \to L_1(\mathbb{T}^d)} \gtrsim \frac{2^{2n|B|} n^{|B|}}{Q\left(2^{2n|B|}\right)}. \tag{8}$$

The fact that

$$L_Q\left(\mathbb{T}^d\right) \nsubseteq L\log^{|B|} L\left(\mathbb{T}^d\right)$$

is equivalent to the condition

$$\lim_{u\to\infty}\sup\frac{u\log^{|B|} u}{Q(u)} = \infty.$$

Thus there exists $\{u_k : k \geqslant 1\}$ such that

$$\lim_{k\to\infty}\frac{u_k\log^{|B|} u_k}{Q(u_k)} = \infty, \qquad u_{k+1} > u_k, \qquad k = 1,2,...,$$

and a monotonically increasing sequence of positive integers $\{r_k : k \geqslant 1\}$ such that

$$2^{2|B|r_k} \leqslant u_k < 2^{2|B|(r_k+1)}.$$

Then we have

$$\frac{2^{2r_k|B|}r_k^{|B|}}{Q\left(2^{2r_k|B|}\right)} \gtrsim \frac{u_k\log^{|B|} u_k}{Q(u_k)} \to \infty.$$

Thus from (8) we conclude that

$$\sup_n \left\|\left(L_{n(B)}\circ R_{n(B')}\right)\right\|_{L_Q\left(\mathbb{T}^d\right)\to L_1\left(\mathbb{T}^d\right)} = \infty.$$

This complete the proof of Theorem 3 a). Part b) follows immediately from part a). □

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